Derived Foliations

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Introduction

One important feature of derived algebraic geometry (in the sense of [**TV08**, **Toe14**, **PV21**]) is the observation that singularities of varieties and schemes encountered in real life are most of the time tamed by natural derived structures. This is particularly visible in moduli theory. In many instances, classical/underived moduli spaces are very singular, but their derived counterparts typically have finite dimensional cotangent complexes, and in the best scenarios, local complete intersection singularities. As a result, derived moduli spaces carry important information of geometric origin which can not be seen directly on their classical truncations. Typical examples are: virtual fundamental classes (see [**STV15**, **Kha19**]), and shifted symplectic structures (see [**PTVV13**]).

The main subject of the present book is guided by a similar principle: foliations encountered in real life often come equipped with natural *derived structures* which tame their singularities. More precisely, there exists a natural notion of *derived foliation*, with a natural truncation functor to classical, possibly singular, foliations, and this notion helps when dealing with foliation singularities in general. The purpose of this book is to make precise the notion of derived foliation, and to prove results that support this principle.

From classical foliations to graded mixed algebras. Before presenting the main definition of this book, we would like to contemplate the classical situation. For a smooth algebraic complex variety X, a (algebraic) foliation $\mathcal{F}$ on X is usually defined as a sub-bundle[1] $\mathbb{T}_{\mathcal{F}} \subset \mathbb{T}_X$ of the tangent bundle $\mathbb{T}_X$, which is furthermore stable by the Lie bracket. To avoid possible confusions, we will say that such a $\mathbb{T}_{\mathcal{F}} \subset \mathbb{T}_X$ is a *classical smooth foliation on X*. By duality, the sub-bundle $\mathbb{T}_{\mathcal{F}}$ defines a quotient bundle

$$p : \Omega^1_X \twoheadrightarrow \Omega^1_{\mathcal{F}} = \mathbb{T}^\vee_{\mathcal{F}}$$

where Ω^1_X is the sheaf of Kähler differential forms on X. Using the following standard formula relating the Lie bracket $[-,-]$ of vector fields and the de Rham differential dR (for ω a 1-form, and u and v vector fields on X)

$$dR(\omega)(u,v) = u(\omega(v)) - v(\omega(u)) + \omega([u,v])$$

we see that the kernel $K_{\mathcal{F}} \subset \Omega^1_X$ of the above quotient map p is a differential ideal: the (sheaf) image of $dR : K_{\mathcal{F}} \to \Omega^2_X$ is contained in $K_{\mathcal{F}} \wedge \Omega^1_X$, where $K_{\mathcal{F}} \wedge \Omega^1_X$ denotes the (sheaf) image of the composition

$$K_{\mathcal{F}} \otimes \Omega^1_X \longrightarrow \Omega^1 \otimes \Omega^1_X \xrightarrow{\wedge} \Omega^2_X.$$

Conversely, any differential ideal $K_{\mathcal{F}}$, i.e. a sub-bundle with the property that $dR(K_{\mathcal{F}}) \subset K_{\mathcal{F}} \wedge \Omega^1_X \subset \Omega^2_X$, is of the form $\mathbb{T}^\vee_{\mathcal{F}}$ for a sub-bundle $T_{\mathcal{F}} \subset \mathbb{T}_X$ stable by the Lie bracket. More importantly for our purposes, the differential ideal property of $K_{\mathcal{F}} \subset \Omega^1_X$ is also equivalent to

[1] We warn the reader here that all along this book the expression *sub-bundle* of a vector bundle V means a sub-coherent sheaf $W \subset V$ such that both coherent sheaves W and V/W are in fact vector bundles.

the property that the de Rham differential dR on the graded ring of forms Ω^*_X, descends to a differential $dR_{\mathcal{F}}$ on the quotient graded ring

$$\Omega^*_{\mathcal{F}} = (\Omega^*_X / K_{\mathcal{F}}.\Omega^*_X).$$

It is easy to check the association $\mathcal{F} \mapsto (K_{\mathcal{F}}, dR_{\mathcal{F}})$ provides a one-to-one correspondence between classical smooth foliations $\mathcal{F}$ on X and sub-bundles $K_{\mathcal{F}} \subset \Omega^1_X$ together with a multiplicative differential $dR_{\mathcal{F}}$ on $(\Omega^*_X / K_{\mathcal{F}}.\Omega^*_X)$ compatible with the de Rham differential via the quotient map $\Omega^*_X \twoheadrightarrow \Omega^*_{\mathcal{F}}$.

We now want to consider the sheaf Ω^1_X as sitting in cohomological degree -1, and we thus consider $\Omega^1_X[*] = \oplus_p \Omega^p_X[p]$. This unusual convention, as opposed to Ω^1_X being considered in cohomological degrees 1, is explained by the geometric interpretation of forms in terms of *derived loop spaces* and for which we refer to our sections §2.5.2 and §7.2. The sheaf $\Omega^*_{\mathcal{F}}[*]$ is now a sheaf of bi-graded commutative rings, were the bi-degree is given on the one side by the degree of differential forms, and on the other side by the cohomological degree. The differential $dR_{\mathcal{F}}$ on $\Omega^*_{\mathcal{F}}[*]$ is now a multiplicative differential of bi-degree $(1, -1)$: it increases the degree of forms by 1 but decreases the cohomological degree by 1. Moreover, with these conventions we have

$$\Omega^*_{\mathcal{F}}[*] \simeq Sym_{\mathcal{O}_X}(\Omega^1_X[1]),$$

making the $\Omega^*_{\mathcal{F}}[*]$ into the free bi-graded commutative ring over $\Omega^1_{\mathcal{F}}[1]$. Again, the association $\mathcal{F} \mapsto (\Omega^*_{\mathcal{F}}[*], dR_{\mathcal{F}})$ can be seen to define a one-to-one correspondence between classical smooth foliations $\mathcal{F}$ on X and quotient bi-graded commutative rings $\Omega^*_X[*] \twoheadrightarrow \Omega^*_{\mathcal{F}}[*]$ of the form $Sym_{\mathcal{O}_X}(\Omega^1_{\mathcal{F}}[1])$ such that the de Rham differential descends to a compatible differential on $\Omega^*_{\mathcal{F}}[*]$.

This somehow complicated alternative point of view on classical smooth foliations on X has the advantage of introducing (and motivating) one of the major actor of this book: *graded mixed commutative differential graded algebras*, or in short *graded mixed cdga's*. These are bi-graded commutative $\mathbb{C}$-algebras $A^*(*)$, together with two differentials

$$d : A^*(*) \to A^{*+1}(*) \qquad \epsilon : A^*(*) \to A^{*-1}(* + 1)$$

satisfying natural compatibilities (see Definition 1.4.0.1). In the example above arising from a smooth classical foliation on X, $\Omega^*_{\mathcal{F}}[*]$ is a graded mixed cdga with $d = 0$ and $\epsilon = dR_{\mathcal{F}}$. Clearly, smooth classical foliations on X can be understood as quotients of $\Omega^*_X[*]$ in the category of graded mixed cdga's. This observation, which is not very helpful when dealing only with classical smooth foliations, is central in the definition of derived foliations in general.

The general notion of derived foliations. We turn to a more general situation where X is an affine derived scheme, say over some base field of characteristic zero k. Such a derived scheme $X = \mathbf{Spec}\, A$ is the spectrum of a connective k-linear cdga A. The cdga A possesses a cotangent complex $\mathbb{L}_{A/k}$ (relative to k) which comes equipped with a universal derivation $dR : A \to \mathbb{L}_{A/k}$ (see Appendix §A.2). This derivation can be extended uniquely to an extra differential ϵ on the graded cdga $Sym_A(\mathbb{L}_{A/k}[1])$, making it into a graded mixed cdga. This graded mixed cdga is denoted by $\mathbf{DR}(X/k)$ or $\mathbf{DR}(A/k)$ and is called the *de Rham graded mixed cdga* associated to X. The underlying bi-graded alegbra is $Sym_A(\mathbb{L}_{A/k}[1])$ where the bi-degrees are the cohomological degrees on one hand, and the natural power degree in the symmetric algebra on the other. The two differentials d and ϵ are respectively induced by the natural cohomological differential of A and by the universal de Rham derivation $dR : A \to \mathbb{L}_{A/k}$.

We can now state the major definition of this book (see Definition 2.1.1.1).

Definition. A *derived foliation $\mathcal{F}$ on X (relative to k)* consists of a graded mixed cdga $\mathbf{DR}(\mathcal{F})$ together with a morphism of graded mixed cdga's $\mathbf{DR}(X/k) \to \mathbf{DR}(\mathcal{F})$, such that the underlying graded cdga of $\mathbf{DR}(\mathcal{F})$ is quasi-isomorphic to $Sym_A(\mathbb{L}_{\mathcal{F}}[1])$, for some A-dg-module $\mathbb{L}_{\mathcal{F}}$.

In the definition above the A-dg-module $\mathbb{L}_{\mathcal{F}}$ is called the *cotangent complex* of the derived foliation $\mathcal{F}$; it comes together with an anchor map $\mathbb{L}_{X/k} \to \mathbb{L}_{\mathcal{F}}$ and plays the role of the quotient $\Omega^1_X \twoheadrightarrow \Omega^1_{\mathcal{F}}$ previously discussed in the setting of classical smooth foliations. The mixed structure on $\mathbf{DR}(\mathcal{F})$ together with the quasi-isomorphism $Sym_A(\mathbb{L}_{\mathcal{F}}[1]) \simeq DR(\mathcal{F})$ of *graded* cdga's, define on $Sym_A(\mathbb{L}_{\mathcal{F}}[1])$ a de Rham-like differential

$$\epsilon : \Lambda^i_A \mathbb{L}_{\mathcal{F}} \to \Lambda^{i+1}_A \mathbb{L}_{\mathcal{F}}$$

which is well defined as a morphism in the ∞-category of k-dg-modules. This de Rham differential carries (an infinite number of) homotopy coherent structures expressing that it squares to 0, that is multiplicative and compatible with the de Rham differential on $\mathbf{DR}(X/k)$. These homotopy coherent structures are encapsulated and hidden in the full graded mixed cdga structure $\mathbf{DR}(\mathcal{F})$ and are implicit in our approach, as the de Rham differential ϵ exists by definition on $\mathbf{DR}(\mathcal{F})$ (which is only quasi-isomophic to $Sym_A(\mathbb{L}_{\mathcal{F}}[1])$), and does not exist strictly speaking on $Sym_A(\mathbb{L}_{\mathcal{F}}[1])$ itself. We refer to Corollary 1.6.0.4 for more on the shape of homotopy coherences involved, but the point of our definition of derived foliations, and more generally the point of view of this book is to avoid, as much as possible, to work with explicit homotopy coherences and try to keep them implicit in the picture.

This being said, the derived foliations on X as defined above form an ∞-category $\mathcal{F}ol(X/k)$. The assignment $X \mapsto \mathcal{F}ol(X/k)$ is functorial in X in the sense that a derived foliation $\mathcal{F}$ on X can be pulled back along any morphism $f : Y \to X$ of derived affine schemes. This pull-back operation is of fundamental importance, as even when Y and X are smooth schemes, and $\mathcal{F}$ is a classical smooth foliation on X, its pull-back $f^*(\mathcal{F})$ is a derived foliation on Y which is, in general, not a classical foliation anymore. As a result, pull-backs offer a way to *create* interesting examples of derived foliations starting from a classical situation. This is, of course, parallel to the fact that the (derived) fiber product of smooth schemes usually is a non-trivial derived scheme.

The functoriality $X \mapsto \mathcal{F}ol(X/k)$, on derived affine schemes X, can be used in order to provide a definition of derived foliation on any *derived stack* X, simply by imposing the usual descent condition

$$\mathcal{F}ol(X/k) := \lim_{\mathbf{Spec}\, A \to X} \mathcal{F}ol(\mathbf{Spec}\, A/k).$$

However, the behaviour of derived foliations on general derived stacks brings some difficulties, as for instance cotangent complexes of derived foliations are not stable by pull-backs. As a result the existence of well defined global cotangent complex of a derived foliation over a general derived stack is not obvious. In that direction we prove several descent results showing that, in spite of that, cotangent complexes still have a reasonable meaning when working with derived Artin stacks.

Finally, a given derived foliation $\mathcal{F}$ has an attached *de Rham cohomology*, that is the generalisation of the well known de Rham cohomology along the leaves, or *foliated de Rham cohomology*. Within our approach, this foliated de Rham cohomology is very easy and natural to define by means of the *Tate-realization functor*, relating graded mixed cdga's and complete filtered cdga's (see Corollary 1.3.4.4). The image of $\mathbf{DR}(\mathcal{F})$ by this realization, $\widehat{C}^*_{DR}(\mathcal{F})$, is by definition the commutative dg-algebra of *de Rham cohomology along* $\mathcal{F}$, and the filtration is here of course the natural Hodge filtration. This de Rham cohomology is however only a piece of a refined cohomological invariant called the *Hodge filtration* associated to a derived foliation $\mathcal{F}$. The terminology is here unfortunate, as it should not be confused with the filtration on $\widehat{C}^*_{DR}(\mathcal{F})$. The Hodge filtration associated with $\mathcal{F}$ is indeed a filtration on the de Rham cohomology of X (suitably derived and completed) $\widehat{C}^*_{DR}(X)$, whose first graded piece recovers $\widehat{C}^*_{DR}(\mathcal{F})$. More generally, the higher graded pieces can also be described as foliated de Rham cohomology with coefficients in the powers of the normal complex (see § 3.3). The existence of this Hodge filtration seems to us more important than the coarser notion of foliated de Rham cohomology, and it will turn out to be very helpful in the study of characteristic classes associated to derived foliations (see Corollary 7.4.0.3).

Formal aspects of derived foliations. One specific and important aspect of derived foliations is their *formal integrability* property. This is a major difference with the classical situation, where certain singular foliations might not be formally integrable (see Remark 2.4.1.3). It can be shown, under very mild assumptions, that any derived foliation $\mathcal{F}$ on a derived Artin stack X is formally integrable around each point in X (see Corollary 2.3.3.3). In particular, this implies that for a field valued point $x \in X(K)$, there is a well defined notion of *formal leaf passing through* x, as well as a *formal leaf space at* x. As a result, it may happen that a given classical singular foliation cannot be enhanced to a derived foliation (i.e. it is not the truncation of a derived foliation). This can be seen as either a positive result, or a negative result, depending of the situation. Nevertheless, this shows that the situation diverges from what is happening in derived algebraic geometry, where any scheme can be endowed with the trivial derived structure.

The precise connection between classical singular foliations and derived foliations can be made clearer. For a smooth scheme X we have a notion of *quasi-smooth and rigid* derived foliation on X. This is an important class of derived foliations as they can be considered as the derived foliations which are the closest to be classical. These are given by $\mathcal{F} \in \mathcal{F}ol(X/k)$ for which the cotangent complex sits in an exact triangle of perfect complexes on X

$$\mathcal{N}^*_{\mathcal{F}} \longrightarrow \Omega^1_X \longrightarrow \mathbb{L}_{\mathcal{F}}$$

where $\mathcal{N}^*_{\mathcal{F}}$ is a vector bundle on X. In many situation $\mathcal{N}^*_{\mathcal{F}} \to \Omega^1_X$ is a monomorphism, and thus $\mathcal{F}$ is a classical smooth foliation on the open $U \subset X$ where $\mathbb{L}_{\mathcal{F}}$ is a vector bundle. The closed complement of U is by definition the *singular set* of $\mathcal{F}$. The derived foliation $\mathcal{F}$ admits a classical truncation which is the image $\mathcal{N}^*_{\mathcal{F}}$ in Ω^1_X and which defines a differential ideal $K_{\mathcal{F}}$ on X. As we have already mentioned, not all differential ideals arise by this construction. In fact we can show that the differential ideals obtained this way are the one for which an infinite *Godbillon-Vey* sequence exists (see §2.4). That of a Godbillon-Vey sequence is a classical notion in foliation theory and therefore this result sheds some light of how derived foliations are related to classicial singular foliations.

Derived foliations in the holomorphic setting. Most of this book is written in the algebraic setting. However, except in some extremely specific situations, classical foliations *cannot* be integrated algebraically, and *analytic* integration is in most situations the best one can hope. It is therefore important to introduce derived foliations also in the holomorphic context. Our definition of derived foliation as certain graded mixed cdga's can be easily transported in the holomorphic setting, but with an extra technical component in order to keep track of the holomorphic structure. Nevertheless derived foliations on a derived analytic Artin stack makes sense, and formally behave as in the algebraic setting. We develop the strict minimum of the holomorphic theory with two objectives in sight: the *GAGA theorem*, and the existence of an *analytic leaf space* under appropriate assumptions.

The GAGA theorem for derived foliations states that for a proper derived Deligne-Mumford stack X, the analytification functor induces an equivalence between the ∞-categories of algebraic (and almost perfect) derived foliations on X and the ∞-category of holomorphic derived foliations on the derived analytic stack X^h associated to X (see Theorem 5.5.0.4). This is not very surprising as derived foliations are defined in terms of coherent data via the derived de Rham algebra of forms, and comparison results relating algebraic and holomorphic cotangent complexes are well known.

As in the algebraic situation, holomorphic derived foliations are always formally integrable and it is thus natural to ask if they can be integrated locally in the analytic topology. For this, we can use classical results from [**Mal76**,**Mal77**] to provide interesting conditions ensuring that quasi-smooth and rigid holomorphic derived foliations are indeed locally analytically integrable. This is the starting point for a construction of a global holomorphic leaf space and for the global analytic integrability. Indeed, under certain natural assumption, it is possible to show that a quasi-smooth holomorphic derived foliation $\mathcal{F}$ on a smooth complex variety X admits a holomorphic leaf space $\pi : X \to X /\!\!/ \mathcal{F}$ (see Theorem 6.2.2.1). Here $X /\!\!/ \mathcal{F}$ is a complex (highly non-separated) orbifold, π is a flat surjective holomorphic map, and $\mathcal{F}$ is realized as the derived foliation induced by π. The situation is thus very similar to the case of smooth classical foliation for which $X /\!\!/ \mathcal{F}$ is represented by the well known *holonomy groupoid* of $\mathcal{F}$. The novelty here is that π is only flat and thus can have singular fibers, corresponding to the possibly singular leaves of $\mathcal{F}$.

It is interesting to combine together the GAGA theorem and the existence of the analytic leaf space when we start with a derived foliation $\mathcal{F}$ on a smooth and proper complex variety X. As a first application, we prove a *Riemann-Hilbert correspondence* which provides an equivalence between the category of algebraic vector bundles with integrable connections along $\mathcal{F}$, and the category of relative local systems along the fibers of π (see Theorem 6.4.0.4). This is a generalization to the singular setting of the Riemann-Hilbert correspondence appearing in [**Del70**]. In another direction, we can exploit the notion of the *holonomy groups* deduced from the existence of the analytic leaf space, which are, by definition, the stabilizers of the orbifold $X /\!\!/ \mathcal{F}$. Classically holonomy groups have been useful in stability results, such as the famous *Reeb stability* theorem (see [**Ree52**]). In our setting, it is in fact possible to prove that $\mathcal{F}$ is globally algebraic integrable on X, if and only if $\mathcal{F}$ possesses at least one compact leaf with finite holonomy groups (see Theorem 6.5.0.1). This is again an extension to the singular setting of previously known results for classical smooth foliations (see [**Per01**]).

Crystals along derived foliations. We have already mentioned that a given derived foliation has an associated foliated de Rham cohomology. This can be extended by the introduction of a category of coefficients for foliated de Rham cohomology called *crystals along a derived foliation* $\mathcal{F}$. Morally speaking these are quasi-coherent complexes endowed with partial flat connections along $\mathcal{F}$. Concretely, they can be described in terms of graded mixed $\mathbf{DR}(\mathcal{F})$-dg-modules (see Definition 3.1.0.1). They also have another description in terms of modules over a sheaf of associative dg-algebras $\mathcal{D}_{\mathcal{F}}$, called the *sheaf of differential operators along* $\mathcal{F}$ (see Theorem 4.1.2.1). The advantage of the latter description is the existence of a natural filtration on $\mathcal{D}_{\mathcal{F}}$ induced by the order of differential operators. This is useful to define the notion of *good filtration* on a given crystal, analogue to the well known notion of a good filtration on a classical $\mathcal{D}$-module. This notion can then be used in order to define the *singular support*, as well as the *characteristic cycle* of a crystal endowed with a good filtration, which is a K-theory class inside the total space of the foliated cotangent complex of $\mathcal{F}$ (see Definition 4.3.2.3). Even though the notion of good filtration has some drawbacks for derived foliations with singularities (see § 4.3.2.2), it is enough to establish a *Grothendieck-Riemann-Roch* theorem and *index* theorem for crystals along derived foliations (see Theorem 4.4.1.1 and Corollary 4.5.4.2).

Derived foliations over bases of arbitrary characteristics. Classical foliations in positive characteristics is a rich subject (see for instance [**Eke87**,**Bos01**]), and we have therefore decided to include a last chapter on derived foliations over arbitrary base ring. It turns out that there are two, non-equivalent, possible extensions of our notion of derived foliations outside of characteristic zero, which roughly speaking correspond to the two existing notion of differential operators: *crystalline differential operators* and *Grothendieck differential operators*. These are also related to two different extension of de Rham cohomology, namely *crystalline cohomology* and *infinitesimal cohomology*.

The best way to understand the difference between these two possible definitions of derived foliations in arbitrary characteristics is to adopt a geometric point of view on derived foliations based on group actions on derived linear stacks (see Definition 2.5.2.1). We introduce two group stacks $\mathcal{H}$ and $\mathcal{H}_\pi$. The first one, $\mathcal{H}$ is the semi-direct product of $\mathbb{G}_m$ with $B\mathbb{G}_a^\sharp$, where $\mathbb{G}_a^\sharp$ is the additive group with divided powers. The second one, $\mathcal{H}_\pi$ is the semi-direct product of $\mathbb{G}_m$ with $\mathbb{G}_a[-1] = \Omega_0\mathbb{G}_a$ the additive group in degree -1. Both of these groups control graded mixed complexes, in the sense that the symmetric monoidal ∞-categories $\mathbf{QCoh}(B\mathcal{H})$ and $\mathbf{QCoh}(B\mathcal{H}_\pi)$ are both equivalent to the symmetric monoidal ∞-category of graded mixed complexes. However, the two groups $\mathcal{H}$ and $\mathcal{H}_\pi$ carry different informations. A complex E together with an action of $\mathcal{H}$ on the corresponding derived linear stack $\mathbb{V}(E)$ is called a *derived foliation* (see Definition 7.2.0.1). On the other hand, a complex E together with an action of $\mathcal{H}_\pi$ on the corresponding derived linear stack $\mathbb{V}(E)$ is called an *infinitesimal derived foliation* (see Definition 7.6.1.1). Derived foliations are directly related to (derived and Hodge completed) de Rham cohomology, and thus to crystalline cohomology as well (see Proposition 7.5.0.1). On the other hand, infinitesimal derived foliations are closely related to infinitesimal cohomology, suitably derived and Hodge completed, if necessary (see Corollary 7.6.2.2).

Both, derived foliations and infinitesimal derived foliations have a generic behaviour similar to derived foliations in characteristic zero: they have pull-backs, can be defined over general derived stacks, have an associated foliated cohomology theory, etc. However, their difference can be seen in applications. Derived foliations, due to their relation with crystalline cohomology, are useful to study characteristic classes. For instance they can be used to prove a characteristic $p > 0$ version of the vanishing theorem of *Baum-Bott* (and its generalization on the existence of *residues*, see [**BB72a**]). On the other hand infinitesimal derived foliations are always formally integrable, as opposed to derived foliations in general (see Corollary 7.6.3.2). This is a generalization of the fact that not all classical smooth foliations in characteristic $p > 0$ can be formally integrated, because of the well known obstruction associated to p-curvature (see e.g. [**Kat70**, **Eke87**]). In a way, infinitesimal derived foliations in characteristic $p > 0$ should be thought of as *p-restricted derived foliation*, even though we did not try to make this notion precise.

Relations to other works. We finish this introduction by mentioning several relations with other works.

To our knowledge the notion of derived foliation first appears in a talk by Tony Pantev in 2014 (see [**Pan14**]). He explained in particular how derived foliations appear naturally in the setting of shifted symplectic and Poisson structures of [**PTVV13**, **CPT$^+$17**], and in particular how they are useful to reconstruct shifted potentials in order to write a n-shifted symplectic derived stack globally as the derived critical locus of a function of degree $n + 1$. The present book has pursued this direction, and owns a lot to the conversations among the authors of [**PTVV13**, **CPT+17**] on the subject. The reconstruction of potential is not part of this book as we have decided to focus on slightly different aspects of derived foliation theory, but has been carried out in details recently in [**HHR25**, §2.4].

In a similar direction, an idea conveyed by Damien Calaque during the last decade states that shifted Poisson structures in the sense of [**CPT+17**] can be considered as derived foliations with shifted symplectic structures along the leaves (in short a derived symplectic foliation). This is the derived analogue of the fact that a classical Poisson structure provides a symplectic foliation. This has been realized recently in [**Tom25**], where it is shown that a shifted Poisson structure is always given by a Lagrangian thickening, which can be identified with the formal leaf space of the wanted symplectic derived foliation (see also [**CS24**]).

The formal integrability properties of derived foliations studied in this book has been generalized recently in [**BMN25**], including the non-zero characteristic situations. Our results of §2.3.3.3 and §7.6.3.2 are therefore already subsumed by the results of [**BMN25**]. Moreover, the results of the above work also provide a very general construction of a *global formal leaf space* of a derived foliation in the form of a formal thickening. These type of quotients were already mentioned in [**Pan14**] in the characteristic zero case, but are not included in this book.

Many of the filtrations appearing in this book are obtained by means of the equivalence given by the Tate-realization functor, which identifies graded mixed complexes and complete filtered complexes (see §1.3.4.4). Therefore, all the notions of this book could also have been written in terms of complete filtered objects, such as complete filtered cdga or complete filtered *LSym*-algebras. We are aware that Ben Antieau has also developed a theory of derived foliation in the topological context based on this notion of filtered algebra, and that probably some of the definitions of this book also appear in his work.

Mauro Porta has been keeping telling us that graded mixed cdga which are not necessarly graded free (and thus do not correspond, strictly speaking, to derived foliations) are nonetheless interesting. More recently, Philippe Eyssidieux has convinced the authors that there could be indeed an interesting derived version of the notion of *exterior differential system* (EDS for short). By definition, these derived EDS on a variety X are precisely the general graded mixed cdga under the de Rham algebra $\mathbf{DR}(X)$. It is pretty clear that some of the notions and results of this book extend to this more general setting of derived EDS. However, we have not included any content in that direction.

Contents of individual chapters

This book is divided in seven chapters and one appendix. Chapter 1 is devoted to the algebraic preliminaries necessary to approach the study of derived foliations. We have included sections on graded mixed objects, filtered objects and the Tate-realization relating these two notions. These notions will be used throughout the book. The two last sections of this chapter include one incarnation of graded mixed complexes as representations of the group stack $B\mathcal{H}_0$. This geometric interpretation will turn out to be useful, typically to construct natural graded mixed objects as push-forwards along a morphism to $B\mathcal{H}_0$. The last section is devoted to a (partial) description of the homotopy coherences encoded in a graded mixed structure. This is a technical result that will be used occasionally, and does not quite follow the general spirit of the general approach of this book which is to make these coherences implicit as much as possible.

Chapter 2 contains the main definitions and notions on the subject. We define the ∞-category $\mathcal{F}ol(X/k)$ of derived foliations, first on an affine derived k-schemes X, then by descent on general derived k-stacks X. We provide examples, and compare derived foliations in particular with the theory of dg-Lie algebroids already existing in the literature. We present also the two important universal foliations: the initial foliation, also called the zero foliation, and the final foliation, also called the de Rham or tautological foliation. The third section is devoted to the formal properties of derived foliations. We have restricted ourselves to the behaviour formally at a field valued point only, which is a non-trivial restriction. However, we have treated the general case of almost perfect derived foliations on general derived Artin stacks locally of finite presentation. We explain in particular, how derived foliations on the formal completion at a point are governed purely in terms of dg-Lie algebras. This section is central in the sense that it allows to make sense of the notion of *formal leaf* and *formal leaf* space at a given point, whereas those notions are not totally obvious when working over general derived stacks. The subsequent section concerns formal integrability in the more restrictive context of quasi-smooth and rigid derived foliations on smooth varieties. It contains in particular a comparison with classical singular foliations, and provide interpretations of classical notions (such as Godbillon-Vey sequences and transversal jet spaces) in terms of derived foliations. The last section is taken from the work of [**Alf25**], and presents a useful constuction tool of derived foliations by direct images (or Weil restriction). The geometric interpretation of derived foliations which is behind this direct image construction will be useful to understand and motivate Chapter 7 about derived foliations in arbitrary characteristics.

In Chapter 3 we introduce and study foliated de Rham cohomology. We start by the introduction of the categories of crystals along a derived foliations, and their basic functorialities by pull-backs. They give rise to the notion of *foliated de Rham cohomology with coefficients in a crystal* that will be used quite often in the rest of the book. The third section of this chapter introduces the Hodge filtration. This is a filtration on the de Rham cohomology of a derived stack X induced by the datum of a derived foliation $\mathcal{F} \in \mathcal{F}ol(X/k)$. It is obtained by the geometric point of view on derived foliations explained in § 2.5 together with a very general deformation to the normal bundle construction. The final section of this chapter provides some elements of transversal geometry by constructing the formal jets transversal to a given derived foliation. We explain how these jets are endowed with a crystal structure, and how they give rise to an algebraic notion of holonomy and monodromy by means of a Tannakian approach. These include higher homotopical invariants captured in the form of a schematic homotopy type in the sense of [**Toe06**], and closely related to *Galois-differential homotopy theory.*

Chapter 4 is devoted to the study of operations on crystals. We have not included a full 6 operations formalism, which seems to us out of reach at the moment, but have studied direct images and direct images with proper support when they are defined. The first section introduces the sheaf of differential operators along a derived foliation $\mathcal{F}$, which is a sheaf of filtered dg-algebras whose associated graded object is of the form $Sym_{\mathcal{O}_X}(\mathbb{T}_{\mathcal{F}})$, the symmetric algebra over the tangent complex of the derived foliation $\mathcal{F}$. Modules over $\mathcal{D}_{\mathcal{F}}$ are equivalent to crystals along $\mathcal{F}$, but the existence of the filtration on $\mathcal{D}_{\mathcal{F}}$ can be used in order to define good filtrations on crystals and the corresponding notion of characteristic cycles. We prove in particular that characteristic cycles are stable by push-forward, i.e. a foliated version of the Grothendieck-Riemann-Roch formula.

The analytic theory of derived foliation is covered in chapter 5. In its first section we remind the basics of derived analytic geometry based on the notion of holomorphic cdga. In the second section we introduce the *holomorphic graded mixed cdga*, which can be used to defined holomorphic derived foliations along the same lines as in the algebraic setting. The two last sections are devoted to the proof of two main results. First of all we study the analytic integrability, locally in the analytic topology, of quasi-smooth and rigid derived holomorphic foliations. This allow us to make comparisons with previously known integrability results for holomorphic classicial singular foliations, notably the results from [**Mal76**, **Mal77**]. In the last section of the chapter we prove a GAGA theorem for derived foliations on proper derived Deligne-Mumford stacks.

Chapter 6 contains the notion of analytic leaves for holomorphic derived foliations. By restricting to quasi-smooth and rigid derived foliations, we provide simple assumptions ensuring the existence of a nice analytic leaf space whose construction uses techniques arising in the study of *étendues* from $SGA4$, and is a generalization of the construction of the holonomy groupoid of a classical smooth holomorphic foliation. The possible singularities bring new features concerning the leaf space: one instance is the non-triviality of the *relative homotopy type* that encodes the variations of the homotopy types of the leaves. This relative homotopy type is not locally constant (even at the level of germs because of the existence of vanishing cycles). In the last two sections we give two main applications of the existence of an analytic leaf space and of our previously established GAGA theorem. The first application is a Riemann-Hibert correspondence, and the second application is an algebraic integrability result based on the classical Reeb stability theorem.

The last chapter of the book is chapter 7. It contains some results on derived foliations in non-zero characteristic situations. Its first two sections introduce an important group stack $\mathcal{H}$, that enables us to define derived foliations as linear derived stacks endowed with an action of $\mathcal{H}$. In the next section we explain how this notion is related to de Rham cohomology, and introduce extensions of the notion of crystals and of the Hodge filtration already encountered in characteristic 0. The fourth section is devoted to the construction of characteristic classes of derived foliations over any base ring, and explain how these interact nicely with the Hodge filtration. We then establish classical vanishing results for the characteristic classes. In the subsequent section of this chapter, we use these results to prove a characteristic $p > 0$ version of the existence of residues of foliations introduced by Baum-Bott in the complex setting. These are proven in crystalline cohomology thanks to the comparison between derived de Rham and crystalline cohomology. Finally, the last section introduces the notion of infinitesimal derived foliations, explains its relation with infinitesimal cohomology, and how infinitesimal derived foliations can be always integrated by formal groupoids.

The Appendix contains some quick and basic reminders on topics in ∞-categories and derived algebraic geometry (global and formal), together with some important descent results that are used in the main text.

Guide to our notations

Most of our notations will be defined in the main text. Here we only fix a few of the most recurring notations.

Throughout the book, k will be a commutative ring, sometimes supposed to be a field: see the beginning of each chapter or section to understand what are the standing hypotheses on k.

We will be freely using the language of model categories and of ∞-categories (see §A.1 for a brief recapitulation about these, and for the corresponding notations we adopted). We will denote by **Top** the ∞-category of spaces ([**Lur09**, §1.2.16]), and by $\mathbf{Cat}_\infty$ the ∞-category of ∞-categories (see [**Lur09**, Chapter 3]).

For a model category (or a category with weak equivalences) C, we will denote by $\mathbf{C}$ its associated ∞-category obtained by inverting in the ∞-categorical sense the weak equivalences (see §A.1 for more details). For a functor $F : \mathsf{C} \to \mathsf{D}$ that induces a functor between the associated ∞-categories (e.g. a functor preserving weak equivalences or a left/right Quillen functor), we will denote with boldface fonts the induced functor $\mathbf{F} : \mathbf{C} \to \mathbf{D}$.

The ∞-category of derived stacks over k (mostly for the étale topology) will be denoted by $\mathbf{dSt}_k$ (see §A.2), and, for G a group scheme or group stack over k, $\mathbf{dSt}_k^G$ will be the ∞-category of derived stacks endowed with a G-action.

CHAPTER 1

Algebraic preliminaries

In this first chapter we gather preliminaries results that we will use all along the book. We fix the cohomological conventions, and present the central notion of graded mixed complexes. We show that it is related to complete filtered dg-modules by means of the Tate-realization functor. In the last two section we present a stacky interpretation of graded mixed complexes as quasi-coherent sheaves on the classifying stack of the group stack $\mathbb{G}_m \ltimes B\mathbb{G}_a$. Finally, we study the moduli spaces of graded mixed structures.

All along this chapter we denoted by k a ground commutative ring. It will be assumed to be a $\mathbb{Q}$-algebra starting from Section 1.4 on, but most of the results of the chapter are independent of the characteristics of the ground ring.

1.1. Homological conventions

In this section k is of arbitrary characteristics. The sole purpose of this section is to fix notations and the conventions used in this book.

1.1.1. Basics. Complexes of k-modules are by definition cohomologically indexed and their differential d will raise the cohomological degree by 1. More explicitly, a complex of k-modules consists of a $\mathbb{Z}$-indexed family of k-modules $\{E^n\}_{n\in\mathbb{Z}}$, together with k-linear morphisms

$$d^n : E^n \to E^{n+1}$$

such that $d^{n+1}d^n = 0$. We will often write symbolically such a complex as a direct sum $E = \oplus_n E^n$, and d as a k-linear endomorphism $d : E \to E$ satisfying $d^2 = 0$.

For two complexes E and F, a morphism $f : E \to F$ consists of a family of k-linear morphisms $f^n : E^n \to F^n$ which commute with the differentials of E and F

$$d^n_F f^n = f^{n+1} d^n_E,$$

where we have denoted by d_E and d_F the differentials of E and F, respectively. The complexes of k-modules and morphisms between them form a category denoted by $C(k)$.

For an object $E \in C(k)$, we define its i-th cohomology k-module as

$$H^i(E) := \frac{(Kerd^i : E^i \to E^{i+1})}{(Imd^{i-1} : E^{i-1} \to E^i)}.$$

This defines functors $H^i(-) : C(k) \longrightarrow k - Mod$, for $i \in \mathbb{Z}$.

1.1.2. Symmetric monoidal structure. The category $C(k)$ is endowed with a symmetric monoidal structure $\otimes_k$ defined as follows. For two objets E and F, we define $E \otimes_k F \in C(k)$ by letting for $n \in \mathbb{Z}$

$$(E \otimes_k F)^n := \bigoplus_{i+j=n} E^i \otimes_k F^j,$$

the differential on $E \otimes_k F$ being defined as the k-linear map sending a tensor $x \otimes y$ to

$$d(x \otimes y) := d(x) \otimes y + (-1)^{|x|} x \otimes d(y)$$

where $|x|$ is the cohomological degree of x, i.e. $x \in E^{|x|}$. In more concrete terms, the differential $(E \otimes_k F)^n \to (E \otimes_k F)^{n+1}$ is the sum of all maps

$$(d_E^i \otimes id, (-1)^i id \otimes d_F^j) : E^i \otimes_k F^j \longrightarrow (E^{i+1} \otimes_k F^j) \oplus (E^i \oplus_k F^{j+1}) \subset \bigoplus_{a+b=n+1} E^a \otimes_k F^b.$$

This defines the functor $- \otimes_k - : C(k) \times C(k) \to C(k)$. It is furthermore endowed with the natural unity and associativity constraints isomorphisms induced from the usual unity and associativity constraints of tensor products of k-modules. The symmetry constraint is itself the usual isomorphism

$$V_{E,F} : E \otimes_k F \simeq F \otimes_k E \tag{1}$$

defined by sending $x \otimes y$ to $(-1)^{ij} y \otimes x$ for $x \in E^i$ and $y \in F^j$. With these definitions, $(C(k), \otimes_k)$ is a closed symmetric monoidal category.

1.1.3. Shifts. We denote by $k[1]$ the object of $C(k)$ which is defined by

$$(k[1])^n = \begin{cases} k & \text{if } n = -1 \\ 0 & \text{if } n \neq -1 \end{cases}$$

obviously endowed with the zero differential. The shift endofunctor on $C(k)$ is defined to be

$$E \mapsto E[1] := k[1] \otimes_k E.$$

Unfolding the definitions of the tensor product we see that $E[1]$ is explicitly given, up to a natural isomorphism, by $(E[1])^n = E^{n+1}$ with differential

$$-d^{n+1} : (E[1])^n = E^{n+1} \longrightarrow (E[1])^{n+2} = E^{n+2}.$$

The endofunctor $E \mapsto E[1]$ is an auto-equivalence of the category $C(k)$ and its inverse will be denoted by $E \mapsto E[-1]$. Finally, we set, for all $i \in \mathbb{Z}$ and $E \in C(k)$

$$E[i] := (k[1])^{\otimes i} \otimes_k E$$

the n-th composite of either $E \mapsto E[1]$ if $i \geq 0$, or $E \mapsto E[-1]$ if $i < 0$. Explicitly, again up to a natural isomorphism, $E[i]$ is given by $(E[i])^n = E^{n+i}$ with differential $(-1)^i d^{n+i} : (E[i])^n \to$

$(E[i])^{n+1}$.

1.1.4. Model category structure. We remind that $C(k)$ possesses a model category structure whose fibrations are the morphisms $f : E \to F$ with $f^n : E^n \to F^n$ surjective for all $n \in \mathbb{Z}$, and weak equivalences are the quasi-isomorphisms, i.e. the morphisms $f : E \to F$ such that for all i the induced map $H^i(f) : H^i(E) \to H^i(F)$ is bijective. The cofibrations are defined by the left lifting property with respect to the class of fibrations that are also quasi-isomorphisms. This model category structure is suitably compatible with the symmetric monoidal structure and thus makes $C(k)$ into a *symmetric monoidal model category* in the sense of [**Hov99**, §4]. It satisfies moreover the monoid axiom of [**SS00**].

As explained in the Appendix A.1, the model category $C(k)$ gives rise to an ∞-category by localization along the weak equivalences. This ∞-category is called the ∞-*category of complexes*, and will be denoted by

$$\mathbf{dg}_k := L(C(k)).$$

A concrete model for $\mathbf{dg}_k$ is given by considering the category of cofibrant[1] objects $C(k)^c \subset C(k)$. Between two such cofibrant objects E and F, we can define a simplicial set of maps $Hom^{\Delta}(E, F) \in sSets$, whose set of n-dimensional simplices is given by the formula

$$Hom^{\Delta}(E, F)_n := Hom_{C(k)}(\mathsf{C}(\Delta^n) \otimes_k E, F)$$

where $\mathsf{C}(\Delta^n) \in C(k)$ is the normalized chain complex of homology of the simplicial set Δ^n. More precisely, if $\mathsf{N} : sMod_k \to C^-(k)$ is the Dold-Kan normalization functor (see ...), $C(\Delta^\bullet) := \mathsf{N}(\Delta^\bullet)$ is a cosimplicial object in $C^-(k) \subset C(k)$, hence

$$Hom^{\Delta}(E, F) := Hom_{C(k)}(\mathsf{C}(\Delta^\bullet) \otimes_k E, F) \in sSets.$$

It is easy to see that these data organize themselves into an $\mathbb{S}$-category (i.e. a category enriched in $sSets$) which is a concrete model for $\mathbf{dg}_k$ (see §A.1 for more details).

Finally, the symmetric monoidal model structure on $C(k)$ induces a natural symmetric monoidal structure on the ∞-category $\mathbf{dg}_k$ (see §A.1). It will be again denoted by $\otimes_k$. Note however, that the localization ∞-functor

$$C(k) \longrightarrow \mathbf{dg}_k$$

is not a monoidal ∞-functor, even though its restriction to cofibrant objects is so. Therefore, $E \otimes_k F$ is an ambiguous notation, whose meaning depends on whether E and F are considered as objects in $C(k)$ or as objects in $\mathbf{dg}_k$. We hope however that the context will provide the necessary extra information, each time we will use such a notation in the text.

For an associative and unital dg-algebra A, we will also denote by $Mod(A)$ the category of (left) A-dg-modules. It is endowed with an induced model category structure whose weak equivalences (resp. fibrations) are morphisms which are quasi-isomorphisms (resp. epimorphisms) on the underlying complexes (see e.g. [**SS00**]). The corresponding ∞-category will be

[1]Note that all objects in $C(k)$ are, obviously, fibrant.

denoted by $\mathbf{dg}_A$. For a morphism of dg-algebras $\varphi : A \to B$, we have the usual base change functor $B \otimes_A - : Mod(A) \to Mod(B)$, which is a left Quillen functor. It induces an adjunction on the corresponding ∞-categories

$$B \otimes_A - : \mathbf{dg}_A \leftrightarrows \mathbf{dg}_B : (-)_{[\varphi]}$$

When φ will be clear from the context, the functor $(-)_{[\varphi]}$ (restriction of scalars) will be simply omitted from our notations.

1.2. Graded mixed complexes

In this section k is of arbitrary characteristics.

1.2.1. Graded complexes. We consider $C(k)^{gr}$ the category of $\mathbb{Z}$-graded objects in $C(k)$. An object in $C(k)^{gr}$ is by definition a collection $E = \{E^{(n)}\}_{n\in\mathbb{Z}}$ of complexes $E^{(n)}$. Morphisms $f : E \to F$ in $C(k)^{gr}$ are defined levelwise as families of morphisms of complexes

$$f^{(n)} : E^{(n)} \longrightarrow F^{(n)}.$$

In order to avoid confusions with the notion of cohomological degree we refer to $E^{(n)}$ as the *part of weight n of E*, and in general refer to *weights* for the degree corresponding to the extra $\mathbb{Z}$-grading on objects in $C(k)^{gr}$.

The category $C(k)^{gr}$ is endowed with the usual monoidal structure of graded objects. For two objects E and F of $C(k)^{gr}$ we set

$$(E \otimes_k F)^{(n)} := \bigoplus_{i+j=n} E^{(i)} \otimes_k F^{(j)}.$$

This defines a monoidal structure on the category $C(k)^{gr}$. We define a symmetry constraints on $C(k)^{gr}$ induced from the symmetry constraint of the monoidal structure on $C(k)$. In formula, this is the family of isomorphisms $(E \otimes_k F)^{(n)} \simeq (F \otimes_k F)^{(n)}$ which is the sum over $i + j = n$ of the symmetry maps of complexes reminded in (1) on page 18

$$V_{E^{(i)},F^{(j)}} : E^{(i)} \otimes_k F^{(j)} \longrightarrow F^{(j)} \otimes_k E^{(i)}$$

We warn the reader here that the signs rule only involves the cohomological degrees and *not* the weights themselves. With these definitions the category $C(k)^{gr}$ is endowed with a symmetric monoidal structure $\otimes_k$.

The model category structure on $C(k)$ defines a natural model category structure on $C(k)^{gr}$ by defining fibrations, equivalences and cofibrations of graded complexes levelwise on their weight pieces. A morphism $f : E \to F$ in $C(k)^{gr}$ is thus a fibration (resp. cofibration, resp. weak equivalence) if for each $n \in \mathbb{Z}$ the morphism $f^{(n)} : E^{(n)} \to F^{(n)}$ is so in $C(k)$. This model category structure is compatible with the symmetric monoidal structure and makes $C(k)^{gr}$ into a symmetric monoidal model category in the sense of [**Hov99**, §4]

1.2.2. Weight twists. As for the shift of complexes, we have weight shifts, which will be rather referred to as *weight twists* . We denote the weight twists of an object E by $E(i)$, and it is formally defined by the formula

$$(E(i))^{(n)} := E^{(n-i)}.$$

We warn the reader about the discrepancy between cohomological shifts and weight shifts: for $F \in C(k)$ and $E \in C(k)^{gr}$ we have

$$(F[i])^n = F^{n+i} \qquad (E(i))^n = E^{(n-i)}.$$

Therefore, with these conventions, if E is pure of weight 0, that is $E^{(i)} = 0$ if $i \neq 0$, then $E(1)$ is pure of weight 1.

1.2.3. Mixed structures on graded complexes. Because of its importance, we display the notion of mixed structure as a definition by its own.

DEFINITION 1.2.3.1. *A* graded mixed structure *on a graded complex $E \in C(k)^{gr}$ consists of a family of morphisms of complexes*

$$\epsilon^{(n)} : E^{(n)} \longrightarrow E^{(n+1)}[-1],$$

for $n \in \mathbb{Z}$, such that

$$(\epsilon^{(n+1)}[-1]) \circ \epsilon^{(n)} : E^{(n)} \to E^{(n+2)}[-2] \tag{2}$$

equals the zero map for all n.

A graded complex E endowed with a graded mixed structure is called a graded mixed complex graded mixed complex.

In order to keep notations simple we will most often write $\epsilon : E^{(n)} \to E^{(n+1)}[-1]$ instead of $\epsilon^{(n)}$, and the structural equation (2) by $\epsilon^2 = 0$. Note that graded mixed complexes can also be considered as bi-graded k-modules (for the cohomological degree and the weight) endowed with two differentials d and ϵ, satisfying the following conditions.

(1) The differential d is of bidegree $(1, 0)$ and ϵ is of bidegree $(-1, 1)$.
(2) We have $d^2 = \epsilon^2 = 0$.
(3) We have $d\epsilon + \epsilon d = 0$.

Therefore, graded mixed complexes can also be considered as bicomplexes, by reindexing the weights in order to get a second differential of bidgree $(0, 1)$. However, we recommend the reader not doing so, as for instance the symmetry constraints we defined on graded complexes do not match the usual sign rules of tensor products of bicomplexes. It is also important, for many purposes in the sequel, to consider ϵ as an *external* differential which should not be treated on the same footing as the *internal* cohomological differential.

Graded mixed complexes form a category in an obvious manner. The morphisms between two graded mixed complexes E and F simply being the morphisms of graded complexes which commute with the mixed structures ϵ on both sides. More explicitly, a morphism from E to F

consists of a family of morphisms $f(n) : E^{(n)} \to F^{(n)}$ in $C(k)$ such that the following diagram commutes

$$\begin{tikzcd} E^{(n)} \arrow[r, "f^{(n)}"] \arrow[d, "\epsilon^{(n)}"'] & F^{(n)} \arrow[d, "\epsilon^{(n)}"] \\ E^{(n+1)}[-1] \arrow[r, "f^{(n+1)}[-1]"'] & F^{(n+1)}[-1]. \end{tikzcd}$$

This category is denoted by $\epsilon - C(k)^{gr}$.

It is useful to present $\epsilon - C(k)^{gr}$ also as a category of modules over a certain algebra in the monoidal category $C(k)^{gr}$ of graded complexes. For this, we introduce $k[\epsilon] \in C(k)^{gr}$, the object defined as

$$k[\epsilon] := k(0) \oplus k(1)[1].$$

According to our conventions with shifts and twists (see section 1.2.2) this means that the weight pieces of $k[\epsilon]$ are given by

$$k[\epsilon]^{(n)} = \begin{cases} k & if\, n = 0 \\ k[1] & if\, n = 1 \\ 0 & if\, n \neq 0, 1. \end{cases}$$

The graded complex $k[\epsilon]$ possesses a natural monoid structure, with respect to the tensor product of graded complexes mentioned in §1.2.1. It is given by a morphism of graded complexes

$$k[\epsilon] \otimes_k k[\epsilon] \longrightarrow k[\epsilon]$$

defined by

$$k[\epsilon] \otimes_k k[\epsilon] \simeq k(0) \oplus k[1](1) \oplus k[1](1) \oplus k[2](2) \longrightarrow k \oplus k[1](1)$$

sending $k[2](2)$ to zero, k isomorphically to k by the identity, and each factor $k[1](1)$ isomorphically to $k[1](1)$ by the identity as well. In other words, as a monoid in $C(k)^{gr}$, $k[\epsilon]$ is the trivial square zero extension of k by $k[1](1)$ (see [**TV08**, §1.2.1]). It is, in particular, an associative, unital and commutative monoid.

We can now rephrase the definition of graded mixed complexes 1.2.3.1 as follows. A graded mixed structure on a graded complex $E \in C(k)^{gr}$ consists of the data of a (unital) $k[\epsilon]$-module structure

$$k[\epsilon] \otimes_k E \longrightarrow E.$$

Indeed, the structure map of such a module structure provides

$$k[\epsilon] \otimes_k E \simeq E \oplus E(1)[1] \longrightarrow E,$$

sum of the identity and a morphism $E \longrightarrow E(-1)[-1]$ of graded complexes. In degree n this defines a morphism of complexes

$$\epsilon^{(n)} : E^{(n)} \longrightarrow E^{(n+1)}[-1].$$

The fact that ϵ defined as above satisfies $\epsilon^2 = 0$ is the consequence of the associativity of the module structure. This construction provides a functor

$$\epsilon - C(k)^{gr} \longrightarrow k[\epsilon] - Mod,$$

which is an isomorphism of categories. Note that the right hand side of this equivalence of categories consists of modules inside the monoidal category $C(k)^{gr}$, which by unfolding the definitions amount to graded dg-modules over the graded dg-algebra $k[\epsilon]$ (endowed with the zero cohomological differential).

The fact that the category $\epsilon - C(k)^{gr}$ can be written as a category of modules $k[\epsilon] - Mod$ possesses an important consequence, namely the existence of model category structures. Indeed, according to [**Kel06**, Thm. 3.2] there exists two model category structures on $\epsilon - C(k)^{gr}$, the projective model structure and the injective model structure, defined using the forgetful functor $\epsilon - C(k)^{gr} \longrightarrow C(k)^{gr}$. The following Proposition summarizes the situation.

PROPOSITON 1.2.3.2. *There exists a projective (resp. an injective) model category structure on $\epsilon - C(k)^{gr}$ whose fibrations (resp. cofibrations) and weak equivalences are defined on the underlying graded complexes obtained by forgetting the mixed structures. The forgetful functor $\epsilon - C(k)^{gr} \to C(k)^{gr}$ is a left (resp. right) Quillen functor for the projective (resp. injective) model category structure.*

The ∞-category associated to the model category $\epsilon - C(k)^{gr}$ will be denoted by $\epsilon - \mathbf{dg}_k^{gr}$. By definition it does not depend on which of the two model category structure we choose as they both share the same notion of weak equivalences.

REMARK 1.2.3.3. In the proposition above, the expression *injective model structure* can be confusing. Indeed, the injective model structure on $\epsilon - C(k)^{gr}$ is here relative to the projective model structure on $C(k)^{gr}$. In particular, it is not the same as the usual (absolute) injective model structure for which cofibrations are defined to be monomorphisms. One important advantage of the injective model structure of proposition 1.2.3.2 is the fact that it remains a monoidal model category (which is not the case for the absolute injective model structure).

The category $\epsilon - C(k)^{gr}$, of graded mixed complexes, carries a natural symmetric monoidal structure denoted by $\otimes_k$ and defined as follows. For two objects E and F of $\epsilon - C(k)^{gr}$, we first form the tensor product $E \otimes_k F$ as a graded complex. We then define a mixed structure on $E \otimes_k F$ whose component of weight n is given as a morphism of complexes

$$\epsilon^{(n)} : \bigoplus_{i+j=n} E^{(i)} \otimes_k F^{(j)} \longrightarrow \bigoplus_{a+b=n+1} E^{(a)} \otimes_k F^{(b)}[-1]$$

sending $x \otimes y \in E^{(i)} \otimes_k F^{(j)}$ to

$$\epsilon(x) \otimes y + (-1)^{|x|} x \otimes \epsilon(y) \in E^{(i+1)} \otimes_k F^{(j)}[-1] \bigoplus E^{(i)} \otimes_k F^{(j+1)}[-1],$$

where as usual $|x|$ denotes the cohomological degree of x. It is straightforward to check that this defines a graded mixed structure on the graded complex $E \otimes_k F$, and thus an object, still denoted by $E \otimes_k F$, in the category $\epsilon - C(k)^{gr}$. From the formula, it is easy to see that the associativity, unital and symmetry constraints form the monoidal structure of graded complexes, are all compatible with this definition, and thus provide an associative, unital and symmetric monoidal structure on $\epsilon - C(k)^{gr}$. By construction, it is such that the faithful functor

$$F : \epsilon - C(k)^{gr} \longrightarrow C(k)^{gr},$$

obtained by forgetting the mixed structure, is a symmetric monoidal functor when endowed with the canonical monoidal structure identifying $F(E \otimes_k F)$ with $F(E) \otimes_k F(F)$ by means of the identity morphism.

The symmetric monoidal structure on $\epsilon - C(k)^{gr}$ can also be described in terms of a commutative and cocommutative Hopf structure on the object $k[\epsilon] \in C(k)^{gr}$. The comultiplication for this Hopf structure is a morphism of monoids in $C(k)^{gr}$

$$\Delta : k[\epsilon] \longrightarrow k[\epsilon] \otimes_k k[\epsilon]$$

obtained by sending ϵ to $\epsilon \otimes 1 + 1 \otimes \epsilon$. It should be noted here that the multiplication on $k[\epsilon] \otimes_k k[\epsilon]$ is defined as the composition

$$k[\epsilon] \otimes_k k[\epsilon] \otimes_k k[\epsilon] \otimes_k k[\epsilon] \xrightarrow{id \otimes V \otimes id} k[\epsilon] \otimes_k k[\epsilon] \otimes_k k[\epsilon] \otimes_k k[\epsilon] \xrightarrow{\mu \otimes \mu} k[\epsilon] \otimes_k k[\epsilon],$$

where $V = V_{k[\epsilon],k[\epsilon]}$ is the volte (see (1) on page 18), and μ the multiplication map. In particular, $(1 \otimes \epsilon).(\epsilon \otimes 1) = -(\epsilon \otimes 1).(1 \otimes \epsilon)$, and thus $(\epsilon \otimes 1 + 1 \otimes \epsilon)^2 = 0$ as required. This comultiplication makes $k[\epsilon]$ into a commutative and cocommutative bialgebra object in $C(k)^{gr}$ and thus induces a symmetric monoidal structure on its category of modules $k[\epsilon] - Mod \simeq \epsilon - C(k)^{gr}$. We leave to the reader that this monoidal structure is naturally isomorphic to the one already discussed above.

Finally, the two model category structures on $\epsilon - C(k)^{gr}$ are compatible with the monoidal structure, and makes $\epsilon - C(k)^{gr}$ into a symmetric monoidal model category, for both the injective and the projective model structures. As a result, the ∞-category $\epsilon - \mathbf{dg}_k^{gr}$ of graded mixed complexes over k carries a canonical symmetric monoidal structure, again denoted by $\otimes_k$ (see A.1.

We would like to finish this section by mentioning that the stable ∞-category $\epsilon - \mathbf{dg}_k^{gr}$ carries a natural t-structure for which an object E is connective if for any $n \in \mathbb{Z}$ the part of weight n $E^{(n)}$ is n-connective: $H^i(E^{(n)}) = 0$ if $i < -n$. This t-structure is directly related to the Beilinson's t-structure on filtered complexes by the equivalence 1.3.4.4 proved in the next section, and therefore will be referred as the Beilinson's t-structure on graded mixed complexes as well.

DEFINITION 1.2.3.4. *The* Beilinson's t-structure *on the ∞-category $\epsilon-\mathbf{dg}_k^{gr}$ of graded mixed complexes is the t-structure for which an object E is connective if $E^{(n)}$ is n-connective for all $n \in \mathbb{Z}$.*

1.3. Filtered complexes

In this section, k is an arbitrary commutative ring.

1.3.1. The homotopy theory of filtered complexes. We remind that a filtered complex E (of k-modules) consists of a sequence of complexes F^nE, indexed by $n \in \mathbb{Z}$, together with morphisms of complexes $F^nE \to F^{n+1}E$ for all n. In another language, E consists of a functor

$$\mathbb{Z} \to C(k),$$

whose values at n is F^nE, and where $\mathbb{Z}$ is the ordered group of integers considered as a category in the usual manner. Filtered complexes form a category in a natural way, for which the morphisms are defined to be the natural transformations of functors $\mathbb{Z} \to C(k)$. We denote by $C(k)^{fil}$ this category, which is therefore the category of all functors from the ordered group $\mathbb{Z}$ to $C(k)$. Note that a morphism $E \to E'$ consists of a family of morphisms of complexes $F^nE \to F^nE'$ for $n \in \mathbb{Z}$, such that all the diagrams

$$\begin{tikzcd} F^nE \arrow[r] \arrow[d] & F^nE' \arrow[d] \\ F^{n+1}E \arrow[r] & F^{n+1}E' \end{tikzcd}$$

are commutative.

REMARK 1.3.1.1. Our conventions for filtered complexes differ from the one of [**Lur15**] and [**Mou21**]. In our setting the filtrations are increasing, whereas the filtrations of these references are decreasing by definition, and therefore correspond to functors $\mathbb{Z}^{op} \to C(k)$ instead. These two notions are equivalent by means of the isomorphism of ordered groups $-1 : \mathbb{Z} \to \mathbb{Z}^{op}$, sending n to $-n$. In more concrete terms, we can turn an increasing filtered complex E into a decreasing one E' by setting $F^nE' := F^{-n}E$ for all $n \in \mathbb{Z}$.

Being a category of diagrams in the model category $C(k)$, the category $C(k)^{fil}$ comes equipped with a natural model category structure, for which the fibrations and the weak equivalence equivalences are defined levelwise (the projective model structure on the category of diagrams in a combinatorial model category). In a more explicit manner, a morphism $f : E \to E'$ is defined to be a weak equivalence (resp. a fibration) if and only if for all $n \in \mathbb{Z}$ the morphism of complexes $F^nE \to F^nE'$ is a quasi-isomorphism (resp. an epimorphism). The corresponding ∞-category (obtained by localization, see A.1) will be denoted by $\mathbf{dg}_k^{fil}$.

The category $C(k)^{fil}$ is also endowed with a symmetric monoidal structure. It can be defined explicitly, for two filtered complexes E and E', by the following formula

$$F^n(E \otimes_k E') := \mathrm{Colim}_{i+j \leq n}(F^i E) \otimes_k (F^j E).$$

The colimit is here taken over the category whose objects are pairs (i, j) of integers with $i + j \leq n$, and for which there is a unique morphism $(i, j) \to (i', j')$ if and only if $i \leq i'$ and $j \leq j'$. In a more functorial manner, this monoidal structure is obtained as the convolution product on the category of functors $Fun(\mathbb{Z}, C(k))$, using that addition in $\mathbb{Z}$ to define the structure of a symmetric monoidal category. For two such functors $E, E' : \mathbb{Z} \to C(k)$, we can form their external product

$$E \boxtimes_k E' : \mathbb{Z} \times \mathbb{Z} \to C(k)$$

sending (i, k) to $F^i E \otimes_k F^j E'$. The addition defines a functor $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, and we can thus form the left Kan extension along $+$ to get the tensor product

$$E \otimes_k E' := (+)_!(E \boxtimes_k E') \in Fun(\mathbb{Z}, C(k)).$$

We refer the reader to [**Lur15**, **Mou21**] for more details on this construction.

The monoidal structure on $C(k)^{fil}$ comes equipped with natural associative, unital and symmetry constraints. Moreover, it is compatible with the model category structure on $C(k)^{fil}$, and makes it into a symmetric monoidal model category. As a consequence the ∞-category $\mathbf{dg}_k^{fil}$ comes equipped with a canonical symmetric monoidal structure (see A.1.

We have two important functors out of the category $C(k)^{fil}$, namely the underlying object and associated graded functors. The associated graded is the functor

$$Gr : C(k)^{fil} \longrightarrow C(k)^{gr}$$

sending a filtered complex E to the graded complex $Gr(E)$ whose component of weight n is given by

$$Gr(E)^{(n)} := Coker(F^{n-1}E \to F^n E) \in C(k).$$

We will often use the more standard notation

$$Gr^n(E) := Gr(E)^{(n)}.$$

This functor is a left adjoint and part of a Quillen adjunction

$$Gr : C(k)^{fil} \rightleftarrows C(k)^{gr} : e.$$

The right adjoint e sends a graded object E to the filtered object $e(E)$ with $F^n(e(E)) := E^{(n)}$, and for which the morphism $E^{(n)} \to E^{(n+1)}$ are all defined to be zero. This adjunction is a Quillen adjunction as e obviously preserves fibrations and equivalences. The left adjoint Gr is moreover endowed with a canonical symmetric monoidal structure

$$Gr(E) \otimes_k Gr(E') \simeq Gr(E \otimes_k E').$$

These functorial isomorphisms are defined by the observation that for all $n \in \mathbb{Z}$ and $a+b<n$, the composite morphism

$$F^a(E) \otimes_k F^b(E') \to F^n(E \otimes_k E') \to Gr^n(E \otimes_k E')$$

is the zero map, which gives rise to natural morphisms

$$Gr^i(E) \otimes_k Gr^j(E') \to Gr^n(E \otimes_k E')$$

for $i+j=n$. We leave the reader to check that the sum of all these morphisms over $i+j=n$ is an isomorphism. The left derived functor of Gr therefore induces a well defined symmetric monoidal ∞-functor (see A.1)

$$\mathbf{Gr} : \mathbf{dg}_k^{fil} \longrightarrow \mathbf{dg}_k^{gr}.$$

Remark 1.3.1.2. Note that the functor $Gr : C(k)^{fil} \to C(k)^{gr}$ and its left derived ∞-functor $\mathbf{Gr} : \mathbf{dg}_k^{fil} \to \mathbf{dg}_k^{gr}$ are related by the following diagram of ∞-functors

$$\begin{tikzcd} C(k)^{fil} \arrow[r, "Gr"] \arrow[d] & C(k)^{gr} \arrow[d] \\ \mathbf{dg}_k^{fil} \arrow[r, "\mathbf{Gr}"'] \arrow[ur, Rightarrow, "h"] & \mathbf{dg}_k^{gr}. \end{tikzcd}$$

where h is a *non-invertible* natural transformation between ∞-functors $C(k)^{fil} \to \mathbf{dg}_k^{gr}$ (measuring the non-commutativity of Gr with the vertical localization functors). We refer to A.1 for details on the general situation of deriving symmetric monoidal ∞-functors.

The second important functor out of the category $C(k)^{fil}$ is the *underlying object functor* . It is the functor

$$(-)^u : C(k)^{fil} \longrightarrow C(k)$$

sending E to the complex

$$E^u := \mathrm{Colim}_{n \in \mathbb{Z}} F^n E.$$

This functor is again the left adjoint of a Quillen adjunction. The right adjoint to this adjunction sends a complex E to the constant diagram $F^n E = E$, where all morphisms $F^n E \to F^{n+1} E$ are identities. We will denote this filtered object by E_∞. Note that in particular we have $Gr(E_\infty) \simeq 0$, whereas $(E_\infty)^u \simeq E$ by the counit of the adjunction. In particular $E \to E_\infty$ is fully faithful and $(-)^u$ is a localization functor.

The functor $(-)^u$ is again endowed with a natural symmetric monoidal structure

$$(E)^u \otimes_k (E')^u \simeq (E \otimes_k E')^u.$$

Both of these functors preserve quasi-isomorphisms, and after localization this adjunction induces an adjunction on the corresponding ∞-categories

$$(-)^u : \mathbf{dg}_k^{fil} \leftrightarrows \mathbf{dg}_k : (-)_\infty.$$

Remark 1.3.1.3. The ∞-category $\mathbf{dg}_k^{fil}$ can be canonically identified with the ∞-category $\mathbf{QCoh}([\mathbb{A}^1/\mathbb{G}_m])$ of quasi-coherent complexes on the (geometric) quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$. This idea goes back to C. Simpson (see [Mou21] for a recent account). Analogously (and more classically), the ∞-categoty $\mathbf{dg}_k^{gr}$ can be canonically identified with the ∞-category $\mathbf{QCoh}(B\mathbb{G}_m)$ of quasi-coherent complexes on the (geometric) classifying stack $B\mathbb{G}_m$ of $\mathbb{G}_m$. Through these identifications, the underlying object functor $(-)^u : \mathbf{dg}_k^{fil} \to \mathbf{dg}_k$ becomes the pullback on quasi-coherent complexes along the inclusion $\mathbf{Spec}\, k \simeq [\mathbb{G}_m/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$, while the associated graded object functor $\mathbf{Gr} : \mathbf{dg}_k^{fil} \to \mathbf{dg}_k^{gr}$ is viewed as the pullback along the map $0 : B\mathbb{G}_m \to [\mathbb{A}^1/\mathbb{G}_m]$ corresponding to the $\mathbb{G}_m$-equivariant inclusion of the origin in $\mathbb{A}^1$. The interested reader is referred to [Mou21] for details.

1.3.2. Graded mixed complexes and filtered complexes. We are now ready to compare filtered complexes with graded mixed complexes. This comparison states the existence of a Quillen adjunction between $C(k)^{fil}$ and $\epsilon - C(k)^{gr}$, which fully faithfully embeds the ∞-category of graded mixed complexes into the ∞-category of filtered complexes. The image of this embedding consists of *complete* filtered complexes, and thus we can state that the ∞-category of graded mixed complexes is equivalent to the ∞-category of complete filtered complexes. This equivalence can be enhanced into an equivalence of symmetric monoidal ∞-categories once the monoidal structure on filtered complexes is suitably completed.

We start by describing the *Tate realization functor*

$$|-|^t : \epsilon - C(k)^{gr} \longrightarrow C(k)^{fil}.$$

For a given graded mixed complex E the i-th layer of the filtered complex $|E|^t$ is given by

$$F^i|E|^t := \prod_{p\geq -i} E^{(p)}[-2p], \tag{3}$$

where the infinite product on the right is endowed with the total differential $D := d + \epsilon$. In formulas, for a family of elements $x := \{x_p\}$ in $\prod_{p\geq -i} E^{(p)}[-2p]$, the element $D(x) := \{D(x)_p\}$ is explicitly given by

$$D(x)_p := d(x_p) + \epsilon(x_{p-1}),$$

where by convention $x_p = 0$ if $p < -i$. The morphism $F^i|E|^t \hookrightarrow F^{i+1}|E|^t$ is itself defined to be the natural inclusion of infinite products

$$\prod_{p\geq -i} E^{(p)}[-2p] \hookrightarrow \prod_{p\geq -i-1} E^{(p)}[-2p],$$

that sets the first component to be zero.

This defines in a functor

$$|-|^t : \epsilon - C(k)^{gr} \longrightarrow C(k)^{fil}.$$

This functor commutes with arbitrary limits, and therefore by Freyd adjoint theorem (see [**TV15**, Prop. 2.9]), it admits a left adjoint denoted by ψ. The functor ψ can be made explicit using complicated formula, but we will not need this and for our purpose its existence will be enough. Moreover, by definition of the various model structures, it is clear that $|-|^t$ preserves fibrations. The following lemma shows that it furthermore preserves equivalences.

LEMMA 1.3.2.1. *The functor* $|-|^t : \epsilon - C(k)^{gr} \longrightarrow C(k)^{fil}$ *sends quasi-isomorphisms of graded mixed complexes to quasi-isomorphisms of filtered complexes.*

Proof. Let $E \to E'$ be a quasi-isomorphism of graded mixed complexes. We must show that for all $i \in \mathbb{Z}$ the induced morphism of complexes

$$F^i|E|^t \to F^i|E'|^t$$

is a quasi-isomorphism. For this, we use that sequence of subcomplexes $F^j|E|^t \subset F^i|E|^t$, for $j \leq i$. Each successive quotient morphism

$$(F^i|E|^t)/F^{j+1}|E|^t) \longrightarrow (F^i|E|^t)/(F^j|E|^t)$$

is a fibration of complexes, and by the explicit formula for $|E|^t$ ((3) on page 28) the natural map

$$F^i|E|^t \longrightarrow \lim_{j\leq i}(F^i|E|^t)/(F^j|E|^t)$$

is an isomorphism of complexes. In particular, the canonical map

$$F^i|E|^t \longrightarrow \mathrm{Holim}_{j\leq i}(F^i|E|^t)/(F^j|E|^t)$$

is itself an quasi-isomorphism of complexes. As a consequence, the lemma will follow if we can prove that for all $j \leq i$ fixed, the induced morphisme

$$(F^i|E|^t)/(F^j|E|^t) \longrightarrow (F^i|E'|^t)/(F^j|E'|^t)$$

is a quasi-isomorphism. But this can be proved by induction on j using the short exact sequences of complexes

$$0 \longrightarrow (F^j|E|^t)/(F^{j-1}|E|^t) \longrightarrow (F^i|E|^t)/(F^{j-1}|E|^t) \longrightarrow (F^i|E|^t)/(F^j|E|^t) \longrightarrow 0,$$

together with the identification of complexes $(F^j|E|^t)/(F^{j-1}|E|^t) \simeq E^{(-j)}[-2j]$. $\square$

An important consequence of the lemma 1.3.2.1 is that $|-|^t$ is a right Quillen functor. In particular, we obtain this way an induced adjunction of ∞-categories

$$\psi : \mathbf{dg}_k^{fil} \leftrightarrows \epsilon - \mathbf{dg}_k^{gr} : |-|^t.$$

Note that the associated graded of the realization can be described explicitly as (beware of the change of signs in the gradings !)

$$Gr^p(|E|) \simeq E^{-p}[-2p]. \tag{4}$$

The importance of the realization ∞-functor is the following result, that will be used many times during this book.

Propositon 1.3.2.2. *The ∞-functor*

$$|-|^t : \epsilon - \mathbf{dg}_k^{gr} \longrightarrow \mathbf{dg}_k^{fil}$$

is fully faithful.

Proof. We must show that for any object $E \in \epsilon - \mathbf{dg}_k^{gr}$, the counit morphism

$$\alpha : \psi(|E|^t) \longrightarrow E$$

is an equivalence in $\epsilon - \mathbf{dg}_k^{gr}$. For this, we combine the adjunction $(\psi, |-|^t)$ with the adjunction

$$\mathsf{oub} : \epsilon - \mathbf{dg}_k^{gr} \leftrightarrows \mathbf{dg}_k^{fil} : F,$$

where the left adjoint oub is the forgetful ∞-functor sending a graded mixed complex to its underlying graded complex. The fact that oub is a left adjoint follows formally from the description of $\epsilon - \mathbf{dg}_k^{gr}$ as the ∞-category of graded dg-modules over $k[\epsilon]$ (see §1.2.3). The ∞-functor oub is conservative, and thus in order to prove that α is an equivalence, it is enough to show the following fact: for any graded complex $M \in \mathbf{dg}_k^{gr}$, the induced morphism

$$\mathbf{Map}(E, F(M)) \longrightarrow \mathbf{Map}(\psi(|E|^t), F(M))$$

is an equivalence of spaces. Using the adjunction property this is equivalent to the fact that for all $M \in \mathbf{dg}_k^{gr}$, the natural morphism

$$\mathbf{Map}(\mathsf{oub}(E), M) \longrightarrow \mathbf{Map}(|E|^t, |F(M)|^t)$$

is an equivalence of spaces.

To prove this last statement on mapping spaces, it will be enough to have an explicit description of the filtered complex $|F(M)|^t$. As a start, $F(M)$ is the graded mixed complex whose piece of weight p is $M^{(p)} \oplus M^{(p+1)}[-1]$. The mixed structure is defined by the obvious map

$$\epsilon : M^{(p)} \oplus M^{(p+1)}[-1] \longrightarrow M^{(p+1)}[-1] \oplus M^{(p+1)}[-2]$$

which is the identity on the factor $M^{(p+1)}[-1]$ and zero on other factors. Using that M can be written as an infinite product $M \simeq \prod_p M^{(p)}(p)$ inside the ∞-category $\mathbf{dg}_k^{gr}$, we see that $|F(M)|^t$ is itself given, up to an equivalence, as an infinite product of filtered complexes

$$|F(M)|^t \simeq \prod_{p \in \mathbb{Z}} W_p.$$

Here, each W_p is the Tate realization of the graded mixed complex $F(M^{(p)}(p))$. Note that $F(M^{(p)}(p))$ only possesses non-zero weight pieces in weight p and $p-1$, respectively $M^{(p)}$ and $M^{(p)}[-1]$. The mixed structure is itself non-zero only in weight $p-1$, for which it is the identity

$$F(M^{(p)}(p))(p-1) = M^{(p)}[-1] \longrightarrow F(M^{(p)}(p))(p)[-1] = M^{(p)}[-1].$$

Using the explicit formula for the Tate realization $|-|^t$ (see (3) on page 28), we deduce that W_p is, up to a natural equivalence, the filtered complex given by

$$F^iW_p = \left\{ \begin{array}{cc} 0 & if\, i \neq p \\ M^{(p)}[-2p] & if\, i = -p. \end{array} \right.$$

As a result, for any $F' \in \mathbf{dg}_k^{fil}$, we have functorial equivalences in $M \in \mathbf{dg}_k^{gr}$

$$\mathbf{Map}_{\mathbf{dg}_k^{fil}}(F', |F(M)|^t) \simeq \prod_{p\in\mathbb{Z}} \mathbf{Map}_{\mathbf{dg}_k}(Gr^{-p}(F')[2p], M^{(p)}).$$

On the other hand, for $E \in \epsilon - \mathbf{dg}_k^{gr}$ we have canonical equivalences of complexes (see (4) above)

$$E^{(-p)}[-2p] \simeq Gr^p(|E|^t).$$

Therefore, we have canonical equivalences of spaces

$$\mathbf{Map}_{\mathbf{dg}_k^{gr}}(\mathsf{oub}(E), M) \simeq \prod_{p\in\mathbb{Z}} \mathbf{Map}_{\mathbf{dg}_k}(E^{(p)}, M^{(p)}) \simeq \mathbf{Map}_{\mathbf{dg}_k^{fil}}(|E|^t, |F(M)|^t).$$

By construction, this equivalence is the natural morphism $Map(\mathsf{oub}(E), M) \longrightarrow Map(|E|^t, |F(M)|^t)$ induced by adjunction. This concludes the proof that the counit α is itself an equivalence, and thus the proof of the Proposition. □

1.3.3. Graded mixed complexes as complete filtered complexes. The full embedding of Proposition 1.3.2.2 can be refined into an equivalence of ∞-categories, by characterizing the essential image of the ∞-functor ψ. For this, we introduce the notion of *completeness* for objects in $\mathbf{dg}_k^{fil}$.

DEFINITION 1.3.3.1. *An object $E \in \mathbf{dg}_k^{fil}$ is* complete *if we have an equivalence in $\mathbf{dg}_k$*

$$\lim_{i\leq 0} F^iE \simeq 0.$$

Note that the limit and the equivalence of the above definition happen inside the ∞-category $\mathbf{dg}_k$ of complexes. In particular, $\lim_{i\leq 0} F^iE$ is here the homotopy limit of the projective system of complexes

$$\dots \longrightarrow F^iE \longrightarrow F^{i-1}E \longrightarrow \dots \longrightarrow F^{-1}E \longrightarrow F^0E.$$

On the level of cohomology, this homotopy limit also involves the derived functor $\lim^1$. In particular, if V is a k-module endowed with a filtration by submodules $F^iV \subset V$, being complete in the sense of Definition 1.3.3.1 is stronger than the naive separateness condition stating that $\cap_i F^iV = 0$. Indeed, if we denote by $F^0\hat{V} := \lim_{i\leq 0} F^0V/F^iV$ the naive completion of k-modules, we have natural isomorphisms

$$H^0(\mathrm{Holim}_i F^iV) \simeq \cap_i F^iV \qquad H^1(\mathrm{Holim}_i F^iV) \simeq (F^0\hat{V})/(F^0V),$$

and $H^i(\mathrm{Holim}_i F^i V) \simeq 0$ for all $i \neq 0, 1$. This implies that a filtered k-module V is complete in the sense of Definition 1.3.3.1 if and only if it satisfies the following two conditions

- $\cap_{i\leq 0} F^i V = 0$
- the inclusion map $F^0 V \to F^0 \hat{V}$ is bijective.

REMARK 1.3.3.2. The completeness condition of Definition 1.3.3.1 can also be stated as follows. For a filtered complex $E \in \mathbf{dg}_k^{fil}$, and any $j \in \mathbb{Z}$ we can form the natural morphism of complexes

$$F^j E \longrightarrow \lim_{i\leq j} (F^i E)/(F^j E).$$

Then, E is complete if and only if for all j the above morphism is a quasi-isomorphism. Indeed, the fiber of the above morphism, taken in $\mathbf{dg}_k$, is equivalent to $\lim_{i\leq j} F^i E$. Moreover, if $j \geq 0$, the natural projection $\lim_{i\leq j} F^i E \longrightarrow \lim_{i\leq 0} F^i E$ is a quasi-isomorphism, as the inclusion of posets $\{i \leq 0\} \subset \{i \leq j\}$ is final. In the same manner, if $j \leq 0$, the projection $\lim_{i\leq 0} F^i E \longrightarrow \lim_{i\leq j} F^i E$ is an equivalence.

PROPOSITON 1.3.3.3. *The essential image of the full embedding*

$$| - |^t : \epsilon - \mathbf{dg}_k^{gr} \hookrightarrow \mathbf{dg}_k^{fil}$$

consists of all complete filtered complexes in the sense of Definition 1.3.3.1.

Proof. By the Proposition 1.3.2.2, the ∞-functor $| - |^t$ identifies $\epsilon - \mathbf{dg}_k^{gr}$ with a full reflexive sub-∞-category $\mathcal{C}$ of $\mathbf{dg}_k^{fil}$. We have to show that an object $E \in \mathbf{dg}_k^{fil}$ lies in this sub-∞-category if and only if it is complete.

As a start, from the explicit form of the Tate realization functor (see (3) on page 28) we see that $|E|^t$ is a complete filtered complex for any graded mixed complex E. Therefore, all objects of $\mathcal{C}$ are complete filtered complexes. Conversely, let E be a complete filtered complex. By definition, E lies in $\mathcal{C}$ if and only if the adjuntion unit morphism $E \longrightarrow |\psi(E)|^t$ is a quasi-isomorphism of filtered complexes. During the proof of Proposition 1.3.2.2, we have seen that the composition of ψ with the forgetful ∞-functor $\mathsf{oub} : \epsilon - \mathbf{dg}_k^{gr} \to \mathbf{dg}_k^{gr}$, is naturally equivalent to $E \mapsto Gr^{-*}(E)[2*]$. Here, $Gr^{-*}(E)[2*]$ denotes the graded complexe whose weight p piece is $Gr^{-p}(E)[2p]$. We deduce from this the following lemma.

LEMMA 1.3.3.4. *The ∞-functor*

$$\psi : \mathbf{dg}_k^{fil} \to \epsilon - \mathbf{dg}_k^{gr},$$

restricted to the full sub-∞-category of complete filtered complexes, is conservative.

Proof of the lemma. As the forgetful ∞-functor oub is itself conservative, it is enough to prove the conservativity of the composition $\mathsf{oub} \circ \psi : \mathbf{dg}_k^{fil} \to \mathbf{dg}_k^{gr}$, when restricted to complete filtered complexes. By the formula $\circ oub \psi \simeq Gr^{-*}[2*]$ we see that it is even enough to show that the family of ∞-functors Gr^i for $i \in \mathbb{Z}$ is a conservative family of ∞-functors when restricted to complete filtered complexes.

Let $E \to E'$ be a morphism of complete filtered complexes, and assume that for all $i \in \mathbb{Z}$ the induced morphism $Gr^i(E) \to Gr^i(E')$ is a quasi-isomorphism. For all $i \in \mathbb{Z}$, we have a canonical morphism of complexes

$$\alpha_i : F^iE \longrightarrow \lim_{j \leq i}(F^iE/F^jE).$$

The fiber of this morphism is naturally equivalent to $\lim_{j\leq i} F^jE$, which is the zero object by completeness of E. Therefore, for all i, the morphism α_i is an equivalence of complexes. Moreover, for all $j \leq i$ we have a commutative diagram of complexes, whose horizontal rows are fibration sequences of complexes

$$\begin{tikzcd}
Gr^{-j}(E)[2j] \arrow[r] \arrow[d] & (F^i|E|^t)/(F^{j-1}|E|^t) \arrow[r] \arrow[d] & (F^i|E|^t)/(F^j|E|^t) \arrow[d] \\
Gr^{-j}(E')[2j] \arrow[r] & (F^i|E'|^t)/(F^{j-1}|E'|^t) \arrow[r] & (F^i|E|^t)/(F^j|E|^t).
\end{tikzcd}$$

We therefore see by induction on j, than for all $j \leq i$ the induced morphism

$$(F^i|E|^t)/(F^j|E|^t) \longrightarrow (F^i|E'|^t)/(F^j|E'|^t)$$

is a quasi-isomorphism. Passing to the limit along $j \leq i$ and using completeness as above we have that the induced morphism

$$F^iE \simeq \lim_{j\leq i}(F^iE/F^jE) \longrightarrow F^iE' \simeq \lim_{j\leq i}(F^iE'/F^jE')$$

is a quasi-isomorphism for all $i \in \mathbb{Z}$. □

The lemma implies that complete filtered complexes are objects in $\mathcal{C}$, and thus finishes the proof of the Proposition. □

1.3.4. The completed monoidal structure. By Proposition 1.3.3.3 we know that the Tate-realization ∞-functor $|-|^t : \epsilon-\mathbf{dg}_k^{gr} \longrightarrow \mathbf{dg}_k^{fil}$ is fully faithful, and that its essential image consists of all complete filtered complexes in the sense of Definition 1.3.3.1. We will denote by $\widehat{\mathbf{dg}}_k^{fil} \subset \mathbf{dg}_k^{fil}$ the full sub-∞-category of complete filtered complexes, so that $|-|^t$ induces an equivalence of ∞-categories

$$|-|^t : \epsilon - \mathbf{dg}_k^{gr} \simeq \widehat{\mathbf{dg}}_k^{fil}.$$

We will now study the monoidal behaviour of the above equivalence. For this, we start by considering the right Quillen functor $|-|^t : \epsilon - C(k) \longrightarrow C(k)^{fil}$, and show that it carries a natural lax monoidal structure. For two graded mixed complexes E and F, we construct a natural morphism

$$\alpha_{E,F} : |E|^t \otimes |F|^t \longrightarrow |E \otimes F|^t \tag{5}$$

as follows. Such a morphism is determined by a compatible family of maps

$$\alpha_{i,j} : F^i|E|^t \otimes F^j|F|^t \longrightarrow F^{i+j}|E \otimes F|^t$$

for all integers i and j. By the explicit formula for $|-|^t$ the morphism $\alpha_{i,j}$ is a morphism of complexes

$$(\prod_{p\geq -i} E^{(p)}[-2p]) \otimes (\prod_{q\geq -j} F^{(q)}[-2q]) \longrightarrow \prod_{n\geq -i-j} (\bigoplus_{a+b=n} E^{(a)} \otimes F^{(b)}[-2n]).$$

On the level of graded modules, the above morphism is given by a family of maps

$$\alpha_n : (\prod_{p\geq -i} E^{(p)}[-2p]) \otimes (\prod_{q\geq -j} F^{(q)}[-2q]) \longrightarrow \bigoplus_{a+b=n} E^{(a)} \otimes F^{(b)}[-2n].$$

By definition, the morphism α_n symbolically sends a tensor of the form $\{x_p\} \otimes \{y_q\}$ to the element $\sum_{p+q=n} x_p \otimes y_q$ (note that this sum is finite as p and q are bounded below by $-i$ and $-j$). The family of α_n for $n \in \mathbb{Z}$ provides the morphism of complexes

$$\alpha_{i,j} : F^i|E|^t \otimes F^j|F|^t \longrightarrow F^{i+j}|E \otimes F|^t.$$

These morphisms are associative and symmetric with respect to the associativity and unit constraints. Moreover, the image by $|-|^t$ of the unit object $k(0) \in \epsilon - C(k)^{gr}$ is canonically isomorphic to k, considered as a filtered complex by

$$F^i k = \begin{cases} 0 & if\; i < 0 \\ k & if\; i \geq 0 \end{cases}$$

In other words, there is a canonical isomorphism $u : |k(0)|^t \simeq k$, where k is the unit object in $C(k)^{fil}$. All together, the $\alpha_{i,j}$ and the isomorphism u defines a lax monoidal structure on the functor $|-|^t$, which is unital, associative and symmetric.

By adjunction, the lax monoidal structure on $|-|^t$ defines a unital, associative and symmetric colax monoidal structure on the left adjoint functor $\psi : C(k)^{fil} \to \epsilon - C(k)^{gr}$ as follows. We start by E and F two filtered complexes, and we consider the adjunction morphisms

$$E \to |\psi(E)|^t \qquad F \to |\psi(F)|^t.$$

Using the lax monoidal structure on $|-|^t$ constructed above (see (5) on page 33), we define a morphism

$$E \otimes F \longrightarrow |\psi(E)|^t \otimes |\psi(F)|^t \longrightarrow |\psi(E) \otimes \psi(F)|^t.$$

Its image by ψ, composed with the counit $\psi|-|^t \Rightarrow id$, gives a morphism

$$\psi(E \otimes F) \longrightarrow \psi(E) \otimes \psi(F),$$

which is a unital, associative and symmetric colax monoidal structure on the functor ψ.

LEMMA 1.3.4.1. *The induced colax monoidal structure on the ∞-functor*

$$\psi : \mathbf{dg}_k^{fil} \longrightarrow \epsilon - \mathbf{dg}_k^{gr}$$

is a monoidal structure. In other words, for all E and F objects in $\mathbf{dg}_k^{fil}$, the natural morphism

$$\beta_{E,F} : \psi(E \otimes F) \longrightarrow \psi(E) \otimes \psi(F)$$

is an equivalence of graded mixed complexes.

Proof. To check that $\beta_{E,F}$ is a equivalence in $\epsilon - \mathbf{dg}_k^{gr}$ we can use the conservative ∞-functor $\mathsf{oub} : \epsilon - \mathbf{dg}_k^{gr} \to \mathbf{dg}_k^{gr}$ to graded complexes. Moreover, we know that $\mathsf{oub} \circ \psi$ is naturally equivalent to $Gr^{-*}[-2*]$ (see the proof of Proposition 1.3.2.2). By observation, we the image of $\beta_{E,F}$ by oub induces a morphism

$$Gr^{-*}(E \otimes F)[-2*] \longrightarrow Gr^{-*}(E)[-2*] \otimes Gr^{-*}(F)[-2*]$$

which is nothing else than the standard monoidal structure on the ∞-functor $Gr^{-*}[-2*]$. It is therefore an equivalence, proving that $\beta_{E,F}$ is indeed an equivalence of graded mixed complexes.
$\square$

The consequence of the above lemma is that the colax monoidal structure on the ∞-functor ψ is a symmetric monoidal structure. Therefore, we can use the equivalence

$$| - |^t : \epsilon - \mathbf{dg}_k^{gr} \simeq \widehat{\mathbf{dg}}_k^{fil},$$

in order to transport the monoidal structure of $\epsilon - \mathbf{dg}_k^{gr}$ to a symmetric monoidal structure on $\widehat{\mathbf{dg}}_k^{fil}$. It will be denoted by $\hat{\otimes}$, and called the *completed tensor product* . The previous lemma, implies that the ∞-functor ψ provides a symmetric monoidal ∞-functor

$$\psi : \mathbf{dg}_k^{fil} \longrightarrow \widehat{\mathbf{dg}}_k^{fil} \simeq \epsilon - \mathbf{dg}_k^{gr}.$$

This exhibits $\widehat{\mathbf{dg}}_k^{fil}$ has a symmetric monoidal localization of $\mathbf{dg}_k^{fil}$, obtained by inverting the *graded quasi-isomorphisms* (i.e. morphisms of filtered complexes inducing quasi-isomorphisms on the associated graded). The localization ∞-functor ψ will also be called the *completion ∞-functor*, and denoted by

$$\widehat{(-)} : \mathbf{dg}_k^{fil} \longrightarrow \widehat{\mathbf{dg}}_k^{fil}.$$

Note that by construction we have canonical equivalences of complete filtered complexes

$$(\widehat{E \otimes F}) \simeq \widehat{E} \hat{\otimes} \widehat{F}.$$

Note also, that the completion ∞-functor naturally commutes with forming the associated graded

$$Gr^*(\widehat{E}) \simeq Gr^*(E),$$

as can be seen directly from the identification between $\mathsf{oub} \circ \psi$ and $Gr^*[-2*]$.

On the level on notations, $|E|^t$ will always refer to the filtered complex associated to the graded mixed complex E. We will also use the following notation

DEFINITION 1.3.4.2. *We define the (non Tate)* realization *∞-functor*

$$|-|:\epsilon-\mathbf{dg}_k^{gr}\to\mathbf{dg}_k\,,\;E\longmapsto|E|:=F^0|E|^t$$

sending a mixed graded module E to the 0-th layer filtration of its Tate realization.

REMARK 1.3.4.3. It is easy to verify that the realization functor $|-|$ is equivalent to $\mathbb{R}\underline{Hom}(k(0),-)$, the derived enriched Hom complex from the unit graded mixed object $k(0)$. Note also that, when E has only non-negative weight pieces, we have

$$|E|\simeq F^0|E|^t\simeq(|E|^t)^u$$

so $|E|$ is in fact the underlying complex of the filtered complexe $|E|^t$.

We can subsume our comparison results as follows.

COROLLARY 1.3.4.4. *The Tate realization ∞-functor*

$$|-|^t:\epsilon-\mathbf{dg}_k^{gr}\longrightarrow\widehat{\mathbf{dg}}_k^{fil}$$

is a symmetric monoidal equivalence when $\widehat{\mathbf{dg}}_k^{fil}$ is endowed with the complete tensor product $\hat{\otimes}_k$. It induces an equivalence on the ∞-categories of commutative algebras

$$\mathbf{CAlg}(\epsilon-\mathbf{dg}_k^{gr})\simeq\mathbf{CAlg}(\widehat{\mathbf{dg}}_k^{fil}).$$

We note here that the ∞-categories of commutative algebras of the previous corollary can be described in simple terms as localizations of model categories of strict commutative algebra objects, thanks to the rectification result of [**Lur22a**, Prop. 4.5.4.6]. In particular, $\mathbf{CAlg}(\epsilon-\mathbf{dg}_k^{gr})$ is canonically equivalent to $\epsilon-\mathbf{cdga}_k^{gr}$, the ∞-category of graded mixed cdga's described in the next section. In the same manner, $\mathbf{CAlg}(\widehat{\mathbf{dg}}_k^{fil})$ can be identified with the localization of the model category of complete filtered cdgas (i.e. commutative algebra objects in the model category $C(k)^{fil}$ of filtered complexes, whose underlying filtered complex is complete).

REMARK 1.3.4.5. It is easy to check that the (non Tate) realization functor $|-|$ of Definition 1.3.4.2 induces a functor

$$|-|:\mathbf{CAlg}(\epsilon-\mathbf{dg}_k^{gr})\longrightarrow\mathbf{cdga}_k.$$

1.4. Graded mixed cdga's and derived de Rham theory

In this section we show how graded mixed complexes provide a setting for de Rham theory of algebraic varieties and schemes, and more generally derived stacks. For this we assume further more that our base ring k is a $\mathbb{Q}$-algebra. The extension to non-zero characteristics require substantial modification and will be treated separately in last chapter of this book (see chapter 7).

We remind the symmetric monoidal model category $\epsilon - C(k)^{gr}$ of graded mixed complexes, and its corresponding symmetric monoidal ∞-category $\epsilon - \mathbf{dg}_k^{gr}$. We will use in the sequel the category of commutative (unital and associative) monoids in $\epsilon - C(k)^{gr}$, for which we introduce the following notation

$$\epsilon - CAlg(k)^{gr} := Comm(\epsilon - C(k)^{gr}).$$

Objects in $\epsilon - CAlg(k)^{gr}$ will be referred to as *graded mixed cdga's*. In a more explicit form, a graded mixed cdga A consists of the data of a graded mixed complex A, together with

- (multiplication) morphisms of complexes
$$(.) : A^{(i)} \otimes_k A^{(j)} \longrightarrow A^{(i+j)},$$
- (unit) a cocycle $1 \in A^{(0)}$ of cohomological degree 0.

The multiplication and the unit are required to satisfy the obvious associativity, unity and commutativity axioms. Moreover, the multiplication and the unit must be compatible with the mixed structure ϵ in the following sense:

$$\epsilon(1) = 0 \qquad \epsilon(x.y) = \epsilon(x).y + (-1)^{|x|}x.\epsilon(y),$$

for all elements x and y in A with x of cohomological degree $|x| \in \mathbb{Z}$. We note in particular that the sign appearing the above formula does not depend on the weights of x and y, but only on their cohomological degrees.

We can use the forgetful functor $\epsilon - CAlg(k)^{gr} \longrightarrow \epsilon - C(k)^{gr}$, which forgets the multiplicative structure, in order to create a model category structure on $\epsilon - CAlg(k)^{gr}$ for which the fibrations and weak equivalences are defined in $\epsilon - C(k)^{gr}$. The existence of this model category structure uses in an essential manner that k is assumed to be a $\mathbb{Q}$-algebra, and can be deduced for instance from the general statement [**Lur22a**, Prop. 4.5.4.6]. The homotopy theory of graded mixed cdga's is central in the definition and study of derived foliations, and we thus set the following important definition and notations.

Definition 1.4.0.1. *The* model category of graded mixed cdga's *is $\epsilon - CAlg(k)^{gr}$ defined above. The corresponding ∞-category will be denoted by $\epsilon - \mathbf{cdga}_k^{gr}$ and will be called the* ∞-category of graded mixed cdga's over *k.*

The model category $\epsilon - CAlg(k)^{gr}$ comes equipped with two important Quillen adjunctions, which will be used often all along the text. As a start, we have the adjunction induced by the forgetful functor

$$Sym^{\epsilon} : \epsilon - C(k)^{gr} \rightleftarrows \epsilon - CAlg(k)^{gr} : \mathsf{oub}.$$

Here the left adjoint sends a graded mixed complex E to the usual symmetric algebra object

$$Sym^{\epsilon}(E) := \bigoplus_{n \geq 0} E^{\otimes n}/\Sigma_n,$$

where the sum, tensor products and quotients by symmetric groups are all understood in the category $\epsilon - C(k)^{gr}$ of graded mixed complexes. We have denoted this functor by Sym^{ϵ} to avoid confusions with other Sym constructions that we will use later.

The second important Quillen adjunction is denoted by

$$DR : CAlg(k) \rightleftarrows \epsilon - CAlg(k)^{gr} : (-)^{(0)},$$

where $CAlg(k)$ denotes the model category of (non-graded and non-mixed) commutative dg-algebras. The right adjoint of this adjunction sends a graded mixed cdga A to its weight 0 part $A^{(0)}$, which comes equipped with an induced commutative dg-algebra structure. The left adjoint is called the *de Rham functor* and will be studied in detailed in the next two paragraphs.

The two mentioned above Quillen adjunctions of course induce natural adjunctions of ∞-categories (as usual by A.1

$$Sym^{\epsilon} : \epsilon - \mathbf{dg}_k^{gr} \rightleftarrows \epsilon - \mathbf{cdga}_k^{gr} : \mathsf{oub} \qquad \mathbf{DR} : \mathbf{cdga}_k \rightleftarrows \epsilon - \mathbf{cdga}_k^{gr} : (-)^{(0)}.$$

1.4.1. Derivations and the cotangent complex. For a commutative dg-algebra $A \in CAlg(k)$, and a (left) A-dg-module M, we can form the trivial square zero extension of A by M in the usual manner. It is a new commutative dg-algebra, denoted by $A \oplus M$ or $A[M]$, whose underlying complex is the direct sum $A \oplus M$. The multiplication on $A \oplus M$ is the morphism

$$\mu : \ (A \oplus M) \otimes (A \oplus M) \simeq (A \otimes A) \oplus (A \otimes M) \oplus (M \otimes A) \oplus (M \otimes M) \longrightarrow A \oplus M,$$

defined as follows. On the factor $A \otimes A$ the morphism μ is the multiplication in A, followed by the natural inclusion $A \hookrightarrow A \oplus M$. On the factor $M \otimes M$ the morphism μ is the zero map. On the factor $A \otimes M$, μ is the given by the left A-dg-module structure on M, followed by the natural inclusion $M \hookrightarrow A \oplus M$. Finally, on the factor $M \otimes A$, μ is given by the right A-dg-module structure on A followed by the inclusion $M \hookrightarrow A \oplus M$. We note here that the right A-dg-module structure on M is defined using the symmetry constraints on complexes (see (1) on page 18), and is given explicitly for $a \in A^i$ and $m \in M^j$ by $m.a = (-1)^{i+j} a.m$. Unfolding the above description of the multiplication on $A \oplus M$, we can also write the following formula, for homogeneous elements $(a, m) \in A \oplus M$ and $(a', m') \in A \oplus M$

$$(a, m).(a', m') = (a.a', am' + (-1)^{|a'||m|} a'm).$$

With this definition, a derivation on A with values in M is a section of the natural projection $A \oplus M \longrightarrow A$ inside the category of commutative dg-algebras (see [**TV08**, §1.2.1]). Such a section is of the form $(id_A, u) : A \longrightarrow (A \oplus M)$, with $u : A \longrightarrow M$ a morphism of complexes. The morphism u must also satisfy the usual condition

$$u(a.b) = u(a).b + a.u(b) = a.u(b) + (-1)^{|a||b|} b.u(a)$$

for homogeneous elements a and b in A.

For any commutative dg-algebra A, there exists a universal derivation

$$d : A \longrightarrow \Omega^1_{A/k}.$$

Here $\Omega^1_{A/k}$ is the dg-module of Kähler differentials on the commutative dg-algebra A (relative to k), which can be explicitly presented as I/I^2 where $I \subset A \otimes A$ is the kernel of the multiplication map $A \otimes A \longrightarrow A$. The universal property here states that for any left A-dg-module M and any derivation $u : A \to M$, there exists a unique morphism of A-dg-modules $f : \Omega^1_{A/k} \to M$ such that $f \circ d = u$

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "u"'] & \Omega^1_{A/k} \arrow[d, "\exists!"] \\ & M. \end{tikzcd}$$

The construction $A \mapsto \Omega^1_{A/k}$ is functorial in A, and can be seen to be a left adjoint of a Quillen adjunction as follows. We form the category $CAlgMod(k)$, whose objects are pairs (A, M), consisting of a commutative dg-algebra A and a left A-dg-module M. The morphisms from (A, M) to (A', M') are pairs (u, f), where $u : A \to A'$ is a morphism of commutative dg-algebras and $f : M \to M'$ is a morphism of A-dg-modules (where M' is considered as a left A-dg-module via u). The category $CAlgMod(k)$ admits a model category structure for which a morphism (u, f) as above is a weak equivalence (respectively a fibration) if u and f are quasi-isomorphisms of complexes (respectively an epimorphism of complexes). This model category is a particular example of a general construction of a relative model category structure, as explained in [**HP15**, Cor. 6.3.16].

The construction $(A, M) \mapsto A \oplus M$ defines a functor $CAlgMod(k) \longrightarrow CAlg(k)$. This functor is a right Quillen functor and its left adjoint sends a commutative dg-algebra A to the pair $(A, \Omega^1_{A/k})$. We obtain this way an induced adjunction on the corresponding ∞-categories obtained by inversting quasi-isomorphisms

$$\mathbf{cdga}_k \leftrightarrows \mathbf{cdgamod}_k.$$

The left adjoint sends a commutative dg-algebra A to an object of the form $(A, \mathbb{L}_{A/k})$, where $\mathbb{L}_{A/k}$ is a left A-dg-module called the *cotangent complex of* A . An explicit model for the A-dg-module $\mathbb{L}_{A/k} \in \mathbf{dg}_A$ is given by

$$\mathbb{L}_{A/k} \simeq A' \otimes_A \Omega^1_{A'/k},$$

where $A' \overset{\sim}{\twoheadrightarrow} A$ is a cofibrant model of A.

Being a left adjoint, the ∞-functor $A \mapsto (A, \mathbb{L}_{A/k})$ commutes with arbitrary colimits. If $A = \mathrm{Colim}_i A_i$ inside the ∞-category $\mathbf{cdga}_k$, then we have a canonical equivalence of A-dg-modules

$$\mathbb{L}_{A/k} \simeq \mathrm{Colim}_i (A \otimes_{A_i} \mathbb{L}_{A_i/k}).$$

1.4.2. The derived de Rham complex as a left adjoint. We consider the functor $E \mapsto E^{(0)}$, from graded mixed complexes to complexes, defined by keeping the sole weight 0 piece. This functor possesses a natural symmetric lax monoidal structure and therefore naturally extends to a functor $A \mapsto A^{(0)}$ from the category of graded mixed cdga's to the category of cdga's. This functor $CAlg(\epsilon - C(k)^{gr}) \longrightarrow CAlg(C(k))$ is the is the right adjoint of a Quillen adjunction whose left adjoint is called the de Rham functor

$$DR(-/k) : CAlg(C(k)) \longrightarrow CAlg(\epsilon - C(k)^{gr}).$$

Explicitly, the functor DR sends a commutative dg-algebra A to the graded mixed cdga whose weight n piece is $Sym^n_A(\Omega^1_{A/k}[1])$. The multiplicative structure on $DR(A)$ is the standard multiplication induced on a symmetric algebra. The mixed structure is itself induced from the usual de Rham differential $dR : A \longrightarrow \Omega^1_{A/k}$, and its natural multiplicative extension to the whole symmetric algebra $Sym_A(\Omega^1_{A/k}[1])$. In short we write $DR(A/k) = Sym_A(\Omega^1_{A/k}[1])$. We leave to the reader to check that this construction $A \mapsto DR(A/k)$ does indeed define a left adjoint to $(-)^0$, by observing that for any graded mixed cdga B the morphism $\epsilon : B^{(0)} \to B(1)$ is derivation on $B^{(0)}$ with coefficients in the $B^{(0)}$-dg-module $B^{(1)}$, and using the univeral property of $\Omega^1_{A/k}$.

The functor $A \mapsto DR(A/k)$ being left Quillen it induces an ∞-functor on the corresponding ∞-categories (see §A.1)

$$\mathbf{DR} : \mathbf{cdga}_k \longrightarrow \epsilon - \mathbf{cdga}^{gr}_k.$$

This ∞-functor is left adjoint to $A \mapsto A^{(0)}$, sending a graded mixed cdga A to its peice of weight 0. The cotangent complexes being obtained as the left derived functor of $A \mapsto \Omega^1_{A/k}$ we see that on the level of ∞-categories we have canonical equivalences of graded cdga's

$$\mathbf{DR}(A) \simeq Sym_A(\mathbb{L}_{A/k}[1])$$

where the symmetric algebra is understood in the symmetric monoidal ∞-category $\mathbf{dg}^{gr}_k$ of graded complexes. In more concrete terms a model for $\mathbf{DR}(A/k)$ is $DR(A'/k)$ where A' is a cofibrant model of A.

Definition 1.4.2.1. *(1) The* (derived) de Rham algebra *of a commutative dg-algebra A (over k) is the graded mixed commutative dg-algebra*

$$\mathbf{DR}(A) \simeq Sym_A(\mathbb{L}^1_{A/k}[1]) \in \epsilon - \mathbf{dg}^{gr}_k$$

defined above.

(2) The (Hodge-completed and derived) de Rham complex *of a commutative dg-algebra A (over k) is the filtered complex*

$$\widehat{C}_{DR}(A/k) := |\mathbf{DR}(A/k)|^t \in \mathbf{dg}^{fil}_k.$$

By definition the derived de Rham complex $\widehat{C}_{DR}(A/k)$ is a complete filtered complex, and its filtration is the Hodge filtration (hence the use of the expression *Hodge-completed* in the definition above). Its associated graded object is the graded cdga $Sym_A(\mathbb{L}_{A/k}[-1])$, where $\mathbb{L}_{A/k}$ sits in weight -1. It also comes equipped with a canonical commutative multiplicative structure, because the functor $|-|^t$ is endowed with a canonical symmetric lax monoidal structure (see corollary 1.3.4.4). Therefore, $\widehat{C}_{DR}(A/k)$ can be promoted to a filtered and complete commutative dg-algebra over k, but we will use this highly structured version rarely in the sequel.

In the particular case where A is a smooth k-algebra, considered as a dg-algebra concentrated in degree 0, $\widehat{C}_{DR}(A/k)$ turns out to be canonically equivalent to the usual algebraic de Rham complex of A relative to k

$$\widehat{C}_{DR}(A/k) \simeq (\ A \longrightarrow \Omega^1_{A/k} \longrightarrow \Omega^2_{A/k} \longrightarrow \dots \longrightarrow \Omega^i_{A/k} \longrightarrow \dots\ )$$

Indeed, when A is smooth the natural morphism $\mathbb{L}_{A/k} \to \Omega^1_{A/k}$ is a quasi-isomorphism. Therefore, for a cofibrant replacement $A' \stackrel{\sim}{\twoheadrightarrow} A$, the the canonical projection of graded mixed cdga's

$$Sym_{A'}(\mathbb{L}_{A'/k}[1]) \to Sym_A(\Omega^1_{A/k}[1])$$

is a quasi-isomoprhism. Passing to the Tate realizations, we get an equivalence of filtered and complete cdga's

$$\widehat{C}_{DR}(A/k) \simeq C^*_{DR}(A/k)$$

where the right hand side is the usual, *underived* de Rham complex of A over k. This is a bounded complex concentrated in degrees $[0, d]$, where d is the relative dimension of A over k. This complex is filtered by the usual Hodge filtration, and is complete because the filtration is finite. As, by convention, all our filtrations are *increasing*, this Hodge filtration is indexed in a non-standard way by non-positive integers $p \leq 0$ as follows

$$F^p\widehat{C}_{DR}(A/k) = (\ \Omega^{-p}_{A/k} \longrightarrow \Omega^{-p+1}_{A/k} \longrightarrow \dots\ ) \subset \widehat{C}_{DR}(A/k).$$

The associated graded $Gr^*(\widehat{C}_{DR}(A/k))$ is thus the usual Hodge cohomology $\oplus\Omega^i_{A/k}[-i]$, with $\Omega^i_{A/k}$ of weight $-i$.

To finish this part we mention below two well known important properties of derived de Rham cohomology.

Propositon 1.4.2.2. *Let $X = \mathbf{Spec}\, A$ be a derived affine scheme of essentially of finite presentation over $\mathbb{C}$ and $X(\mathbb{C})$ its space of $\mathbb{C}$-points endowed with its transcendent topology. Then there exists a natural equivalence of cdga's*

$$\widehat{C}^*_{DR}(A/\mathbb{C}) \simeq C^*(X(\mathbb{C}), \mathbb{C}),$$

where the right hand side denotes sheaf cohomology with coefficients in the constant sheaf $\mathbb{C}$.

Proof. This results is proven in [**Bha12**]. We can provide a direct proof as follows. As a first statement, the right hand side is well known to be functorially equivalent to algebraic de Rham cohomology of X. Algebraic de Rham cohomology can be computed by chosing a closed embedding $X \to Y = Spec\, B$ for Y a smooth affine scheme over $\mathbb{C}$, and then considering the formal de Rham complex $C^*_{DR}(\hat{Y}_X)$ of Y along X. The formal de Rham complex is obtained by definition from $DR(Y)$ by completion along X

$$C^*_{DR}(\hat{Y}_X) = \ \hat{B} \longrightarrow \hat{\Omega}^1_B \longrightarrow \dots \longrightarrow \hat{\Omega}^r_B \longrightarrow \dots$$

where $\hat{B} = lim_n B/I^n$ with $I \subset B$ the ideal of definition of X, and $\hat{\Omega}^r_B := \hat{B} \otimes_B \Omega^r_B$.

From the results reminded in A.6 the de Rham complex $C^*_{DR}(\hat{Y}_X)$ is also given by

$$C^*_{DR}(\hat{Y}_X) \simeq lim_n \widehat{C}_{DR}(Y_n)$$

where $Y_n = B/I^n$. But the Hodge completed derived de Rham cohomology is invariant by infinitesimal thickening (this can be see along the same lines as A.8.0.3, see A.8.0.4), so we have

$$C^*_{DR}(\hat{Y}_X) \simeq \widehat{C}_{DR}(t_0(X)) \simeq \widehat{C}_{DR}(X)$$

via the pull-backs along the closed immersions $t_0(X) \to X \to Y$. □

For the next Proposition, recall that for any connective cdga A over k, the structural morphism $k \to \mathbf{DR}(A/k)$ is a morphism of graded mixed cdga (where k is endowed with the zero mixed structure). By passing to realizations we get this way a natural morphism of cdga

$$k \to |\mathbf{DR}^*(A/k)| \simeq \widehat{C}^*_{DR}(A/k).$$

Suppose now that A is a discrete commutative $\mathbb{Q}$-algebra and $I \subset A$ an ideal. We can apply the previous morphism to the augmentation $A \to A/I$, and get this way a natural morphism of cdga's

$$A \to \widehat{C}^*_{DR}((A/I)/A).$$

Moreover, as this is obtained by realization from a morphism of graded mixed cdgas, this morphism is compatible with the augmentations to A/I. In particular, this morphism is compatible with the filtrations: the I-adic filtration on A and the natural filtration on $\widehat{C}^*_{DR}((A/I)/A)$. As $\widehat{C}^*_{DR}((A/I)/A)$ is already known to be complete it induces a morphism of completed filtered cdga's

$$\hat{A} \to \widehat{C}^*_{DR}((A/I)/A)$$

where $\hat{A}$ is the I-adic completion of A.

Propositon 1.4.2.3. *Let A be a commutative k-algebra of finite type and $I \subset A$ an ideal. Then, the canonical morphism*

$$\hat{A} \to \widehat{C}^*_{DR}((A/I)/A)$$

is an equivalence. In particular

$$H^0(\widehat{C}^*_{DR}((A/I)/A)) \simeq \hat{A} \qquad H^i(\widehat{C}^*_{DR}((A/I)/A)) \simeq 0 \;\forall\, i \neq 0.$$

Proof. See [**Bha12**, Thm. 4.10]. □

1.5. The geometry of graded mixed objects

We recall here some results from [**MRT22**] and [**Alf25**] which provide a geometric interpretation of graded mixed complexes. These will be used for some specific arguments and are sometimes very useful (see for instance §2.5).

We consider the multiplicative group scheme $\mathbb{G}_m$. It acts by standard weight -1 on the additive group scheme $\mathbb{G}_a$, and thus on the group stack $B\mathbb{G}_a$. Here the weight of the $\mathbb{G}_m$ action is understood geometrically, the induced action of $\mathbb{G}_m$ on functions $\mathcal{O}(\mathbb{G}_a) = k[T]$ is such that T has weight 1.

Definition 1.5.0.1. *We define the group stack $\mathcal{H}_0$ as the semi-direct product*

$$\mathcal{H}_0 := B\mathbb{G}_a \rtimes \mathbb{G}_m$$

We can now consider complexes of k-modules endowed with an action of $\mathcal{H}_0$. They form an ∞-category which is formally defined as $\mathbf{QCoh}(B\mathcal{H}_0)$, the ∞-category of quasi-coherent complexes over the classifying stack $B\mathcal{H}_0$ of $\mathcal{H}_0$. The canonical projection $\mathcal{H}_0 \to \mathbb{G}_m$ possesses a group section $s : \mathbb{G}_m \to \mathcal{H}_0$, which provides a natural morphism $s : B\mathbb{G}_m \to B\mathcal{H}_0$. The fiber of the projection $\mathcal{H}_0 \to \mathbb{G}_m$ identifies with $B\mathbb{G}_a$, which is also naturally identified with the fiber of the section s. Therefore, s exhibits $B\mathbb{G}_a$ as a $\mathcal{H}_0$-equivariant stack.

By writing $B\mathcal{H}_0$ as the standard colimit $B\mathcal{H}_0 = \mathrm{colim}_{[n]\in\Delta^{op}}\mathcal{H}_0^n$, we can write

$$\mathbf{QCoh}(B\mathcal{H}_0) \simeq \lim_{[n]\in\Delta} \mathbf{QCoh}(\mathcal{H}_0^n).$$

Now, for each n, the stack $\mathcal{H}_0^n$ is an affine stack with cdga of functions $\mathcal{O}(\mathcal{H}_0^n) \simeq \mathcal{O}(\mathcal{H}_0)^{\otimes n} \simeq (k[t,t^{-1}]\oplus k[t,t^{-1}][-1](1))^{\otimes n}$. The category of graded mixed complexes $\epsilon - C(k)^{gr}$, before localization by quasi-isomorphisms, can be understood as the category of $(k[t,t^{-1}]\oplus k[t,t^{-1}][-1](1))$-dg-comodules, where $(k[t,t^{-1}] \oplus k[t,t^{-1}][-1](1))$ is the commutative and dg-Hopf algebra obtained by semi-direct product of the multiplicative Hopf algebra $k[t,t^{-1}]$ with $k \oplus k[-1](1) = \mathcal{O}(B\mathbb{G}_a)$. In other words, we can write down an equivalence of genuine categories (where the limit is computed in the 2-category of categories, ad is taken before localization by quasi-isomorphisms)

$$\epsilon - C(k)^{gr} \simeq \lim_{[n]\in\Delta} (k[t,t^{-1}] \oplus k[t,t^{-1}][-1](1))^{\otimes n} - dg - mod.$$

When we localize along quasi-isomorphisms, we get a functor ∞-functor

$$\epsilon - \mathbf{dg}_k^{gr} \longrightarrow \mathbf{QCoh}(B\mathcal{H}_0).$$

By observation, this ∞-functor is also endowed with a canonical symmetric monoidal structure. In general localizations does not preserve limits, and there are no obvious reasons why the previous ∞-functor should be an equivalence. However the following result is proven in [**MRT22**, Prop. 4.2.3].

PROPOSITON 1.5.0.2. *The ∞-functor constructed above induces a symmetric monoidal equivalence*

$$\epsilon - \mathbf{dg}_k^{gr} \simeq \mathbf{QCoh}(B\mathcal{H}_0).$$

A simple corollary of the above Proposition, combined with the rectification results of [**Lur22a**, Prop. 4.5.4.6], is the following interpretation of graded mixed cdga's.

COROLLARY 1.5.0.3. *There exists an natural equivalence of ∞-categories*

$$\epsilon - \mathbf{cdga}_k^{gr} \simeq \mathbf{CAlg}(\mathbf{QCoh}(B\mathcal{H}_0)),$$

*where the right hand side is the ∞-category of commutative algebras in $\mathbf{QCoh}(B\mathcal{H}_0)$ in the sense of [**Lur22a**].*

Proof of the Corollary. The equivalence of Proposition 1.5.0.2 being symmetric monoidal, we get an equivalence on the corresponding ∞-categories of commutative algebra objects $\mathbf{CAlg}(\epsilon - \mathbf{dg}_k^{gr}) \simeq \mathbf{CAlg}(\mathbf{QCoh}(B\mathcal{H}_0))$. To finish the proof we invoke the rectification result of [**Lur22a**, Prop. 4.5.4.6], which can be applied here as k is assumed of characteristic zero (and thus $\epsilon - C(k)^{gr}$ is a *well powered* symmetric monoidal model category). We thus get that the natural ∞-functor $L(\mathbf{CAlg}(\epsilon - C(k)^{gr})) = \epsilon - \mathbf{cdga}_k^{gr} \to \mathbf{CAlg}(\epsilon - \mathbf{cdga}_k^{gr})$ is an equivalence. Combining these two equivalences of ∞-categories finishes the proof of the Corollary. □

REMARK 1.5.0.4. Using Proposition 1.5.0.2 and the group section $s : \mathbb{G}_m \to \mathcal{H}$ introduced at the beginning of this subsection, we may realize geometrically the forgetful functor $\epsilon - \mathbf{dg}_k^{gr} \to \mathbf{dg}_k^{gr}$ as the pullback on quasi-coherent complexes along the map $Bs : B\mathbb{G}_m \to B\mathcal{H}_0$.

The previous description allows for the following easy generalization where $\mathbf{Spec}\, k$ is replaced by an arbitrary derived (pre)stack X

$$\epsilon - \mathbf{dg}_X^{gr} := \mathbf{QCoh}(B\mathcal{H}_0 \times X) \qquad \epsilon - \mathbf{cdga}_X^{gr} \simeq \mathbf{CAlg}(\mathbf{QCoh}(B\mathcal{H}_0 \times X)) \tag{6}$$

(where the indicated products are fiber products over $\mathbf{Spec}\, k$). This equivalence for general derived stacks follows directly from Corollary 1.5.0.3 by descent.

1.6. The classifying spaces of graded mixed structures

In this section we study the *space of graded mixed structures* on a *fixed* graded cdga. We first provide a precisely definition of such a space.

We consider the forgetful ∞-functor

$$\mathsf{oub}_\epsilon : \epsilon - \mathbf{cdga}_k^{gr} \to \mathbf{cdga}_k^{gr}$$

obtained by forgetting the mixed structure. As oub_ϵ is conservative, the fiber ∞-category $oub_\epsilon^{-1}(B)$ taken at any point $B \in \mathbf{cdga}_k^{gr}$ is an ∞-groupoid, whose geometric realization will be denoted by $|oub_\epsilon^{-1}(B)|$.

DEFINITION 1.6.0.1. *For a given graded cdga* $B \in \mathbf{cdga}_k^{gr}$, the classifying space of compatible graded mixed structures on B *is the simplicial set*

$$\mathcal{M}_\epsilon(B) := |\mathsf{oub}_\epsilon^{-1}(B)| \in \mathbb{T}.$$

Such spaces of algebraic structures have been heavily studied in the literature. We start by recalling the well known fact that they can be computed, in our specific case, as mapping spaces between graded dg-Lie algebras. For this, let $B \in \mathbf{cdga}_k^{gr}$ be a fixed graded cdga over k. We can form $Der^{gr}(B)$ the graded dg-Lie algebra of graded derivations from B with values in B. It can be define using a strict cofibrant model $\widetilde{B}$ of B, and considering strict graded derivations of $\widetilde{B}$ into $\widetilde{B}$ as a model for $Der^{gr}(B)$. More explicitly, the piece of weight n of $Der^{gr}(B)$ is the complex of derivations $\widetilde{B} \to \widetilde{B}(n)$, and the Lie bracket is the usual commutator of derivations. The space $\mathcal{M}_\epsilon(B)$ can then be understood as a mapping space in the ∞-category $\mathbf{dgLie}_k^{gr}$ of graded deg-Lie algebras (i.e. Lie algebra object in $\mathbf{dg}_k^{gr}$).

PROPOSITON 1.6.0.2. *There is a canonical equivalence* $Map_{\mathbf{dgLie}^{gr}}(k(1)[1], Der^{gr}(B)) \to \mathcal{M}_\epsilon(B)$.

Proof. The proposition is a particular case of a more general result. Let L be any graded dg-Lie algebra. We have a symmetric monoidal model category $L-Mod^{gr}$ of graded L-dg-modules (for which equivalences and fibrations are defined on the underlying graded complexes), and a corresponding symmetric monoidal ∞-category $L - \mathbf{dg}_k^{gr}$. Commutative algebras in $L - \mathbf{dg}_k^{gr}$ form a new ∞-category $L - \mathbf{cdga}_k^{gr}$, whose objects can be called *graded L-cdga's*. It comes equipped with a forgetful ∞-functor

$$\mathsf{oub}_L : L - \mathbf{cdga}_k^{gr} \to \mathbf{cdga}_k^{gr}.$$

We claim that, for any fixed $B \in \mathbf{cdga}_k^{gr}$, the two spaces

$$\mathsf{oub}_L^{-1}(B) \qquad \mathbf{Map}_{\mathbf{dgLie}_k^{gr}}(L, Der^{gr}(B))$$

are functorially equivalent. The proposition would follow from the special case where $L = k(1)[1]$, as $k(1)[1] - \mathbf{cdga}_k^{gr}$ is nothing else that the ∞-category of graded mixed cdga's.

We fix $B \in \mathbf{cdga}_k^{gr}$, and chose cofibrant model for it. We thus have two ∞-functors

$$L \mapsto \mathsf{oub}_L^{-1}(B) \qquad L \mapsto \mathbf{Map}_{\mathbf{dgLie}_k^{gr}}(L, Der^{gr}(B)).$$

These ∞-functor can be represented as strict ∞-functors from the category of cofibrant graded dg-Lie algebras to the category of spaces. The first one sends L to the nerve of the category $C(L, B)$ whose objects are pairs (B', u), with $B' \in L - \mathbf{cdga}_k^{gr}$ and $u : B \to \mathsf{oub}_L(B')$ is a quasi-isomorphism. The second functor sends L to the simplicial set $Hom(c^*(L), Der^{gr}(B))$, where $c^* : \mathbf{dgLie}_k^{gr} \to (\mathbf{dgLie}_k^{gr})^\Delta$ is a left framing in the sense of [**Hov99**, Def. 5.2.7].

We can construct a natural transformation between these two ∞-functors. Indeed, there is a canonical map from the set $Hom(L, Der^{gr}(B))$ to the nerve of $C(L, B)$, sending a morphism $\rho : L \to Der^{gr}(B)$ to the pair (B_ρ, id), where B_ρ is the object in $L - \mathbf{cdga}_k^{gr}$ defined by the moprhism ρ (and whose underlying object equals B). This map is functorial in L, and thus it defines a morphism of simplicial sets

$$Hom(c^*(L), Der^{gr}(B)) \to |[n] \to N(C(c^n(L), B))|.$$

Now, as quasi-isomorphisms between graded dg-Lie induces Quillen equivalences between the corresponding ∞-categories of graded cdga's, we have that all the morphism in the simplicial diagram $[n] \mapsto N(C(c^n(L), B))$ are weak equivalences, and thus the natural morphism

$$N(C(L, B)) \to |[n] \to N(C(c^n(L), B))|$$

is also a weak equivalence. We have thus defined a well defined natural transformation of ∞-functors

$$\psi_L : Hom(c^*(L), Der^{gr}(B)) \to |[n] \to N(C(c^n(L), B))| \simeq N(C(L, B)).$$

It remains to show that ψ_L is an equivalence for all L. For this we use the following steps.

(1) The induced map

$$\psi_L : \pi_0(Hom(c^*(L), Der^{gr}(B))) \to \pi_0(N(C(L, B)))$$

is bijective for all L.

(2) Both ∞-functors $L \mapsto Hom(c^*(L), Der^{gr}(B))$ and $L \mapsto N(C(L, B))$ sends colimis (in the argument L) to limits.

As a first observation, setps (1) and (2) above finishes the proof of the proposition. Indeed, we can apply (1) to graded dg-Lie algebras of the form $K \otimes L$, for K an arbitrary simplicial set, which by (2) will imply that the morphism ψ_L induces a bijection

$$[K, Hom(c^*(L), Der^{gr}(B))] \to [K, N(C(L, B))].$$

As this is true for all K, Yoneda implies that ψ_L is an isomorphism in $Ho(\mathbf{Top})$ and thus is an equivalence.

The point (1) is a direct exercise in model category theory, and is left to the reader. For point (2), we use that if $L = colim_i L_i$ is a colimit in the ∞-category $\mathbf{dgLie}_k^{gr}$, then the natural ∞-functor

$$L - \mathbf{cdga}_k^{gr} \to lim_i L_i - \mathbf{cdga}_k^{gr}$$

is an equivalence of ∞-categories. Indeed, this can be seen easily by first considering the case of graded dg-modules

$$L - \mathbf{dg}_k^{gr} \to lim_i L_i - \mathbf{dg}_k^{gr}.$$

The fact that this ∞-functor is an equivalence follows directly from the equivalence between dg-modules over L and dg-modules over the universal enveloping algebra $U(L)$, and the fact that $L \mapsto U(L)$ preserves colimits. Indeed, for a colimit in graded dg-algebras $U = colim_i U_i$, the ∞-category of graded U-dg-modules is equivalent to the limit of the ∞-categories of graded U_i-dg-modules as shown in the next lemma.

LEMMA 1.6.0.3. *Let $U = colim_i U_i$ be a colimit in the ∞-category $\mathbf{dga}_k^{gr}$ graded (associative and unital) dg-algebras over k. The canonical ∞-functor on ∞-categories of graded dg-modules*

$$U - \mathbf{dg}_k^{gr} \to lim_i U_i - \mathbf{dg}_k^{gr}$$

is an equivalence.

Proof of the lemma. We have a fibration sequence

$$\mathbf{Map}_{\mathbf{dga}_k^{gr}}(U, T(x,x)) \to \mathbf{Map}_{\mathbf{grdgcat_k}}(BU, T) \to \mathbf{Map}_{\mathbf{grdgcat_k}}(\nVdash, T),$$

where here T is a graded dg-category, the fiber is taken over the unique morphism $\nVdash \to T$ point an object $x \in T$, and BU is the graded dg-category with a unique object determined by U. This is the graded version of the fibration sequence appearing in the proof of [**TV07**, Lem. 2.11], and can proven the exact same manner using the homotopy theory of graded dg-categories. This fibration sequence implies the lemma as $\mathbf{Map}(_{\mathbf{dga}_k^{gr}}(-, T(x,x))$ transforms colimits in limits. ☐

The lemma concludes the proof of item (2) above, and thus of the proposition. ☐

Proposition 1.6.0.2 is useful as the mapping space $Map_{\mathbf{dgLie}^{gr}}(k(1)[1], Der^{gr}(B))$ can be described using an explicit cofibrant model for the graded dg-Lie algebra $k(1)[1]$, introduced in [**CPT⁺17**]. We let $\mathcal{L}$ be the dg-Lie algebra freely generated by elements p_i, for $i > 0$, such that p_i sits in cohomological degree $1 - 2i$ and in weight i, with the relations

$$d(p_n) + \sum_{a+b=n} [p_q, p_b] = 0.$$

We consider the natural projection $\mathcal{L} \to k(1)[1]$ sending p_1 to ϵ and p_i to zero for $i > 1$. As explained in [**PTVV13**,], this projection is a trivial fibration. As $\mathcal{L}$ is clearly cofibrant (as being a cell object), this shows that $\mathcal{L}$ is a cofibrant model for $k(1)[1]$. Therefore, we have a diagram of spaces

$$Hom_{\mathbf{dgLie}^{gr}}(\mathcal{L}, Der^{gr}(B)) \longrightarrow Map_{\mathbf{dgLie}^{gr}}(\mathcal{L}, Der^{gr}(B)) \xleftarrow{\simeq} Map_{\mathbf{dgLie}^{gr}}(k(1)[1], Der^{gr}(B))$$

where $Hom_{\mathbf{dgLie}^{gr}}(\mathcal{L}, Der^{gr}(B))$ is the set of morphisms of graded dg-Lie algebras (which depends on the choice of a cofibrant model for B). Applying π_0 provides a natural surjective map $Hom_{\mathbf{dgLie}^{gr}}(\mathcal{L}, Der^{gr}(B)) \to \pi_0(Map_{\mathbf{dgLie}^{gr}}(k(1)[1], Der^{gr}(B)))$. Combined with the Proposition 1.6.0.2 we obtain the following corollary.

COROLLARY 1.6.0.4. *Let B be a fixed cofibrant graded cdga. Then, there exists a natural surjective map*

$$Hom_{\mathbf{dgLie}^{gr}}(\mathcal{L}, Der^{gr}(B)) \to \pi_0(M_\epsilon(B)).$$

The previous results can be made a bit more explicit. Indeed, by the definition of $\mathcal{L}$ we have that an element in $Hom_{\mathbf{dgLie}^{gr}}(\mathcal{L}, Der^{gr}(B))$ is given by a sequence of k-linear graded derivations

$$d_i : B \to B(i)[1-2i]$$

satisfying the system of equations

$$d(d_n) + \sum_{a+b=n} [d_q, d_b] = 0.$$

Suppose moreover that B is explicitly given by $Sym_A(E[1])$, where A is a cofibrant connective cdga, and E is a cofibrant perfect A-dg-module. Then, each d_i consists itself as a pair

$$\alpha_i : E \longrightarrow \wedge_A^{i+1} E[1-i] \qquad \beta_i : A \longrightarrow \wedge_A^{i} E[1-i],$$

such that β_i is a k-linear derivation, and α_i is a morphism which is A-linear with respect to β_i

$$\alpha_i(a.m) = a\alpha_i(m) + (-1)^? \beta_i(a) \wedge m$$

(where ? is the adequate sign). This description provides an explicit way to write down elements in the set $Hom_{\mathbf{dgLie}^{gr}}(\mathcal{L}, Der^{gr}(B))$, and thus also an explicit manner to write down all the elements in $\pi_0(M_\epsilon(B))$ by Corollary 1.6.0.4. We also refer to [**Mon22**] for another closely related, but slightly different, way to describe these classifying spaces of graded mixed structures.

CHAPTER 2

Derived foliations

In this chapter we give the basic definitions and study the general properties of derived foliations. We provide examples, and a comparison with the theory of dg-Lie algebroids. One of the main result, in the third section, states that derived foliations are always formally integrable, which is a major difference with the situation of underived possibly singular foliations. We provide two versions of this result, a general version for derived foliations on general derived Artin stacks, and a more specific version for a class of derived foliations called *quasi-smooth and rigid* on smooth varieties. This second version is useful to study existence of derived enhancement of underived singular foliations, and to relate them to the classical notion of Godbillon-Vey sequences.

As before, unless other is specified, all structures (algebras, modules, schemes ...) are over our base commutative ring k. We assume that k is furthermore a $\mathbb{Q}$-algebra. The notion of derived foliations in arbitrary characteristics situations will be treated in a separate later chapter of this book, as it requires some extra care and more evolved notions (see §7).

2.1. First definitions and examples

We are now ready to define the notion of derived foliations. We will first consider the situation for an affine derived scheme, and then proceed by gluing in order to define derived foliations on more global objects such as derived schemes and derived Artin stacks.

2.1.1. Definition in the affine setting. Let $X = \mathbf{Spec}\, A$ be a derived affine scheme over k. Recall that this implicitly means that A is a connective cdga over k, and thus it is cohomologically concentrated in non-positive degrees (see appendix §A.2). We will allow ourselves to implicitly identify the ∞-categories of A-dg-modules and of quasi-coherent complexes of $\mathcal{O}_X$-modules. We will also use the derived de Rham algebra $\mathbf{DR}(A/k)$, that will also be denoted by $\mathbf{DR}(X/k)$.

A derived foliation $\mathcal{F}$ on X will be given, by definition, a graded mixed cdga $\mathbf{DR}(\mathcal{F})$, together with a morphism $u : \mathbf{DR}(A/k) \to \mathbf{DR}(\mathcal{F})$ of graded mixed cdga's over k, satisfying some extra conditions. In order to phrase these conditions, note that the morphism u induces a morphism of graded cdga's

$$A \to \mathbf{DR}(A/k) \to \mathbf{DR}(\mathcal{F}),$$

where the first of this morphism is the inclusion of the weight 0 part. This endows $\mathbf{DR}(\mathcal{F})$ with a canonical structure of a graded A-linear cdga structure. In particular, the inclusion of the weight 1 part $\mathbf{DR}(\mathcal{F})^{(1)} \subset \mathbf{DR}(\mathcal{F})$ induces a canonical morphism of graded A-linear cdga's

$$\phi_u : Sym_A(\mathbf{DR}(\mathcal{F})^{(1)}) \longrightarrow \mathbf{DR}(\mathcal{F}).$$

Note that, the morphism ϕ_u above, of course, depends on the original morphism u.

Definition 2.1.1.1. *Let $X = \mathbf{Spec}\, A$ be a derived affine scheme over k. A* derived foliation $\mathcal{F}$ over X relative to k *is a pair* $(\mathbf{DR}(\mathcal{F}), u)$*, which consists of a graded mixed cdga* $\mathbf{DR}(\mathcal{F})$*, and a morphism of graded mixed cdga's*

$$u : \mathbf{DR}(A/k) \longrightarrow \mathbf{DR}(\mathcal{F}),$$

such that the induced morphism

$$\phi_u : Sym_A(\mathbf{DR}(\mathcal{F})^{(1)}) \longrightarrow \mathbf{DR}(\mathcal{F})$$

is a quasi-isomorphism.

As a first comment, it is important to note that the condition that ϕ_u being a quasi-isomorphism includes in particular the condition that the morphism u induces a quasi-isomorphism in weight 0 $A \simeq \mathbf{DR}(\mathcal{F})^{(0)}$. As a second observation, the universal property of the derived de Rham algebra $\mathbf{DR}(A/k)$ (see Definition 1.4.2.1), the morphism u is completely characterized by the morphism of graded cdga's (without any mixed structure)

$$A \to \mathbf{DR}(A/k) \to \mathbf{DR}(\mathcal{F}).$$

Therefore, the definition of derived foliations above could also be phrased as follows: a derived foliation $\mathcal{F}$ on X over k consists of a graded mixed cdga $\mathbf{DR}(\mathcal{F})$, together with a quasi-isomorphism of cdga's $A \to \mathbf{DR}(\mathcal{F})^{(0)}$, and such that the canonical morphism

$$Sym_A(\mathbf{DR}(\mathcal{F})^{(1)}) \longrightarrow \mathbf{DR}(\mathcal{F})$$

is a quasi-isomorphism.

The idea behind Definition 2.1.1.1 is that $\mathbf{DR}(\mathcal{F})$ is the de Rham algebra of forms *along the leaves* of $\mathcal{F}$. Therefore, this A-linear dg-module $\mathbf{DR}(\mathcal{F})^{(1)}$ should be understood as the (shifted by -1) cotangent sheaf, or rather cotangent complex, classifying *differential forms along the leaves of* $\mathcal{F}$. This complex is one of the fundamental data contained in a derived foliation, and we will therefore give a specific name, and introduce a specific notation for it.

Definition 2.1.1.2. *Let $\mathcal{F}$ be a derived foliation on a derived affine scheme $X = \mathbf{Spec}\, A$. The* cotangent complex of $\mathcal{F}$ relative to k *is defined to be*

$$\mathbb{L}_{\mathcal{F}/k} := \mathbf{DR}(\mathcal{F})^{(1)}[-1].$$

When the ground ring k is unambiguous it is simple denoted by $\mathbb{L}_{\mathcal{F}}$.

An important remark is that the cotangent complex $\mathbb{L}_{\mathcal{F}/k}$ of a derived foliation $\mathcal{F}$ always comes equipped with a canonical morphism of A-dg-modules

$$\mathbb{L}_{A/k} \longrightarrow \mathbb{L}_{\mathcal{F}/k},$$

which is induced by the (shifted) weight 1 part of the morphism $u : \mathbf{DR}(A/k) \to \mathbf{DR}(\mathcal{F})$. This is often referred to as the *anchor map*.

Using the notation and the terminology above, a derived foliation $\mathcal{F}$ on X, can be thought of as a graded mixed structure ϵ on $Sym_A(\mathbb{L}_{\mathcal{F}/k}[1])$. This graded mixed structure is itself the de Rham differential on differential forms along $\mathcal{F}$. However, one important point in Definition 2.1.1.1 is that ϵ does not strictly exists on $Sym_A(\mathbb{L}_{\mathcal{F}/k}[1])$, but rather on $\mathbf{DR}(\mathcal{F})$ which is only quasi-isomorphic $Sym_A(\mathbb{L}_{\mathcal{F}/k}[1])$. Trying to transport the mixed structure along this quasi-isomorphism provides a derivation ϵ on $Sym_A(\mathbb{L}_{\mathcal{F}/k}[1])$, but ϵ^2 will now only be homotopic to zero (as opposed to strictly equal to zero). We can use the quasi-isomorphism to show that there exists a canonical such homotopy, which itself should satisfy some higher coherences and so on and so forth. Despite its naive form, the reader must be warned that Definition 2.1.1.1 hides in fact an infinite number of rather evolved homotopy coherences, very similar to the one being found in shifted symplectic and Poisson structures of [**CPT+17**]. We will come back to this point later on in this book and will make these coherence explicit. The reader can already notice that by construction ϵ is a point in the moduli of compatible graded mixed structures on $Sym_A(\mathbb{L}_{\mathcal{F}/k}[1])$ in the sense of Definition 1.6.0.1, and thus can be described by the coherence of homotopies given by corollary 1.6.0.4.

Keeping this in mind, a derived foliation $\mathcal{F}$ has a cotangent complex $\mathbb{L}_{\mathcal{F}/k}$, which comes equipped with de Rham differentials

$$A \to \mathbb{L}_{\mathcal{F}/k} \to \wedge^2_A \mathbb{L}_{\mathcal{F}/k} \to \dots$$

These differential satisfy all the required formula (such as $d^2 = 0$, multiplicativity ect), but only up to natural homotopy coherences. As far as we know, it is in general not possible to make these identities to hold strictly on the noze.

Before showing several basic examples of derived foliations, we introduce some terminology. Derived foliations will be characterized according to the *size* of their cotangent complexes.

Definition 2.1.1.3. *Let $\mathcal{F}$ be a derived foliation on a derived affine scheme $X = \mathbf{Spec}\, A$.*

(1) The derived foliation $\mathcal{F}$ is perfect *(resp.* almost perfect*) if $\mathbb{L}_{\mathcal{F}/k}$ is a perfect (resp.* almost perfect*) A-dg-module.*

(2) The derived foliation $\mathcal{F}$ is n-connective*, for some integer $n \in \mathbb{Z}$, if $H^i(\mathbb{L}_{\mathcal{F}/k}) = 0$ for all $i > n$ (i.e. if $\mathbb{L}_{\mathcal{F}/k}[n]$ is connective).*

(3) The derived foliation $\mathcal{F}$ is smooth *if it is perfect and if $\mathbb{L}_{\mathcal{F}/k}$ is of Tor amplitude contained in $[0, \infty[$.*

(4) The derived foliation $\mathcal{F}$ is quasi-smooth *if it is perfect and if $\mathbb{L}_{\mathcal{F}/k}$ is of Tor amplitude contained in $[-1, \infty[$.*

2.1.2. Basic examples of derived foliations. We gather in this subsection the very first examples of derived foliations. More evolved examples will be given later all along this book.

EXAMPLE 2.1.2.1. **Classical foliations on smooth affine schemes.** We start by the most basic examples of how usual foliations define derived foliations in our sense. For this, we let $X = \mathbf{Spec}\, A$ be a smooth affine scheme over k. Let $V \subset T_X$ be a subbundle of the tangent bundle of X, which is stable by the Lie bracket of T_X. We consider the dual bundle $V^\vee$ and $Sym_A(V^\vee[1])$. The inclusion $V \subset T_X$ defines a quotient map $\Omega^1_{X/k} \to V^\vee$, and thus a morphism of graded cdga's (with zero cohomological differential)

$$\mathbf{DR}(A/k) = Sym_A(\Omega^1_{A/k}[1]) \longrightarrow Sym_A(V^\vee[1]).$$

As V is stable by the Lie bracket of vector fields, the de Rham differential $dR : A \to \Omega^1_{A/k}$ descends to a de Rham differential $dR : A \to V^\vee$, which can be extended multiplicatively to a graded mixed structure on the graded mixed cdga $Sym_A(V^\vee[1])$. We thus have defined a morphism of graded mixed cdga's

$$\mathbf{DR}(A/k) \longrightarrow Sym_A(V^\vee[1]).$$

This morphism defines a derived foliation $\mathcal{F}$ on X relative to k according to the Definition 2.1.1.1, such that $\mathbf{DR}(\mathcal{F}) = Sym_A(V^\vee[1])$ endowed with its induced de Rham differential. This derived foliation has $V^\vee$ as its cotangent complex, and thus is smooth in the sense of Definition 2.1.1.3.

EXAMPLE 2.1.2.2. **Lie algebroids on smooth affine schemes.** The previous example possesses a slight generalization. We let $X = \mathbf{Spec}\, A$ be a smooth affine scheme over k. Let now V be a vector bundle on X endowed with the structure of a Lie algebroid (see §2.2 for more on the relations to Lie algebroids). We remind that this consists of a pair $(a, [,])$, where $a : V \to T_X$ is a morphism of vector bundles on X, and $[,]$ is a k-linear Lie bracket on sections of V. These data are furthermore assumed to satisfy two compatibility conditions. First we have the standard Liebniz rule: for any sections s and $t \in V$ and any function $f \in A$ on X, we have

$$[f.s, t] = f.[s, t] + a(t)(f).s,$$

where $a(t)(f)$ is the value of the derivation $a(t) \in T_X$ on the function f. The second condition states that the morphism a is compatible with the Lie brackets on T_X: $a([s, t]) = [a(s), a(t)]$ for any pair of section (s, t) of V. Classical foliations on X are therefore Lie algebroids $(V, a, [,])$ such that a identifies V with a subbundle of T_X.

For a Lie algebroid $(V, a, [,])$ as above, we can form its de Rham algebra $Sym_A(V^\vee[1])$. This is a graded cdga which comes equipped with an extension of the de Rham differential

$$A \longrightarrow V^\vee \longrightarrow \wedge^2_A(V^\vee) \longrightarrow \dots \longrightarrow \wedge^r_A(V^\vee) \longrightarrow \dots.$$

The morphism $A \to V^\vee$ is given by composing the univeral derivation $A \to \Omega^1_{A/k}$ with the A-linear dual of the map a. The differential $dR : V^\vee \to \wedge^2_A(V^\vee)$ is itself dual to the Lie bracket $[,]$ on V in the following sense: for a and b sections of V, and $v \in V^\vee$, we have

$$dR(v)(s \wedge t) = a(s)(v(t)) - a(t)(v(s)) + v([s,t]).$$

This differential extends to a canonical graded mixed structure on $Sym_A(V^\vee[1])$, and defines a graded mixed cdga, which we denote by $\mathbf{DR}(V)$. By construction, the A-linear dual of a defines a morphism of graded mixed cdga's $\mathbf{DR}(A/k) \to \mathbf{DR}(V)$. The conditions of Definition 2.1.1.1 are satisfied and therefore this construction defines a derived foliation on X. We refer to §2.2 for a more general construction as well as for a precise comparison between derived foliations and objects of Lie-types.

Example 2.1.2.3. **Tautological derived foliations.** For a given derived affine scheme $X = \mathbf{Spec}\, A$ there are two tautological derived foliations on X of particular importance. The first one is called the 0-*foliation*, or the *pointwise foliation*, as morally its leaves are points of X. In terms of graded mixed cdga's it is given by the augmentation morphism

$$\mathbf{DR}(A/k) \longrightarrow A,$$

inducing the identity on A and zero on all piece of higher weights. Here A is considered as a graded mixed cdga with zero mixed structure, and A is considered in weight 0. This augmentation morphism obviously satisfies the conditions of Definition 2.1.1.1, and here the cotangent complex is simply given by 0. This derived foliation will be denoted by 0_X, and is the initial object of the ∞-category of derived foliations on X (see §2.1.4.7 for a more general statement).

The second tautological foliation is the extreme opposite, and is called the *de Rham foliation*, because of its direct relation with the de Rham complex. Morally it has only one leave, namely X itself. In terms of graded mixed cdga's it is given by the identity morphism

$$\mathbf{DR}(A/k) \longrightarrow \mathbf{DR}(A/k).$$

Here the cotangent complex of this derived foliation is $\mathbb{L}_{X/k}$ and the anchor map is the identity map. It is denoted by $*_X$ and we will see later on that it is the final object in the ∞-category of derived foliations over X.

Example 2.1.2.4. **Derived foliations induced by a morphism.** We finish by a very last basic example, the relative de Rham derived foliation. This is again the derived foliation $*_X$ described above, except that the base ring k is now replaced by a base cdga B mapping to A.
Let $f : X = \mathbf{Spec}\, A \longrightarrow Y = \mathbf{Spec}\, B$ be a morphism of derived affine schemes. This morphism induces a morphism of cdga $B \to A$ and thus of graded mixed cdga's $\mathbf{DR}(B/k) \to \mathbf{DR}(A/k)$. We define the graded mixed cdga's $\mathbf{DR}(A/B)$, of A relative to B, by

$$\mathbf{DR}(A/B) := \mathbf{DR}(A/k) \otimes_{\mathbf{DR}(B/k)} B.$$

The tensor product above is taken inside the ∞-category $\epsilon-\mathbf{cdga}_k^{gr}$ of graded mixed cdga's over k, where the morphism $\mathbf{DR}(B/k) \to B$ is the augmentation morphism. The natural induced morphism $\mathbf{DR}(A/k) \to \mathbf{DR}(A/B)$ defines a derived foliation on X in the sense of Definition 2.1.1.1. The cotangent complex of this derived foliation is here $\mathbb{L}_{A/B}$ the relative cotangent complex of A over B, which sits in a exact triangle of A-dg-modules

$$\mathbb{L}_{B/k} \otimes_B A \longrightarrow \mathbb{L}_{A/k} \longrightarrow \mathbb{L}_{A/B}.$$

This is the relative de Rham foliation on X over Y, and is denoted by $*_{X/Y}$. This is the very basic example of algebraically integrable derived foliations, and we will see later on that it is the pull-back along f of the tautological pointwise foliation 0_Y (see §2.1.3).

2.1.3. Global definition. We now turn to the global notion of derived foliations on derived Artin stacks in the sense of §A.2. For this, we will use the standard procedure by performing a left Kan extension from derived affine schemes to all derived stacks.

We start by defining an ∞-functor

$$\mathcal{F}ol(-/k) : \mathbf{dAff}_k^{op} \longrightarrow \mathbf{Cat}_\infty,$$

from the ∞-category of derived affine schemes to the opposite of the ∞-category of ∞-categories. For an affine derived scheme $X = \mathbf{Spec}\, A$, $F(X/k)$ is defined to be the ∞-category of graded mixed $\mathbf{DR}(X/k)$-cdga's which satisfies the conditions of Definition 2.1.1.1. For an object $\mathcal{F} \in \mathcal{F}ol(X/k)$, we will use the notation $\mathbf{DR}(\mathcal{F})$ for the corresponding graded mixed $\mathbf{DR}(X/k)$-cdga.

For a morphism of derived affine schemes $f : Y = \mathbf{Spec}\, B \to X = \mathbf{Spec}\, A$, we define an ∞-functor $f^* : \mathcal{F}ol(X/k) \to \mathcal{F}ol(Y/k)$, by sending a graded mixed $\mathbf{DR}(X/k)$-cdga $\mathbf{DR}(\mathcal{F})$ (corresponding to an object $\mathcal{F} \in \mathcal{F}ol(Y/k)$) to $\mathbf{DR}(Y/k) \otimes_{\mathbf{DR}(X/k)} \mathbf{DR}(\mathcal{F})$, which is a graded mixed $\mathbf{DR}(Y/k)$-cdga. We note here that $\mathbf{DR}(Y/k) \otimes_{\mathbf{DR}(X/k)} \mathbf{DR}(\mathcal{F})$ also satisfies the conditions of Definition 2.1.1.1 and thus defines an object $f^*(\mathcal{F}) \in \mathcal{F}ol(Y/k)$: if as a graded A-linear cdga we have $\mathbf{DR}(\mathcal{F}) \simeq Sym_A(\mathbb{L}[1])$, then $\mathbf{DR}(Y/k) \otimes_{\mathbf{DR}(X/k)} \mathbf{DR}(\mathcal{F})$ is of the form $Sym_B(\mathbb{L}'[1])$, where $\mathbb{L}'$ sits in a push-out diagram of B-dg-modules

$$\begin{tikzcd}
B \otimes_A \mathbb{L}_{A/k}[1] \arrow[r] \arrow[d] & B \otimes_A \mathbb{L}[1] \arrow[d] \\
\mathbb{L}_{B/k}[1] \arrow[r] & \mathbb{L}'[1].
\end{tikzcd}$$

Indeed, after applying the Sym_B ∞-functor, the above push-out diagram of B-dg-modules is transformed to the push-out diagram of graded cdgas defining $\mathbf{DR}(Y/k) \otimes_{\mathbf{DR}(X/k)} \mathbf{DR}(\mathcal{F})$

$$\begin{tikzcd} B \otimes_A \mathbf{DR}(A/k) \arrow[r] \arrow[d] & B \otimes_A \mathbf{DR}(\mathcal{F}) \arrow[d] \\ \mathbf{DR}(B/k) \arrow[r] & Sym_B(\mathbb{L}'[1]). \end{tikzcd}$$

We thus have defined an ∞-functor

$$\mathcal{F}ol(-/k) : \mathbf{dAff}_k^{op} \longrightarrow \mathbf{Cat}_\infty,$$

which sends $X = \mathbf{Spec}\, A$ to $\mathcal{F}ol(X/k)$ and $f : Y = \mathbf{Spec}\, B \to X = \mathbf{Spec}\, A$ to $f^* : \mathcal{F}ol(X/k) \to \mathcal{F}ol(Y/k)$. It is important to notice here that if $\mathcal{F} \in \mathcal{F}ol(X/k)$ is a derived foliation with $\mathbb{L}_F$ as a cotangent complex, then $f^*(\mathcal{F})$ is a derived foliation on Y with a cotangent complex $\mathbb{L}_{f^*(\mathcal{F})}$ defined by the push-out of B-dg-modules

$$\begin{tikzcd} B \otimes_A \mathbb{L}_{A/k}[1] \arrow[r] \arrow[d] & B \otimes_A \mathbb{L}_{\mathcal{F}}[1] \arrow[d] \\ \mathbb{L}_{B/k}[1] \arrow[r] & \mathbb{L}_{f^*(\mathcal{F})}[1]. \end{tikzcd}$$

In particular, even though there exists a canonical map $f^*(\mathbb{L}_{\mathcal{F}}) \to \mathbb{L}_{f^*(\mathcal{F})}$, of quasi-coherent complexes on Y, this map is not an equivalence in general (unless the map f is itself formally étale). In short: cotangent complexes of derived foliations are not stable by pull-backs. This fact will make more complicated the definition of a global cotangent complex when working with global derived stacks, as the gluing procedure can not be straightforward. Note that we will also sometimes use the notation $\mathcal{F}ol(A/k)$ for $\mathcal{F}ol(X/k)$ (when $X = \mathbf{Spec}\, A$).

Our first observation is that the ∞-functor $\mathcal{F}ol(-/k)$ is a derived stack for the étale topology, showing that derived foliations is a local notion for the étale topology.

PROPOSITON 2.1.3.1. *The ∞-functor $\mathcal{F}ol(-/k) : \mathbf{dAff}_k^{op} \longrightarrow \mathbf{Cat}_\infty$ is a (hypercomplete) stack for the étale topology.*

Proof. We start by noticing that $X \mapsto \mathbf{DR}(X/k)$ defines a hypercomplete stack for the étale topology on $\mathbf{dAff}_k^{op}$.

LEMMA 2.1.3.2. *The ∞-functor $\mathbf{DR}(-/k) : \mathbf{dAff}_k^{op} \to \epsilon - \mathbf{cdga}_k^{gr}$ is a hypercomplete stack for the étale topology.*

Proof of the lemma. We consider the forgetful ∞-functor $G : \epsilon - \mathbf{cdga}_k^{gr} \longrightarrow \mathbf{dg}_k^{gr}$, which forgets the algebra and the mixed structures and only retains the underlying graded complex. This ∞-functor is conservative and commutes with all limits. Therefore, in order to prove the lemma it is enough to check that the composed ∞-functor

$$G \circ \mathbf{DR}(-/k) : \mathbf{dAff}_k^{op} \to \mathbf{dg}_k^{gr}$$

is a hypercomplete stack. This last statement is equivalent to the fact that for all $n \in \mathbb{Z}$, the weight n part $X \mapsto \mathbf{DR}(X/k)^{(n)}$ is a hypercomplete stack of complexes. Moreover, this last ∞-functor sends, by definition, a derived affine scheme $X = \mathbf{Spec}\, A$ to the complex $\wedge^n_A \mathbb{L}_{A/k}[n]$ (where $\wedge^n$ is declared zero for $n < 0$ by convention).

For fixed $n \in \mathbb{Z}$, and fixed $X = \mathbf{Spec}\, A$, the ∞-functor $(\mathbf{Spec}\, A' \to X) \mapsto \wedge^n_{A'} \mathbb{L}_{A'/k}[n]$ is a quasi-coherent complex of $\mathcal{O}_X$-modules when restricted to the small étale site of X. Indeed, for $\mathbf{Spec}\, A' \to \mathbf{Spec}\, A$ any étale morphism, the canonical morphism of A'-dg-modules

$$\mathbb{L}_{A/k} \otimes_A A' \to \mathbb{L}_{A'/k}$$

is a quasi-isomorpism, and thus so are all the induced morphisms on wedge powers

$$(\wedge^n_A \mathbb{L}_{A/k}) \otimes_A A' \to \wedge^n_{A'} \mathbb{L}_{A'/k}.$$

However, it is well know that quasi-coherent $\mathcal{O}_X$-modules satisfies hypercomplete étale descent (see [TV08, Prop. 2.2.2.13]). We thus have proven that $G \circ \mathbf{DR}(-/k)$ is indeed an étale hyperstack, and thus have proven the lemma. □

To prove that $\mathcal{F}ol(-/k)$ is a derived stack we use the criterion from [TV08, Cor. 1.3.2.4], and thus we have to check that $\mathcal{F}ol(-/k)$ preserves finite products and étale hypercoverings of connective cdga's.

Compatibility with finite products. Let A be a connective cdga and suppose that it can be written as a product of cdga's $A = \prod_{i \in I} A_i$, with I a finite set. We can write $X = \mathbf{Spec}\, A \simeq \coprod_i X_i$ with $X_i = \mathbf{Spec}\, A_i$. We have a canonical ∞-functor

$$\phi : \mathcal{F}ol(X/k) \longrightarrow \prod_{i \in I} \mathcal{F}ol(X_i/k),$$

simply obtained as the product of the pull-backs ∞-functors $j_i^* : \mathcal{F}ol(X/k) \to \mathcal{F}ol(X_i/k)$, where $j_i : X_i \hookrightarrow X$ is the canonical inclusion. The ∞-functor ϕ possesses an inverse defined as follows. For a family $\mathcal{F}_i \in \mathcal{F}ol(X_i)$, corresponding to graded mixed $\mathbf{DR}(A_i)$-cdgas $\mathbf{DR}(\mathcal{F}_i)$, we can consider the graded mixed cdga $\prod_{i \in I} \mathbf{DR}(\mathcal{F}_i)$. It weight 0 part is $A = \prod_{i \in I} A_i$, and thus it is naturally equipped with a structure of a graded mixed $\mathbf{DR}(A/k)$-cdga. Moreover, as each graded cdga $\mathbf{DR}(\mathcal{F}_i)$ is of the form $Sym_{A_i}(\mathbb{L}_i)$, we have an equivalence of graded cdga's

$$\prod_{i \in I} \mathbf{DR}(\mathcal{F}_i) \simeq Sym_A(\prod_{i \in I} \mathbb{L}_i).$$

Therefore, we have defined an ∞-functor sending a family $\mathcal{F}_i \in \mathcal{F}ol(X_i)$ to the derived foliation on X defined by the graded mixed $\mathbf{DR}(X/k)$-cdga $\prod_{i \in I}(\mathbf{DR}(\mathcal{F}_i))$. Let us denote by

$$\psi : \prod_{i \in I} \mathcal{F}ol(X_i/k) \longrightarrow \mathcal{F}ol(X/k)$$

this ∞-functor.

We claim that ψ and ϕ are inverse to each others. Indeed, for $\mathcal{F} \in \mathcal{F}ol(X/k)$, the canonical morphism of graded mixed $\mathbf{DR}(X/k)$-cdga's

$$\mathbf{DR}(\mathcal{F}) \longrightarrow \prod_{i \in I} (\mathbf{DR}(X_i/k) \otimes_{\mathbf{DR}(X/k)} \mathbf{DR}(\mathcal{F}))$$

induces a natural morphism $\psi\phi(\mathcal{F}) \to \mathcal{F}$ in $\mathcal{F}ol(X/k)$. Because of the lemma 2.1.3.2 we know that $\mathbf{DR}(X/k) \simeq \prod_{i \in I} \mathbf{DR}(X_i/k)$, and thus this natural morphism is in fact an equivalence of derived foliations over X. This shows that $\psi\phi$ is equivalent to the identity ∞-functor.

In the other direction, for $\{\mathcal{F}_i\} \in \prod_{i \in I} \mathcal{F}ol(X_i/k)$, we have a morphism $\{\mathcal{F}_i\} \to \phi\psi(\{\mathcal{F}_i\})$ corresponding to the canonical projections

$$(\prod_{i \in I} \mathbf{DR}(\mathcal{F}_i)) \otimes_{\mathbf{DR}(X/k)} \mathbf{DR}(X_j/k) \longrightarrow \mathbf{DR}(\mathcal{F}_j)$$

for $j \in I$. Again, as $\mathbf{DR}(X/k) \simeq \prod_{i \in I} \mathbf{DR}(X_i/k)$ this morphism is an equivalence in $\prod_{i \in I} \mathcal{F}ol(X_i/k)$. This shows that $\phi\psi$ is also equivalent to the identity ∞-functor.

Compatibility with hypercoverings. Let now $A \longrightarrow B_*$ be an etale hypercovering of connective cdga's as in [**TV08**, Cor. 1.3.4]. This defines a coaugmented cosimplicial graded mixed cdga $\mathbf{DR}(A/k) \longrightarrow \mathbf{DR}(B_*/k)$, which by lemma 2.1.3.2 induces an equivalence $\mathbf{DR}(A/k) \simeq \mathrm{Lim}_{i \in \Delta} \mathbf{DR}(B_i/k)$. By base change we get an ∞-functor on the ∞-categories of graded mixed modules

$$\phi : \mathbf{DR}(A/k) - \epsilon - \mathbf{dg}^{gr} \longrightarrow \mathrm{Lim}_{i \in \Delta} \mathbf{DR}(B_i/k) - \epsilon - \mathbf{dg}^{gr}.$$

This ∞-functor possesses a right adjoint

$$\psi : \mathrm{Lim}_{i \in \Delta} \mathbf{DR}(B_i/k) - \epsilon - \mathbf{dg}^{gr} \longrightarrow \mathbf{DR}(A/k) - \epsilon - \mathbf{dg}^{gr}.$$

which sends a cosimplicial graded mixed $\mathbf{DR}(B_*/k)$-dg-module M_*, to

$$\psi(M_*) = \mathrm{Lim}_{i \in \Delta} M_i \in \mathbf{DR}(A/k) - \epsilon - \mathbf{dg}^{gr},$$

where each M_i is considered as a graded mixed module over $\mathbf{DR}(A/k)$ via the coaugmentation $\mathbf{DR}(A/k) \to \mathbf{DR}(B_i/k)$.

Let $\mathcal{F} \in \mathcal{F}ol(A/k)$ and let us consider the adjunction morphism

$$\mathbf{DR}(\mathcal{F}) \longrightarrow \mathrm{Lim}_{i \in \Delta} (\mathbf{DR}(B_i/k) \otimes_{DR(A/k)} \mathbf{DR}(\mathcal{F})).$$

To prove that this morphism is an equivalence, we can forget the mixed and algebra structures, as the corresponding forgetful ∞-functor commutes with limits and is conservative, and thus only consider the above morphism as a morphism of graded complexes. Moreover, as each map $A \to B_i$ is étale, we have $\mathbf{DR}(B_i/k) \simeq B_i \otimes_A \mathbf{DR}(A/k)$, and thus $\mathbf{DR}(B_i/k) \otimes_{DR(A/k)} \mathbf{DR}(\mathcal{F}) \simeq B_i \otimes_A \mathbf{DR}(\mathcal{F})$. Therefore, the above adjunction morphism, when considered as a morphism of graded complexes, is equivalent to the natural morphism

$$\mathbf{DR}(\mathcal{F}) \longrightarrow \mathrm{Lim}_{i \in \Delta} (B_i \otimes_A \mathbf{DR}(\mathcal{F})).$$

By étale descent for modules (see [**TV08**, Lem. 2.2.2.13]), this is an equivalence. This shows that the counit of the adjunction $\psi\phi \Rightarrow id$ is an equivalence.

We consider now an object $\mathcal{F}_* \in \mathrm{Lim}_i \mathcal{F}ol(B_i/k)$, and the unit of the adjunction $\phi\psi(\mathbf{DR}(\mathcal{F}_*)) \to \mathbf{DR}(\mathcal{F}_*)$, which is a morphism of cosimplicial graded mixed $\mathbf{DR}(B_*)$-cdga's. It is a general fact that the projection to the degree 0 piece $\mathrm{Lim}_{i\in\Delta}\mathcal{C}_i \longrightarrow \mathcal{C}_0$ is conservative for any limits of ∞-categories along Δ. As a consequence, to prove that the unit morphism is an equivalence it is enough to show that the natural map $\phi\psi(\mathbf{DR}(\mathcal{F}_0)) \to \mathbf{DR}(\mathcal{F}_0)$ is an equivalence of graded mixed cdga's. This morphism is given by the canonical map

$$B_0 \otimes_A (\mathrm{Lim}_{i\in\Delta}\mathbf{DR}(\mathcal{F}_i)) \longrightarrow \mathbf{DR}(\mathcal{F}_0).$$

We can, as above, forget the mixed structures, and consider this as a morphism of graded cdga's. But, as a graded cosimplicial B_*-cdga, $\mathbf{DR}(\mathcal{F}_*)$ is of the form $Sym_{B_*}(\mathbb{L}_{\mathcal{F}_*})$, where $\mathbb{L}_{\mathcal{F}_*}$ is the cosimplicial B_*-module defined by the cotangent complex of $\mathcal{F}_*$. Because all transition maps in B_* are formally étale, $\mathbb{L}_{\mathcal{F}_*}$ is a cartesian cosimplicial B_*-dg-module, and thus by étale descent (see [**TV08**, Lem. 2.2.2.13]) comes from a A-dg-module $\mathbb{L}$. Moreover, because the descent equivalence $A-\mathbf{dg} \simeq \mathrm{Lim}_i B_i - \mathbf{dg}$ is a symmetric monoidal equivalence, we have that the graded B_*-dg-module $\mathbf{DR}(\mathcal{F}_*)$ comes from $Sym_A(\mathbb{L})$ by base change along the coaugmentation $A \to B_*$. Moreover, we have $Sym_A(\mathbb{L}) \simeq \mathrm{Lim}_i(\mathbf{DR}(\mathcal{F}_*))$. In particular, we see that the morphism $B_0 \otimes_A (\mathrm{Lim}_{i\in\Delta}\mathbf{DR}(\mathcal{F}_i)) \longrightarrow \mathbf{DR}(\mathcal{F}_0)$ becomes equivalent to the canonical map

$$B_0 \otimes A Sym_A(\mathbb{L}) \longrightarrow Sym_{B_0}(B_0 \otimes_A \mathbb{L})$$

which is an equivalence. This shows that $\mathbf{DR}(\mathcal{F}_*)$ comes by pull-back from a graded mixed $\mathbf{DR}(A/k)$-cdga $\mathbf{DR}(\mathcal{F})$, but also that as a graded A-linear cdga $\mathbf{DR}(\mathcal{F})$ is of the form $Sym_A(\mathbb{L})$.

To conclude, we have shown that the adjunction (ϕ,ψ) restricts to an equivalence of ∞-categories of derived foliations

$$\mathcal{F}ol(A/k) \simeq \mathrm{Lim}_{i\in\Delta}\mathcal{F}ol(B_i/k).$$

□

Proposition 2.1.3.1 implies that $\mathcal{F}ol$ can be considered itself as a derived stack in ∞-categories. As such, for any derived stack $X \in \mathbf{dSt}_k$, we can form the ∞-category of morphisms from X to $\mathcal{F}ol(-/k)$

$$\mathcal{F}ol(X/k) := Map_{\mathbf{dSt}_k}(X, \mathcal{F}ol(-/k)) \in \mathbf{Cat}_\infty. \tag{7}$$

In more concrete terms, we have

$$\mathcal{F}ol(X/k) \simeq \mathrm{Lim}_{Spec\, A \to X}\mathcal{F}ol(A/k),$$

where the limit is taken over the (opposite) ∞-category $(\mathbf{dAff}_k/X)^{op}$ of derived affine schemes over X.

DEFINITION 2.1.3.3. *For a derived stack X over k, the* ∞-category of derived foliations of X *(relative to k) is $\mathcal{F}ol(X/k)$ constructed above. When the base ring k is clear from the context, we will simply write $\mathcal{F}ol(X)$ for $\mathcal{F}ol(X/k)$.*

When a derived foliation $\mathcal{F} \in \mathcal{F}ol(X/k)$ lives over a derived affine scheme $X = \mathbf{Spec}\, A$, its cotangent complex $\mathbb{L}_{\mathcal{F}}$ is the A-dg-module such that $\mathbf{DR}(\mathcal{F}) \simeq Sym_A(\mathbb{L}_{\mathcal{F}}[1])$ as graded cdga's. Using the equivalence between quasi-coherent modules over X and A-dg-modules (see [**TV08**, Cor. 1.3.7.3]), $\mathbb{L}_{\mathcal{F}}$ will be considered an an object in $\mathbf{QCoh}(X)$, the ∞-category of quasi-coherent complexes over X.

When X is a general derived stack, and $\mathcal{F} \in \mathcal{F}ol(X/k)$, the global cotangent complex of $\mathcal{F}$ still makes sense but is slightly more complicated to express. We first define an $\mathcal{O}_X$-dg-module as follows. For each $u : \mathbf{Spec}\, A \longrightarrow X$, we have $u^*(\mathcal{F})$, the induced derived foliation on $\mathbf{Spec}\, A$, and thus a quasi-coherent complex $\mathbb{L}_{u^*(\mathcal{F})} \in \mathbf{QCoh}(\mathbf{Spec}\, A)$. For two morphisms

$$\mathbf{Spec}\, B \xrightarrow{f} \mathbf{Spec}\, A \xrightarrow{u} X,$$

there exists a canonical morphism of quasi-coherent complexes over $\mathbf{Spec}\, B$

$$f^*(\mathbb{L}_{u^*(\mathcal{F})}) \longrightarrow \mathbb{L}_{(uf)^*(\mathcal{F})}.$$

In general, the above morphism is not an equivalence, unless the map f is formally étale. Therefore, the rule that associate to $\mathbf{Spec}\, A \to X$ the A-dg-module $\mathbb{L}_{u^*(\mathcal{F})}$ defines a complex of $\mathcal{O}_X$-dg-modules in the sense reminded in §A.4, but which is not quasi-coherent in general. This will be called the *big cotangent complex* of $\mathcal{F}$, or the *fake cotangent complex*. We will denote it by

$$\mathbb{L}_{\mathcal{F}}^{big} \in \mathcal{O}_X - \mathbf{dg}.$$

As reminded in §A.4, the ∞-category of quasi-coherent complexes over X sits inside $\mathcal{O}_X - \mathbf{dg}$ as a full sub-∞-category, and the natural embedding

$$\mathbf{QCoh}(X) \hookrightarrow \mathcal{O}_X - \mathbf{dg}$$

possesses a canonical right adjoint

$$(-)^{qcoh} : \mathcal{O}_X - \mathbf{dg} \longrightarrow \mathbf{QCoh}(X)$$

called the *quasi-coherator*. Note that the natural morphism $\mathbb{L}_{X/k}^{big} \to \mathbb{L}_{\mathcal{F}}^{big}$ induces a morphism on the associated quasi-coherent complexes. The quasi-coherent complex $(\mathbb{L}_{X/k}^{big})^{qcoh}$ identifies naturally with the global cotangent complex $\mathbb{L}_{X/k}$, as reminded in theorem A.5.0.1. Therefore, when X is a derived Artin stack and $\mathcal{F} \in \mathcal{F}ol(X/k)$, the cotangent complex $\mathbb{L}_{\mathcal{F}}$ always comes equipped with a natural map

$$\mathbb{L}_{X/k} \to \mathbb{L}_{\mathcal{F}}.$$

DEFINITION 2.1.3.4. *For any derived Artin stack X and $\mathcal{F} \in \mathcal{F}ol(X)$, the* cotangent complex of $\mathcal{F}$ *is defined to be*

$$\mathbb{L}_{\mathcal{F}} := (\mathbb{L}_{\mathcal{F}}^{big})^{qcoh} \in \mathbf{QCoh}(X).$$

The conormal complex of $\mathcal{F}$ *is defined to be the fiber of the natural morphism* $\mathbb{L}_X \longrightarrow \mathbb{L}_{\mathcal{F}}$*, and is denoted by* $\mathcal{N}^*_{\mathcal{F}/X}$ *.*

We will see examples of global cotangent complexes in our next paragraph. Before this we can mention that there is a special case in which the cotangent complex defined above is easy to understand, precisely when X is a Deligne-Mumford derived stack (see A.2). Indeed, under this extra assumption, $\mathbf{QCoh}(X)$ can be described using quasi-coherent $\mathcal{O}_X$-dg-modules on the small étale site $(\mathbf{dAff}/X)_{et}$, of derived affine schemes étale over X. When restricted to this small étale site, $\mathbb{L}^{big}_{\mathcal{F}}$ is already quasi-coherent, because all the morphisms in $(\mathbf{dAff}/X)_{et}$ are automatically étale. Thus, $\mathbb{L}^{big}_{\mathcal{F}} = \mathbb{L}_{\mathcal{F}}$ as a quasi-coherent $\mathcal{O}_X$-dg-module over the small étale site of X.

Finally, we introduce the global version of the de Rham algebra $\mathbf{DR}(\mathcal{F}/k)$ of a derived foliation $\mathcal{F}$. For a derived stack X we have a stack of graded mixed cdga's on the big étale site of F

$$\mathbf{DR}_{X/k} : (\mathbf{dAff}_k/X)^{op} \longrightarrow \epsilon - \mathbf{cdga}^{gr}_k,$$

which sends $u : \mathbf{Spec}\, A \to X$ to $\mathbf{DR}(A/k)$. More generally, for X as above and $\mathcal{F} \in \mathcal{F}ol(X/k)$, we have a stack

$$\mathbf{DR}_{\mathcal{F}/k} : (\mathbf{dAff}_k/X)^{op} \longrightarrow \epsilon - \mathbf{cdga}^{gr}_k,$$

which sends $u : \mathbf{Spec}\, A \to X$ to $\mathbf{DR}(u^*(\mathcal{F})/k)$. This stack comes moreover equipped with a canonical morphism $\mathbf{DR}_{X/k} \to \mathbf{DR}_{\mathcal{F}/k}$ of stacks of graded mixed cdga's, and thus will be referred to as a *graded mixed* $\mathbf{DR}_{X/k}$*-algebra.*

Definition 2.1.3.5. *Let* $X \in \mathbf{dSt}_k$ *and* $\mathcal{F} \in \mathcal{F}ol(X/k)$*.*

(1) The stack of de Rham algebras of $\mathcal{F}$ *(relative to k) is the stack of graded mixed cdga's* $\mathbf{DR}_{\mathcal{F}/k}$ *defined above.*
(2) The (global) de Rham algera of $\mathcal{F}$ *(relative to k) is*

$$\mathbf{DR}(\mathcal{F}/k) := \Gamma(\mathbf{DR}_{\mathcal{F}/k}) \in \epsilon - \mathbf{cdga}^{gr}_k.$$

With the definition above, it is straightforward to reinterpret derived foliations on X as sheaves of graded mixed $\mathbf{DR}_{X/k}$-algebras. This reinterpretation simply associates to a derived foliation $\mathcal{F}$ the sheaf $\mathbf{DR}_{\mathcal{F}/k}$ of the above definition. We get this way a full embedding of $\mathcal{F}ol(X/k)$ into the ∞-category $\mathbf{DR}_{X/k} - \epsilon - \mathbf{cdga}^{gr}$ of sheaves of graded mixed $\mathbf{DR}_{X/k}$-algebras

$$\mathcal{F}ol(X/k) \hookrightarrow \mathbf{DR}_{X/k} - \epsilon - \mathbf{cdga}^{gr}.$$

The essential image of this full embedding is easy to describe by simply unwinding the definitions.

COROLLARY 2.1.3.6. *The above full embedding identifies $\mathcal{F}ol(X/k)$ with the full sub-∞-category of sheaves of graded mixed $\mathbf{DR}_{X/k}$-algebras $\mathcal{D}$ which are quasi-coherent over $\mathbf{DR}_{X/k}$: for any diagram of derived stacks $\mathbf{Spec}\, B \to \mathbf{Spec}\, A \to X$ the natural morphism*

$$\mathbf{DR}_{X/k}(\mathbf{Spec}\, B) \otimes_{\mathbf{DR}_{X/k}(\mathbf{Spec}\, B)} \mathcal{D}(\mathbf{Spec}\, A) \to \mathcal{D}(\mathbf{Spec}\, B)$$

is an equivalence of graded mixed complexes.

REMARK 2.1.3.7. We warn the reader that in the statement above $\mathbf{DR}_{X/k}$ is itself not quasi-coherent over $\mathcal{O}_X$, even as merely a sheaf of graded $\mathcal{O}_X$-cdga's (by forgetting the mixed structure). In particular, $\mathcal{D}$ is also not quasi-coherent as a graded $\mathcal{O}_X$-algebra over X. It is rather of the form $Sym_{\mathcal{O}_X}(\mathbb{L}[1])$ for an $\mathcal{O}_X$-dg-module $\mathbb{L}$ which is not necessarily quasi-coherent.

When the mixed structure is forgotten, the sheaf $\mathbf{DR}_{\mathcal{F}/k}$ becomes, as a graded $\mathbf{DR}_{X/k}$-cdga, equivalent to $Sym_{\mathcal{O}_X}(\mathbb{L}^{big}_{\mathcal{F}/k}[1])$. It is important to note that the graded cdga $Sym_{\mathcal{O}_X}(\mathbb{L}_{\mathcal{F}/k}[1])$ does not carry any natural compatible mixed structure, at least not before passing to global sections (see Theorem. 2.1.3.8). This is already true for the tautological derived foliation $*_X$ whose de Rham algebra is $\mathbf{DR}_{X/k}$, as the de Rham differential does not exists as a morphism $\mathcal{O}_X \to \mathbb{L}_{X/k}$ but only as a morphism to $\mathbb{L}^{big}_{X/k}$.

We finish by the important descent statement showing that even though the de Rham differential of a derived foliation does not exists as a morphism $\mathcal{O}_X \to \mathbb{L}_{\mathcal{F}/k}$, it does so after having taken global sections. This result can be seen as an extension of the descent statement for closed forms of [**PTVV13**, Prop. 1.14] to the derived foliation setting.

THEOREM 2.1.3.8. *Let $\mathcal{F} \in \mathcal{F}ol(X/k)$ be a derived foliation over a derived Artin stack X. We assume that the conormal complex $\mathcal{N}^*_{\mathcal{F}}$ is a perfect complex on X. Then, the natural morphism $\mathbb{L}_{\mathcal{F}/k} \to \mathbb{L}^{big}_{\mathcal{F}/k}$ induces an equivalence $\mathcal{O}_X$-dg-modules*

$$\wedge^i_{\mathcal{O}_X} \mathbb{L}_{\mathcal{F}/k} \to (\wedge^i_{\mathcal{O}_X} \mathbb{L}^{big}_{\mathcal{F}/k})^{qcoh}.$$

In particular, there exists a natural equivalence of graded (non-mixed) cdga's

$$\Gamma(X, Sym_{\mathcal{O}_X}(\mathbb{L}_{\mathcal{F}/k}[1])) \simeq \Gamma(X, Sym_{\mathcal{O}_X}(\mathbb{L}^{big}_{\mathcal{F}/k}[1])) \simeq \mathbf{DR}(\mathcal{F}/k).$$

Proof. The last statement of the theorem is a direct consequence of the fact that the quasi-coherator, by its definition as the right adjoint of the natural inclusion ∞-functor, does not change global sections. Also, for $i = 0$ the statement is trivial, as for $i = 1$ for which it is true by definition of $\mathbb{L}_{\mathcal{F}/k}$.

We first show the following lemma, for which the assumption that $\mathcal{N}^*_{\mathcal{F}/k}$ is perfect is not needed.

LEMMA 2.1.3.9. *The commutative diagram of $\mathcal{O}_X$-dg-modules*

$$\begin{tikzcd} \mathbb{L}_{X/k} \arrow[r] \arrow[d] & \mathbb{L}_{\mathcal{F}/k} \arrow[d] \\ \mathbb{L}^{big}_{X/k} \arrow[r] & \mathbb{L}^{big}_{\mathcal{F}/k} \end{tikzcd}$$

is homotopy cartesian.

Proof of the lemma. In the diagram under consideration, the vertical morphisms are the adjunction morphism coming from the fact that top horizontal morphism is obtained from the bottom horizontal morphism by applying $(-)^{qcoh}$ (see theorem A.5.0.1). Therefore, as $(-)^{qcoh}$ is a stable ∞-functor, it is enough to show that the cofiber of $\mathbb{L}^{big}_{X/k} \to \mathbb{L}^{big}_{\mathcal{F}/k}$ is already a quasi-coherent $\mathcal{O}_X$-dg-module.

For $u : U = Spec\, A \to X$, the morphism of A-dg-modules $\mathbb{L}^{big}_{X/k}(U) \to \mathbb{L}^{big}_{\mathcal{F}/k}(U)$ is equivalent, by definition, to the natural morphism

$$\mathbb{L}_{A/k} \to \mathbb{L}_{u^*(\mathcal{F})/k}.$$

For any $f : V = \mathbf{Spec}\, B \to U$, with $v = uf : V \to X$, the commutative diagram

$$\begin{tikzcd} B \otimes_A \mathbb{L}_{A/k} \arrow[r] \arrow[d] & B \otimes_A \mathbb{L}_{u^*(\mathcal{F})/k} \arrow[d] \\ \mathbb{L}_{B/k} \arrow[r] & \mathbb{L}_{f^*u^*(\mathcal{F})/k} \end{tikzcd}$$

is cartesian by definition of pull-backs of derived foliations over affine derived schemes. Therefore, if C denotes the cofiber of $\mathbb{L}^{big}_{X/k} \to \mathbb{L}^{big}_{\mathcal{F}/k}$, then the natural morphism of B-dg-modules $B \otimes_A C(U) \to C(V)$ is an equivalence. This is precisely the statement that C is a quasi-coherent $\mathcal{O}_X$-dg-module. $\square$

According to the lemma, we have a commutative diagram with exact rows

$$\begin{tikzcd} \mathbb{L}_{X/k} \arrow[r] \arrow[d] & \mathbb{L}_{\mathcal{F}/k} \arrow[r] \arrow[d] & C \arrow[d, "="] \\ \mathbb{L}^{big}_{X/k} \arrow[r] & \mathbb{L}^{big}_{\mathcal{F}/k} \arrow[r] & C. \end{tikzcd}$$

For all $i > 1$ fixed, these exact rows induces finite filtrations on $\wedge^i_{\mathcal{O}_X} \mathbb{L}_{\mathcal{F}/k}$ and $\wedge^i_{\mathcal{O}_X} \mathbb{L}^{big}_{\mathcal{F}/k}$ whose graded pieces of weight p are respectively

$$\wedge^p_{\mathcal{O}_X} \mathbb{L}_{X/k} \otimes_{\mathcal{O}_X} \wedge^{i-p}_{\mathcal{O}_X} C \qquad \text{and} \qquad \wedge^p_{\mathcal{O}_X} \mathbb{L}^{big}_{X/k} \otimes_{\mathcal{O}_X} \wedge^{i-p}_{\mathcal{O}_X} C.$$

These filtrations are moreover compatible with the vertical morphism in the middle. Therefore, in order to show that the morphism $\wedge^i_{\mathcal{O}_X} \mathbb{L}_{\mathcal{F}/k} \longrightarrow (\wedge^i_{\mathcal{O}_X} \mathbb{L}^{big}_{\mathcal{F}/k})^{qcoh}$ is an equivalence, it is enough to show that it induces an equivalence of those graded pieces.

LEMMA 2.1.3.10. *For any perfect complex E on X and any $E' \in \mathcal{O}_X - \mathbf{dg}$, the natural morphism*

$$E \otimes (E')^{qcoh} \longrightarrow (E \otimes E')^{qcoh}$$

is an equivalence.

Proof of the lemma. We use here that perfect complexes are dualizable objects in $\mathbf{QCoh}(X)$, and thus also in the bigger symmetric monoidal ∞-category $\mathcal{O}_X - \mathbf{dg}$. Therefore, for any quasi-coherent complex E_0, we have functorial equivalences

$$\mathbf{Map}(E_0, E \otimes (E')^{qcoh}) \simeq Map(E_0 \otimes E^\vee, (E')^{qcoh}) \simeq \mathbf{Map}(E_0 \otimes E^\vee, E') \simeq \mathbf{Map}(E_0, E \otimes E').$$

By observation, the composed equivalence $\mathbf{Map}(E_0, E \otimes (E')^{qcoh}) \to \mathbf{Map}(E_0, E \otimes E')$ is induced by the canonical morphism $E \otimes (E')^{qcoh} \longrightarrow (E \otimes E')^{qcoh}$, and the lemma thus follows from Yoneda. $\square$

To finish the proof of the theorem, we invoke theorem A.5.0.1 and the lemma 2.1.3.10. They imply that the natural morphisms

$$\wedge^p_{\mathcal{O}_X} \mathbb{L}_{X/k} \otimes_{\mathcal{O}_X} \wedge^{i-p}_{\mathcal{O}_X} C \to \wedge^p_{\mathcal{O}_X} \mathbb{L}^{big}_{X/k} \otimes_{\mathcal{O}_X} \wedge^{i-p}_{\mathcal{O}_X} C$$

are equivalences (note that C is perfect by assumption), and thus, by what we have already see before, that the morphism $\wedge^i_{\mathcal{O}_X} \mathbb{L}_{\mathcal{F}/k} \to (\wedge^i_{\mathcal{O}_X} \mathbb{L}^{big}_{\mathcal{F}/k})^{qcoh}$ is also an equivalence. $\square$

2.1.4. The ∞-category of derived foliations. By construction, for a derived stack $X \in \mathbf{dSt}_k$, derived foliations on X (relative to k) form an ∞-category $\mathcal{F}ol(X/k)$. We study here two basic properties of these ∞-categories: existence of finite limits and pull-backs functoriality along arbitrary morphisms.

Let $f : X \to Y$ be a morphism of derived stacks. It induces an ∞-functor on the ∞-categories of maps to the derived stack $\mathcal{F}ol(-/k)$

$$f^* : \mathcal{F}ol(Y/k) = Map(Y, \mathcal{F}ol(-,k)) \longrightarrow \mathcal{F}ol(X/k) = Map(X, \mathcal{F}ol(-,k)).$$

This defines an ∞-functor

$$\mathcal{F}ol(-/k) : \mathbf{dSt}_k^{op} \longrightarrow \mathbf{Cat}_\infty,$$

which is the left Kan extension of $\mathcal{F}ol(-/k) : \mathbf{dAff}_k^{op} \longrightarrow \mathbf{Cat}_\infty$ along the Yoneda embedding $\mathbf{dAff}_k \hookrightarrow \mathbf{dSt}_k$ (see [**Toe14**, §2.1]).

Note that the big cotangent complex construction of Definition 2.1.3.4 defines an ∞-functor

$$\mathbb{L}^{big} : \mathcal{F}ol(X/k) \longrightarrow (\mathcal{O}_X - \mathbf{dg})^{op}.$$

Composed with the quasi-coherator, it defines another ∞-functor

$$\mathbb{L} : \mathcal{F}ol(X/k) \longrightarrow (\mathcal{O}_X - \mathbf{dg})^{op} \longrightarrow (\mathbf{QCoh}(X))^{op}.$$

the ∞-functors $\mathbb{L}^{big}$ and $\mathbb{L}$ are not functorial in X and rather defines a *lax natural transformations* between derived stacks of ∞-categories $\mathcal{F}ol(-/k) \longrightarrow (\mathcal{O}_- - \mathbf{dg})^{op} \longrightarrow (\mathbf{QCoh}(-))^{op}$. In other words, for a morphism of derived stacks $f : X \to Y$, and $\mathcal{F} \in \mathcal{F}ol(Y/k)$, there exists a natural morphism of $\mathcal{O}_X$-dg-modules $u : f^*(\mathbb{L}^{big}_{\mathcal{F}}) \longrightarrow \mathbb{L}^{big}_{f^*(\mathcal{F})}$. This morphism u depends functorialy in f and $\mathcal{F}$ in some obvious manner, with all the required homotopy coherences. The formal way to express these coherences is by stating the existence of an ∞-functor on the Grothendieck's integral which commutes with the natural projections to $\mathbf{dSt}_k$ (see [**TV15**, §1.3])

$$\begin{tikzcd} \int_{\mathbf{dSt}_k} \mathcal{F}ol(-/k) \arrow[rr, "\mathbb{L}^{big}"] \arrow[dr] & & \int_{\mathbf{dSt}_k} (\mathcal{O}_- - \mathbf{dg})^{op} \arrow[dl] \\ & \mathbf{dSt}_k. & \end{tikzcd}$$

The next Proposition shows that $\mathcal{F}ol(X/k)$ admits arbitrary limits for any derived stack $X \in \mathbf{dSt}_k$.

PROPOSITON 2.1.4.1. *(1) For any derived stack $X \in \mathbf{dSt}_k$, the ∞-category $\mathcal{F}ol(X/k)$ admits small limits.*

(2) For any morphism of derived stacks $f : X \to Y$, the pull-back ∞-functor $f^* : \mathcal{F}ol(Y/k) \to \mathcal{F}ol(X/k)$ preserves limits.

Proof. (1) The ∞-category $\mathcal{F}ol(X/k)$ can be written as a limit

$$\mathcal{F}ol(X/k) \simeq \mathrm{Lim}_{\mathbf{Spec}\, A \to X} \mathcal{F}ol(A/k).$$

In order to show that $\mathcal{F}ol(X/k)$ admits small limits it is therefore enough to show the following two facts.

(1) For all $\mathbf{Spec}\, A \in \mathbf{dAff}_k$, the ∞-category $\mathcal{F}ol(A/k)$ admits small limits.
(2) For any morphism $f : \mathbf{Spec}\, B \to \mathbf{Spec}\, A$ in $\mathbf{dAff}_k$ the induced ∞-functor $f^* : \mathcal{F}ol(B/k) \to \mathcal{F}ol(A/k)$ preserves small limits.

For the first of these assertions, let $I \to \mathcal{F}ol(A/k)$ be a small diagram. It corresponds to a diagram $D : I^{op} : \mathbf{DR}(A/k) - \epsilon - \mathbf{cdga}^{gr}$, of graded mixed $\mathbf{DR}(A/k)$-cdga's. We consider $C := \mathrm{Colim}_{i \in I^{op}} D(i) \in \mathbf{DR}(A/k) - \epsilon - \mathbf{cdga}^{gr}$. We claim that C is a graded mixed $\mathbf{DR}(A/k)$-cdga satisfying the conditions of Definition 2.1.1.1. Indeed, D induces a diagram on cotangent complexes $i \mapsto \mathbb{L}_{D(i)/k}$, from I^{op} to $A - \mathbf{dg}$, simply by considering the ∞-functor obtained by the weight one piece. This is a diagram of A-dg-modules under $\mathbb{L}_{A/k}$. We know that the forgetful ∞-functor, from graded mixed cdga's to graded cdga's commutes, and even reflects, limits and colimits. Therefore, C is, as a graded $\mathbf{DR}(A/k)$-cdga, of the form

$$C \simeq \mathrm{Colim}_{i \in I^{op}} Sym_A(\mathbb{L}_{D(i)/k}[1]) \simeq Sym_A(\mathbb{L}[1]),$$

where $\mathbb{L} = \mathrm{Colim}_{i\in I^{op}}\mathbb{L}_{D(i)/k}$ is the colimit in the ∞-category $\mathbb{L}_{A/k}/A - \mathbf{dg}$ of A-dg-modules under $\mathbb{L}_{A/K}$. Finally, by definition $\mathcal{F}ol(A/k)$ is a full sub-∞-category of the complete ∞-category $(\mathbf{DR}(A/k) - \epsilon - \mathbf{cdga}^{gr})^{op}$, and we just have seen that this sub-∞-category is stable by arbitrary limits. In particular, $\mathcal{F}ol(A/k)$ possesses limits.

The second of the above assertions is also true, as f^* is defined, on the level of graded mixed cdga's, by the base change $-_{\mathbf{DR}(A/k)}\mathbf{DR}(B/k)$, and thus sends colimits of graded mixed $\mathbf{DR}(A/k)$-cdga's to colimits of graded mixed $\mathbf{DR}(B/k)$-cdga's.

(2) As before, the statement is reduced to morphisms between affine derived schemes, for which we already have seen that pull-backs preserve limits. □

As a corollary of the proof of Proposition 2.1.4.1 we have the following formula for the cotangent complex of a limit of derived foliations. In the statement below we denote by $\mathbb{L}^{big}_{X/k}$ the *big* or *fake* cotangent complex of a derived stack X, defined as an $\mathcal{O}_X$-dg-module on $\mathbf{dAff}_k/\mathbf{dSt}_k$ by sending $\mathbf{Spec}\,A \to X$ to $\mathbb{L}_{A/k}$. Note that its image by the quasi-coherator $\mathcal{O}_X - \mathbf{dg} \to \mathbf{QCoh}(X)$ is the usual cotangent complex $\mathbb{L}_{X/k}$ (see theorem A.5.0.1).

Corollary 2.1.4.2. *Let X be a derived stack, and $\mathcal{F}$ a derived foliation that is expressed as a finite limit $\mathcal{F} \simeq \mathrm{Lim}_{i\in I}\mathcal{F}_i$ in $\mathcal{F}ol(X/k)$. Then, the canonical morphisms*

$$\mathrm{Colim}_{i\in I^{op}}(\mathbb{L}^{big}_{\mathcal{F}_i}) \longrightarrow \mathbb{L}^{big}_{\mathcal{F}} \qquad \mathrm{Colim}_{i\in I^{op}}(\mathbb{L}_{\mathcal{F}_i}) \longrightarrow \mathbb{L}_{\mathcal{F}}$$

are equivalences in the ∞-categories $\mathbb{L}^{big}_{X/k}/\mathcal{O}_X - \mathbf{dg}$ and $\mathbb{L}_{X/k}/\mathbf{QCoh}(X)$ of objects under $\mathbb{L}^{big}_{X/k}$ and $\mathbb{L}_{X/k}$.

It is also important to note that cotangent complexes of pull-backs of derived foliations can be computed explicitly.

Corollary 2.1.4.3. *Let $f : X \to Y$ be a morphism of derived Artin stacks, and $\mathcal{F} \in \mathcal{F}ol(Y/k)$. Then, there exists a cocartesian square of quasi-coherent complexes on X*

$$\begin{array}{ccc} f^*(\mathbb{L}_{Y/k}) & \longrightarrow & f^*(\mathbb{L}_{\mathcal{F}/k}) \\ \downarrow & & \downarrow \\ \mathbb{L}_{X/k} & \longrightarrow & \mathbb{L}_{f^*(\mathcal{F})/k}. \end{array}$$

Proof. We have tautological equivalences of $\mathcal{O}_X$-dg-modules

$$f^*(\mathbb{L}^{big}_{Y/k}) \simeq \mathbb{L}^{big}_{X/k} \qquad f^*(\mathbb{L}^{big}_{\mathcal{F}/k}) \simeq \mathbb{L}^{big}_{f^*(\mathcal{F})/k}.$$

By Lemma 2.1.3.9 applied to $\mathcal{F}$ we have a cocartesian square of $\mathcal{O}_Y$-dg-modules

$$\begin{tikzcd} \mathbb{L}_{Y/k} \arrow[r] \arrow[d] & \mathbb{L}_{\mathcal{F}/k} \arrow[d] \\ \mathbb{L}^{big}_{Y/k} \arrow[r] & \mathbb{L}^{big}_{\mathcal{F}/k}. \end{tikzcd}$$

Applying f^* we get another cocartesian square

$$\begin{tikzcd} f^*(\mathbb{L}_{Y/k}) \arrow[r] \arrow[d] & f^*(\mathbb{L}_{\mathcal{F}/k}) \arrow[d] \\ f^*(\mathbb{L}^{big}_{Y/k}) \arrow[r] & f^*(\mathbb{L}^{big}_{\mathcal{F}/k}). \end{tikzcd}$$

The corollary follows by the tautological identifications

$$f^*(\mathbb{L}^{big}_{Y/k}) = \mathbb{L}^{big}_{X/k} \qquad f^*(\mathbb{L}^{big}_{\mathcal{F}/k}) = \mathbb{L}^{big}_{f^*(\mathcal{F})/k},$$

and an application of the ∞-functor $(-)^{qcoh}$. □

We finish this part with the specific case of derived foliations over *Deligne-Mumford* derived stacks. We remind that $X \in \mathbf{dSt}_k$ is called *Deligne-Mumford* if it is n-geometric for some n in the sense of [**TV08**, Def. 1.3.3.1], and if there are étale morphisms $\mathbf{Spec}\, A_i \to X$ such that $\coprod_i \mathbf{Spec}\, A_i \to X$ is an epimorphism. Derived schemes are in particular Deligne-Mumford, as each of the morphism $\mathbf{Spec}\, A_i \to X$ can be chosen to be a Zariksi open immersion. For such a Deligne-Mumford derived stack X, we can define the small étale site X_{et}, which is the full sub-∞-category of $\mathbf{dAff}_k/X$ consisting of all étale morphisms $\mathbf{Spec}\, A \to X$, endowed with the induced étale topology. Stacks on the ∞-site X_{et} form a sub-∞-topos of $(\mathbf{dAff}_k/X)^{\sim,et}$, called the small étale ∞-topos of X. One important feature of the small étale site is the inclusion of the truncation $j : t_0X \to X$ induces an equivalence of ∞-topos $j^* : X^{\sim}_{et} \simeq (t_0X)^{\sim}_{et}$. In particular the small étale topos does not depend on derived structures, in a similar manner than the small étale topos of a scheme does not depend on non-reduced scheme structures.

It is immediate to check that the big cotangent complex $\mathbb{L}^{big}_{X/k}$ of a derived Deligne-Mumford stack X is quasi-coherent when restricted to the small étale site. More precisely, the adjunction morphism of $\mathcal{O}_X$-dg-modules $\mathbb{L}_{X/k} \to \mathbb{L}^{big}_{X/k}$ restricts on X_{et} to an equivalence. Similarly, if $\mathcal{F} \in \mathcal{F}ol(X/k)$ is a derived foliation on X, then the canonical morphism $\mathbb{L}_{\mathcal{F}/k} \to \mathbb{L}^{big}_{\mathcal{F}/k}$ restricts to an equivalence on X_{et}. Implementing on this, we can prove the following proposition, which describes derived foliations in terms of sheaf theory on the small étale sites. For this we introduce the following notation: if $\mathcal{F} \in \mathcal{F}ol(X/k)$ we denote by $\mathbf{DR}_{\mathcal{F}/k,et}$ the restriction of the sheaf of graded mixed cdga's $\mathbf{DR}_{\mathcal{F}/k}$ of Definition 2.1.3.5 to X_{et}.

PROPOSITON 2.1.4.4. *Let $X \in \mathbf{dSt}_k$ be a derived Deligne-Mumford stack. The ∞-functor*

$$\mathcal{F}ol(X/k) \longrightarrow (\mathbf{DR}_{X/k,et} - \epsilon - \mathbf{cdga}^{gr})^{op}$$

from derived foliations to stacks of graded mixed $\mathbf{DR}_{X/k,et}$-cdga's on the small étale site of X, is fully faithful. Its essential image consists of stacks of graded mixed $\mathbf{DR}_{X/k,et}$-cdga D satisfying the following two properties.

1. *The weight 1 piece $D(1)$ of D is a quasi-coherent $\mathcal{O}_X$-dg-module, for the natural $\mathcal{O}_X$-module structure induced by the graded $\mathbf{DR}_{X/k,et}$-algebra structure on D.*
2. *The natural morphism of stacks of graded $\mathbf{DR}_{X/k,et}$-cdga's $Sym_{\mathcal{O}_X}(D(1)) \to D$ is an equivalence.*

Proof. This is a formal exercise left to the reader. $\square$

The importance of Proposition 2.1.4.4 is that $\mathbf{DR}(\mathcal{F}/k)_{et}$ is, as a stack of graded cdga's, of the form $Sym_{\mathcal{O}_X}(\mathbb{L}_{\mathcal{F}/k}[1])$. In other words, the de Rham differential for $\mathcal{F}$ exists as a morphism of stacks of k-dg-modules $\mathcal{O}_X \to \mathbb{L}_{\mathcal{F}/k}$ globally defined on X_{et}. This is specific property of Deligne-Mumford stacks, for general Artin derived stacks this de Rham differential only exists as a k-linear morphism $\mathcal{O}_X \to \mathbb{L}^{big}_{\mathcal{F}/k}$ and does not factor a priori through the adjunction morphism $\mathbb{L}_{\mathcal{F}/k} \to \mathbb{L}^{big}_{\mathcal{F}/k}$.

Finally, the various notions introduced in the definition for derived foliations of derived affine schemes, extends to derived foliations on derived Artin stacks in a very natural manner.

DEFINITION 2.1.4.5. *Let $\mathcal{F}$ be a derived foliation on a derived stack X.*

1. *The derived foliation $\mathcal{F}$ is* perfect *(resp.* almost perfect*) if $\mathbb{L}_{\mathcal{F}/k}$ is a perfect (resp.* almost perfect*) complex on X.*
2. *The derived foliation $\mathcal{F}$ is* n-connective *, for some integer $n \in \mathbb{Z}$, if the cohomology sheaves $\underline{H}^i(\mathbb{L}_{\mathcal{F}/k}) = 0$ for all $i > n$ (i.e. if $\mathbb{L}_{\mathcal{F}/k}[n]$ is connective).*
3. *The derived foliation $\mathcal{F}$ is* smooth *if it is perfect and if $\mathbb{L}_{\mathcal{F}/k}$ is of Tor amplitude contained in $[0, \infty[$.*
4. *The derived foliation $\mathcal{F}$ is* quasi-smooth *if it is perfect and if $\mathbb{L}_{\mathcal{F}/k}$ is of Tor amplitude contained in $[-1, \infty[$.*

We finish this part by a general method to compute mapping spaces in the ∞-category $\mathcal{F}ol(X/k)$ of derived foliations on a derived stack X, in terms of their cotangent complexes and actions of the group stack $\mathcal{H}_0$ described in §1.5. Let then X be any derived stack (not necessarily assumed to be Artin). We define its global cotangent complex using the quas-coherator construstion of A.5

$$\mathbb{L}_{X/k} := (\mathbb{L}^{big}_{X/k})^{qcoh}\mathbf{QCoh}(X).$$

For a derived foliation $\mathcal{F} \in \mathcal{F}ol(X)$, we also have its big cotangent complex $\mathbb{L}^{big}_{\mathcal{F}/k}$ and its cotangent complex defined the same way

$$\mathbb{L}_{\mathcal{F}/k} := (\mathbb{L}^{big}_{\mathcal{F}/k})^{qcoh} \in \mathbf{QCoh}(X).$$

We consider the exact triangle of $\mathcal{O}_X$-dg-modules

$$\mathcal{N}^*_{\mathcal{F}} \longrightarrow \mathbb{L}^{big}_{X/k} \longrightarrow \mathbb{L}^{big}_{\mathcal{F}/k},$$

where $\mathcal{N}^*_{\mathcal{F}}$ is by definition the conormal complex of $\mathcal{F}$ (relative to k). A simple observation show that $\mathcal{N}^*_{\mathcal{F}}$ is quasi-coherent. As a result, the commutative square

$$\begin{tikzcd} \mathbb{L}^{big}_{X/k} \arrow[r] & \mathbb{L}^{big}_{\mathcal{F}/k} \\ \mathbb{L}_{X/k} \arrow[r] \arrow[u] & \mathbb{L}_{\mathcal{F}/k} \arrow[u] \end{tikzcd}$$

is homotopy cartesian.

For two quasi-coherent complexes E and E' on X, we can form the derived stack of morphism $\underline{\mathbf{Map}}(E, E') \in \mathbf{dSt}_k$. By definition, it sends a connective cdga A to the simplicial set $\mathbf{Map}_{\mathbf{QCoh}(X)}(E, A \otimes E')$, where $A \otimes_k E'$ is the external tensor product of E' by the k-linear complex underlying A. It is also the ∞-functor sending A to $\mathbf{Map}_{\mathbf{QCoh}(X \times \mathbf{Spec}\, A)}(p^*(E), p^*(E'))$, where $p : X \times \mathbf{Spec}\, A \to X$ is the natural projection. The stack $\underline{\mathbf{Map}}(E, E')$ possesses a natural linear structure (i.e. is a stack of $\mathcal{O}$-dg-modules), but we will consider it merely as a space valued stacks and will not keep track of its linear structure in what follows. When E and E' both sits under a given object $a : E_0 \to E$, and $b : E_0 \to E'$, we also have the mapping stack under E_0 $\underline{\mathbf{Map}}_{E_0/}(E, E')$, of morphisms from E to E' that commute with the morphism a and b. By definition, we have a fibration sequence of derived stacks

$$\underline{\mathbf{Map}}_{E_0/}(E, E') \to \underline{\mathbf{Map}}(E, E') \to \underline{\mathbf{Map}}(E_0, E')$$

where the fiber is taken at the point corresponding to b.

Finally, we remind (§1.5) that $\mathcal{H}_0$ is the group stack $\mathbb{G}_m \ltimes B\mathbb{G}_a$, semi-direct product of $B\mathbb{G}_a$ with $\mathbb{G}_m$ (for the natural induced weight 1 action of $\mathbb{G}_m$ on $\mathbb{G}_a$).

PROPOSITON 2.1.4.6. *Let X be a derived stack and $\mathcal{F}, \mathcal{F}' \in \mathcal{F}ol(X/k)$ two derived foliations on X. Then, there exists a stack $\pi : \mathcal{M}(\mathcal{F}, \mathcal{F}') \to B\mathcal{H}_0$ with the following properties.*

(1) The base change $\mathcal{M}(\mathcal{F}, \mathcal{F}') \times_{B\mathcal{H}_0} B\mathbb{G}_m$, is equivalent to

$$\underline{\mathbf{Map}}^{gr}_{\mathbb{L}_{X/k}/}(\mathbb{L}_{\mathcal{F}'/k}, \oplus_{i \geq 1}(\wedge^i_{\mathcal{O}_X} \mathbb{L}^{big}_{\mathcal{F}}[i-1])^{qcoh})$$

the derived stack of graded morphisms from $\mathbb{L}_{\mathcal{F}'/k}$ to $\oplus_{i \geq 0} \wedge^i_{\mathcal{O}_X} \mathbb{L}^{big}_{\mathcal{F}}[i-1]$ under $\mathbb{L}_{X/k}$, whith the natural gradings where $\mathbb{L}_X$ and $\mathbb{L}_{\mathcal{F}'}$ have weight one, and $\wedge^i_{\mathcal{O}_X} \mathbb{L}^{big}_{\mathcal{F}}$ has weight i.

(2) There is a natural equivalence of spaces

$$\mathbb{H}(B\mathcal{H}_0, \mathcal{M}(\mathcal{F}, \mathcal{F}')) := p_*(\mathcal{M}(\mathcal{F}, \mathcal{F}'))(k) \simeq \mathbf{Map}_{\mathcal{F}ol(X)}(\mathcal{F}, \mathcal{F}')$$

where $p : B\mathcal{H}_0 \to *$ is the structural morphism, and $p_* : \mathbf{dSt}_k/B\mathcal{H}_0 \to \mathbf{dSt}_k$ is the direct image ∞-functor.

Proof. We use the equivalence

$$\mathbf{CAlg}(\mathbf{QCoh}(B\mathcal{H}_0)) \simeq \epsilon - \mathbf{cdga}_k^{gr}$$

between commutative algebras in the symmetric monoidal ∞-category $\mathbf{QCoh}(B\mathcal{H}_0)$ and graded mixed cdga's (see §1.5). Using this equivalence, for two graded mixed cdga D and D' the mapping space $Map_{\epsilon-\mathbf{cdga}^{gr}}(D, D')$ can be written as in the Proposition

$$Map_{\epsilon-\mathbf{cdga}_k^{gr}}(D, D') \simeq p_*(\mathcal{M}(D, D'))(k).$$

Here the derived stack $\mathcal{M}(D, D')$ lives over $B\mathcal{H}_0$ and can be defined functorially as follows. For $u : \mathbf{Spec}\, A \to B\mathcal{H}_0$, the space $\mathcal{M}(D, D')(A)$ is given by $\mathbf{Map}_{\mathbf{cdga}_A}(u^*(D), u^*(D'))$, where D and D' are here considered as quasi-coherent cdga's over $B\mathcal{H}_0$ via the equivalence mentioned above. Note that a similar formula remains valid when D and D' are D_0-graded mixed cdga, for a given graded mixed cdga D_0, with $\mathcal{M}(D, D')$ being replace by $\mathcal{M}_{D_0}(D, D')$ the derived stack of morphisms of D_0-cdga's. Assuming further that D (resp. D' and D_0) are, as graded cdga's, of the form $Sym_A(E)$ (resp. $Sym_A(E')$ and $Sym_A(E_0)$) for some A-dg-module E, we can instead of $\mathcal{M}_{D_0}(D, D')$ consider its reduced version

$$\mathcal{M}^*_{D_0}(D, D'),$$

that is the fiber of $\mathcal{M}_{D_0}(D, D') \longrightarrow \mathcal{M}_{D_0}(D, A)$, taken at the natural augmentation $D \to A$.

Now, the mapping space $\mathbf{Map}_{\mathcal{F}ol(X/k)}(\mathcal{F}, \mathcal{F}')$ is given by the limit

$$\mathbf{Map}_{\mathcal{F}ol(X/k)}(\mathcal{F}, \mathcal{F}') \simeq \lim_{\mathbf{Spec}\, A \to X} \mathbf{Map}_{\mathbf{DR}(A/k)-\epsilon-\mathbf{cdga}^{gr}}(\mathbf{DR}_{\mathcal{F}'}(A), \mathbf{DR}_{\mathcal{F}}(A)).$$

Using the previous identification for mapping spaces of graded mixed cdga's, we see that the Proposition (2) is true for a certain derived stack $\mathcal{M}(\mathcal{F}, \mathcal{F}') \to B\mathcal{H}_0$, namely

$$\mathcal{M}(\mathcal{F}, \mathcal{F}') \simeq \lim_{\mathbf{Spec}\, A \to X} \mathcal{M}^*_{\mathbf{DR}(A/k)-\epsilon-\mathbf{cdga}^{gr}}(\mathbf{DR}_{\mathcal{F}'}(A), \mathbf{DR}_{\mathcal{F}}(A)).$$

The base change along $B\mathbb{G}_m \to B\mathcal{H}_0$ is then given by the derived stack

$$\lim_{\mathbf{Spec}\, A} \underline{\mathbf{Map}}^*_{\mathbf{DR}(A/k)-\mathbf{cdga}^{gr}}(\mathbf{DR}_{\mathcal{F}'}(A), \mathbf{DR}_{\mathcal{F}}(A)),$$

limits of the derived stacks $\underline{\mathbf{Map}}^*_{\mathbf{DR}(A/k)-\mathbf{cdga}^{gr}}(\mathbf{DR}_{\mathcal{F}'}(A), \mathbf{DR}_{\mathcal{F}}(A))$ of morphisms of $\mathbf{DR}(A/k)$-linear morphisms of graded cdga's from $\mathbf{DR}_{\mathcal{F}'}(A)$ to $\mathbf{DR}_{\mathcal{F}}(A)$ (obtained by forgetting the mixed structures) compatible with the augmentations to A. Because these cdga's are graded free, we see that this base change can also be written in terms of graded morphisms of complexes

$$\lim_{\mathbf{Spec}\, A} \underline{\mathbf{Map}}^{gr}_{\mathbb{L}_{A/k}/\mathbf{cdga}^{gr}}(\mathbb{L}^{big}_{\mathcal{F}'}(A), \oplus_{i\geq 1} \wedge^i_A \mathbb{L}^{big}_{\mathcal{F}}(A)[i-1])) \simeq \underline{\mathbf{Map}}^{gr}_{\mathbb{L}^{big}_X}(\mathbb{L}^{big}_{\mathcal{F}'}, \oplus_{i\geq 1} \wedge^i_{\mathcal{O}_X} \mathbb{L}^{big}_{\mathcal{F}}[i-1]).$$

(note that the sum does not involve the term $i = 0$ because we use the reduced version of the derived stacks of morphisms of graded cdga's). To finish the proof of the Proposition it thus simply remains to prove that the quasi-coherator construction induces for each $\mathcal{O}_X$-dg-module E with a morphism $\mathbb{L}_X^{big} \to E$, an equivalence

$$\underline{\mathbf{Map}}_{\mathbb{L}_X^{big}}(\mathbb{L}_{\mathcal{F}'}^{big}, E) \longrightarrow \underline{\mathbf{Map}}_{\mathbb{L}_X}(\mathbb{L}_{\mathcal{F}'}, E^{qcoh}),$$

as the result will be obtained by the special cases where $E = \wedge^i \mathbb{L}_{\mathcal{F}}^{big}$.

To establish this last step, by replacing X with $X \times \mathbf{Spec}\, A$ we see that it is enough to check this simply on mapping spaces rather than mapping derived stacks. To check this last step, we consider the commutative diagram

$$\begin{tikzcd} \mathbf{Map}_{\mathbb{L}_X^{big}}(\mathbb{L}_{\mathcal{F}'}^{big}, E) \arrow[r] \arrow[d] & \mathbf{Map}(\mathbb{L}_{\mathcal{F}'}^{big}, E) \arrow[r] \arrow[d] & \mathbf{Map}(\mathbb{L}_X^{big}, E) \arrow[d] \\ \mathbf{Map}_{\mathbb{L}_X}(\mathbb{L}_{\mathcal{F}'}, E^{qcoh}) \arrow[r] & \mathbf{Map}(\mathbb{L}_{\mathcal{F}'}, E^{qcoh}) \arrow[r] & \mathbf{Map}(\mathbb{L}_X, E^{qcoh}). \end{tikzcd}$$

Each row of this diagram is a fibration sequence, for the base point in $\mathbf{Map}(\mathbb{L}_X^{big}, E)$ corresponding to the fixed morphism $\mathbb{L}_X^{big} \to E$. However, we know that the square

$$\begin{tikzcd} \mathbb{L}_X^{big} \arrow[r] & \mathbb{L}_{\mathcal{F}'}^{big} \\ \mathbb{L}_X \arrow[r] \arrow[u] & \mathbb{L}_{\mathcal{F}'} \arrow[u] \end{tikzcd}$$

is cocartesian, and thus the square

$$\begin{tikzcd} \mathbf{Map}(\mathbb{L}_{\mathcal{F}'}^{big}, E) \arrow[r] \arrow[d] & \mathbf{Map}(\mathbb{L}_X^{big}, E) \arrow[d] \\ \mathbf{Map}(\mathbb{L}_{\mathcal{F}'}, E^{qcoh}) \arrow[r] & \mathbf{Map}(\mathbb{L}_X, E^{qcoh}). \end{tikzcd}$$

is cartesian. This implies that the natural morphism $\mathbf{Map}_{\mathbb{L}_X^{big}}(\mathbb{L}_{\mathcal{F}'}^{big}, E) \to \mathbf{Map}_{\mathbb{L}_X}(\mathbb{L}_{\mathcal{F}'}, E^{qcoh})$ is indeed an equivalence as wanted. $\square$

An important corollary of the previous Proposition is obtained when X is moreover a derived *Artin* stack, as we know from A.5.0.1 that $\wedge^i_{\mathcal{O}_X} \mathbb{L}_{\mathcal{F}}^{big}[i-1])^{qcoh} \simeq \wedge^i_{\mathcal{O}_X} \mathbb{L}_{\mathcal{F}}[i-1]$. In particular we get the following important consequence.

COROLLARY 2.1.4.7. *Let X be a derived Artin stack and let $\mathcal{F}$ be a derived foliation such that $\mathbb{L}_{\mathcal{F}} \simeq 0$. Then $\mathcal{F}$ is an initial object in $\mathcal{F}ol(X)$.*

Proof of Corollary. In this case, for any $\mathcal{F}' \in \mathcal{F}ol(X)$, the natural projection $\mathcal{M}(\mathcal{F}, \mathcal{F}') \to \times_{B\mathcal{H}_0} B\mathbb{G}_m$ must be an equivalence and thus $p_*(\mathcal{M}(\mathcal{F}, \mathcal{F}')) = *$ is the punctual derived stack. It follows from the proposition 2.1.4.6 that $\mathbf{Map}_{\mathcal{F}ol(X)}(\mathcal{F}, \mathcal{F}') \simeq *$. $\square$

2.1.5. More examples. We have given several examples of derived foliations on derived affine schemes in Section 2.1.2. We now provide several global examples on derived schemes and stacks.

EXAMPLE 2.1.5.1. **Tautological derived foliations.** For any derived stack X, the ∞-category possesses a final and and sometimes an initial object. The final object is denoted by $*_X$, whose cotangent complex is $\mathbb{L}_{X/k}$. As an ∞-functor $(\mathbf{dAff}_k/X)^{op} \to \epsilon - \mathbf{cdga}^{gr}$ it is given by sending $\mathbf{Spec}\, A \to X$ to $\mathbf{DR}(A/k)$.

The initial object of $\mathcal{F}ol(X/k)$, when it exists, is denoted by 0_X. When X is representable by a derived Artin stack, such an initial object always exists and moreover its cotangent complex is 0. As an ∞-functor $(\mathbf{dAff}_k/X)^{op} \to \epsilon - \mathbf{cdga}^{gr}$, 0_X is given by sending $\mathbf{Spec}\, A \to X$ to $\mathbf{DR}(A/X)$, the relative de Rham algebra of $\mathbf{Spec}\, A$ over X. We need here to provide more definitional details, as X is not assumed to be affine and thus $\mathbf{DR}(A/X)$ has not been defined yet. We define this notion by using descent over X, by the standard formula

$$\mathbf{DR}(A/X) = \mathrm{Lim}_{\mathbf{Spec}\, B \to X} \mathbf{DR}(\mathbf{Spec}\, A \times_{\mathbf{Spec}, B} X/B).$$

In more general terms, for a morphism of derived stacks $f : Y \to Z$, we set

$$\mathbf{DR}(Y/Z) := \mathrm{Lim}\, \mathbf{DR}(A/B) \in \epsilon - \mathbf{cdga}_k^{gr},$$

where the limit is taken over the ∞-category of commutative diagrams of the form

$$\begin{tikzcd} \mathbf{Spec}\, A \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathbf{Spec}\, B \arrow[r] & Z. \end{tikzcd}$$

As the diagonal of X is representable by a derived Artin stack, all the morphisms $\mathbf{Spec}\, A \to X$ are also representable, and thus the relative cotangent complex $\mathbb{L}_{A/X}$ always exits as an A-dg-module. Moreover, by the descent statement theorem 2.1.3.8 we have an equivalence of graded cdga's

$$\mathbf{DR}(A/X) \simeq Sym_A(\mathbb{L}_{A/X}[1]).$$

This shows that 0_X is indeed a stack of graded mixed cdga's on $\mathbf{dAff}/X$ which corresponds to an object in $\mathcal{F}ol(X/k)$. Moreover, the big cotangent complex of 0_X is the $\mathcal{O}_X$-dg-module sending $\mathbf{Spec}\, A \to X$ to $\mathbb{L}_{A/X}$, and we thus have an exact triangle of $\mathcal{O}_X$-dg-modules

$$\mathbb{L}_{X/k} \to \mathbb{L}^{big}_{X/k} \to \mathbb{L}^{big}_{0_X}.$$

By our theorem A.5.0.1 we have $\mathbb{L}_{0_X} = (\mathbb{L}^{big}_{0_X})^{qcoh} \simeq 0$ and thus by Corollary 2.1.4.7 0_X is indeed an initial object.

EXAMPLE 2.1.5.2. **Globally integrable derived foliations.** As in the affine case (see 2.1.2.4, we will say that a derived foliation $\mathcal{F} \in \mathcal{F}ol(X/k)$ is *globally integrable (or simply integrable)* if there is a morphism $f : X \to Y$ with Y a derived Artin stack such that $\mathcal{F} \simeq f^*(0_Y)$. We note that Y is not uniquely defined, as any factorization $f : X \xrightarrow{g} Y' \xrightarrow{p} Y$ with p being étale has the property that $g^*(0'_Y) \simeq f^*(0_Y)$ (simply because $p^*(0_Y) = 0'_Y$ by étaleness and the corollary 2.1.4.7).

EXAMPLE 2.1.5.3. **Relative global derived foliations.** Let $f : X \to Y$ be a morphism in $\mathbf{dSt}_k$ with Y being a derived Artin stack. We define the ∞-category of *derived foliations on X relative to Y* to be the comma ∞-category

$$\mathcal{F}ol(X/Y) := \mathcal{F}ol(X/k)/f^*(0_Y),$$

of derived foliations over $f^*(0_Y)$ (we know 0_Y exists by our previous example). Informally, a derived foliations on X relative to Y is a family of derived foliations in the fibers of f. When $X = \mathbf{Spec}\, A \to Y = \mathbf{Spec}\, B$ is a a morphism of derived affine schemes, $\mathcal{F}ol(X/Y)$ is canonically equivalent to $\mathcal{F}ol(X/B)$. This follows simply from the observation that the ∞-category of graded mixed $\mathbf{DR}(A/B)$-cdga's can be also expressed as the comma ∞-category of graded mixed cdga's under $\mathbf{DR}(A/B)$. By gluing we also deduce that $\mathcal{F}ol(X/Y)$ is equivalent to $\mathcal{F}ol(X/B)$ in the sense of Definition 2.1.3.3 for any derived stack X with a map to $Y = \mathbf{Spec}\, B$.

EXAMPLE 2.1.5.4. **Closed 1-forms.** We recall from [**PTVV13**] the notion of closed p-forms on a derived stack X (relative to k). This notion can be expressed easily in terms of graded mixed complexes. For this, we consider the graded mixed complex $k(p)[p-n]$, which is k sitting in pure weight p and cohomological deree $n-p$ (and both the cohomological differential as well as the mixed structure being zero). The space of closed p-forms on X of degree n is defined to be the mapping space of graded mixed complexes

$$\mathcal{A}^{p,cl}(X/k) := \mathbf{Map}_{\epsilon-\mathbf{dg}^{gr}_k}(k(p)[p-n], \mathbf{DR}(X/k)).$$

We can use here the equivalence between graded mixed complexes and complete filtered complexes in order to express the space of closed p-forms in a more traditional manner in terms of truncated de Rham complex $F^{-p}\widehat{C}_{DR}(X/k) := F^{-p}|\mathbf{DR}(X/k)|^t$ (see Definition 1.4.2.1).

Let now $\omega : k(1)[1-n] \to \mathbf{DR}(X/k)$ be a closed 1-form on X of degree n. As $\mathbf{DR}(X/k)$ is the graded mixed cdga of global sections of $\mathbf{DR}_{X/k}$ (see theorem 2.1.3.8), the form ω can also be considered as a morphism of stacks of graded mixed cdgas on $\mathbf{dAff}/X$

$$Sym_k(k(1)[1-n]) \longrightarrow \mathbf{DR}_{X/k}.$$

We set

$$\mathbf{DR}_\omega := \mathbf{DR}_{X/k} \otimes_{Sym_k(k(1)[1-n])} k,$$

where we use the canonical augmentation morphism $Sym_k(k(1)[1-n])k \to k$. Note that, as a stack of graded cdga's, $\mathbf{DR}_\omega$ is of the form $Sym_{\mathcal{O}_X}(\mathbb{L}_\omega[1]^{big})$ where $\mathbb{L}_\omega[1]^{big}$ is the $\mathcal{O}_X$-dg-module whose values on $\mathbf{Spec}\, A \to X$ is the cone of the morphism $A[-n] \to \mathbb{L}_{A/k}$ given by the restriction of ω as a 1-form of cohomological degree n on $\mathbf{Spec}\, A$. It is clear from this description that $\mathbf{DR}_\omega$ defines an object in $\mathcal{F}ol(X/k)$, denoted by $\mathcal{F}_\omega$, and called the *derived foliation associated to* ω. Note that the cotangent complex of $\mathcal{F}_\omega$ is by construction $(\mathbb{L}_\omega[1]^{big})^{qcoh}$ and thus sits in an exact triangle of quasi-coherent complexes

$$\mathcal{O}_X[-n] \to \mathbb{L}_{X/k} \to \mathbb{L}_{\mathcal{F}_\omega/k}$$

where the first morphism is defined by ω. In particular, when X is a smooth scheme over k and $n=0$, a closed 1-form (of degree 0) on X simply is a section ω of $\Omega^1_{X/k}$ such that $dR(\omega)=0$. The derived foliation associated to ω has a cotangent complex given explicitly by a lengh two complex $\mathcal{O}_X \to \Omega^1_{X/k}$, where $\Omega^1_{X/k}$ sits in degree 0. We see in particular that the derived foliation $\mathcal{F}_\omega$ is smooth in the sense of Definition 2.1.4.5 if and only if ω is nowhere vanishing, or in other words if ω identifies $\mathcal{O}_X$ with a subbundle of $\Omega^1_{X/k}$. When X is moreover proper over k, Hodge theory implies that any global section $\omega \in \Gamma(X, \Omega^1_{X/k})$ is closed for the de Rham differential, and thus defines a derived foliation $\mathcal{F}_\omega$. This provides many examples of derived foliations on smooth proper schemes over k.

EXAMPLE 2.1.5.5. **Vector fields.** Let $X = \mathbf{Spec}\, A$ be a derived affine scheme over k and $\nu : \mathbb{L}_{X/k} \to A$ be a morphism of A-dg-module. Such a morphism will be called a vector field on X (relative to k). Note that this is the same thing as a k-linear derivation $A \to A$. We define a graded mixed cdga $\mathbf{DR}_\nu$ by considering $Sym_A(A[1]) = A \oplus A[1]$ as a graded cdga, and define the mixed structure by the derivation $\nu : A \to A$. The weight zero part is equal here to A, and thus we have a natural induced morphism of graded mixed cdga's $\mathbf{DR}(A/k) \to \mathbf{DR}_\nu$. By construction $\mathbf{DR}_\nu$ defines a derived foliation on X relative to k whose cotangent complex is $\mathcal{O}_X$. By étale gluing this contruction can be extended to any derived Deligne-Mumford stack X and $\nu \in H^0(X, \mathbb{T}_{X/k})$. Extension to derived Artin stacks is trickier, can be done using formal localization techniques from [**CPT⁺17**], but will not be discussed in this book. We can moreover extend this example by starting with a derivation $\nu : \mathbb{L}_{X/k} \to \mathcal{L}$, with $\mathcal{L}$ an arbitrary line bundle. The previous construction provides a derived foliation F_ν whose cotangent complex is $\mathcal{L}$ and whose sheaf of graded mixed cdga is $Sym_{\mathcal{O}_X}(\mathcal{L}[1]) \simeq \mathcal{O}_X \oplus \mathcal{L}[1]$ endowed with the mixed structure defined by ν.

EXAMPLE 2.1.5.6. **Group actions.** Let X be a derived Artin stack and G a smooth group scheme acting on X. We can form the quotient map $p : X \to Y = [X/G]$, which is a morphism of derived Artin stacks. The foliation $p^*(0_Y) \in \mathcal{F}ol(X/Y)$ or rather its image in $\mathcal{F}ol(X/k)$, is called the *foliation induced from the G-action* . Its big cotangent complex has values on $u : \mathbf{Spec}\, A \to X$ the A-dg-module $\mathbb{L}_{A/Y}$, and thus sits in an exact triangle

$$\mathbb{L}_{X/Y} \to \mathbb{L}^{big}_{p^*(0_Y)} \to \mathbb{L}^{big}_{0_X}.$$

Applying the quasi-coherator we find that the natural morphism $\mathbb{L}_{X/Y} \to \mathbb{L}_{p^*(0_Y)}$ is an equivalence. Moreover, the cartesian square of derived stacks

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ \bullet \arrow[r] & BG, \end{tikzcd}$$

implies that $\mathbb{L}_{X/Y}$ is of the form $g^* \otimes_k \mathcal{O}_X$, where $g^* = \mathbb{L}_{*/BG}$ is the dual Lie algebra of G.

2.2. Comparison with derived Lie algebroids

For a commutative k-dg algebra A, $\mathsf{T}_A := \mathsf{Der}_k(A, A)$ is the tangent A-dg module consisting of k-derivations $\partial : A \to A$. Note that the graded commutator $[\partial, \partial'] := \partial \circ \partial' + (-1)^{|\partial||\partial'|+1} \partial' \circ \partial$ endows T_A with the structure of a dg-Lie algebra over k.

Definition 2.2.0.1. *Let A be a commutative k-dg algebra. We denote by* dgLieAlgds$_A$ *the category of* dg-Lie algebroids over A, *whose objects are pairs $(\mathfrak{g}, \rho)$ where $\mathfrak{g}$ is an (a priori, unbounded) A-dg module, endowed with a further structure $[\,,\,]_{\mathfrak{g}} : \mathfrak{g} \otimes_k \mathfrak{g} \to \mathfrak{g}$ of k-dg Lie algebras, and $\rho : \mathfrak{g} \to \mathsf{T}_A$ is both a morphism of A-dg modules and of k-dg-Lie algebras, called the* anchor map, *such that*

$$[X, aY]_{\mathfrak{g}} = (-1)^{|X||a|} a[X, Y]_{\mathfrak{g}} + \rho(X)(a)Y. \tag{8}$$

Morphisms $(\mathfrak{g}, \rho) \to (\mathfrak{g}', \rho')$ in dgLieAlgds$_A$ *are morphisms $\mathfrak{g} \to \mathfrak{g}'$ of A-dg modules and k-dg-Lie algebras commuting with the anchor maps.*

When no confusion is possible, we will simply write $[\,,\,]$ instead of $[\,,\,]_{\mathfrak{g}}$ for the Lie bracket of a dg-Lie algebroid.

Remark 2.2.0.2. Condition (8) can be viewed (and remembered) as a sort of a Leibniz rule, relating the anchor map to the failure of the bracket $[\,,\,]_{\mathfrak{g}}$ to be an A-linear dg-Lie bracket on $\mathfrak{g}$: the anchor map gives the correction to A-linearity of $[\,,\,]_{\mathfrak{g}}$ in the Leibniz rule (8).

Theorem 2.2.0.3. *[**Nui19**, Theorem 3.1] Let A be a commutative k-dg algebra. The category* dgLieAlgds$_A$ *of dg-Lie algebroids over A admits a right proper, tractable, semi-model structure whose fibrations are degreewise surjections, and whose weak equivalences are quasi-isomorphisms.*

For the necessary background on semi-model structures, the reader is referred to [**Nui19**].

Definition 2.2.0.4. *Let A be a commutative k-dg algebra, and QA its cofibrant replacement. We denote by*

- **dgLieAlgds**$_A$ *the ∞-category obtained by Dwyer-Kan localization of* dgLieAlgds$_{QA}$ *along its weak-equivalences of Theorem 2.2.0.3.*

- $\mathbf{dgLieAlgds}_A^{\mathrm{perf},\geq 0}$ *the full* ∞*-subcategory of* $\mathbf{dgLieAlgds}_A$ *consisting of dg-Lie algebroids* $(\mathfrak{g},\rho)$ *where* $\mathfrak{g}$ *is a perfect* A*-dg-module cohomologically concentrated in degrees* $[0,+\infty)$.

For a dg-Lie algebroid $(\mathfrak{g},\rho)$ over A, we will consider its Chevalley-Eilenberg complex $\mathsf{CE}(\mathfrak{g},\rho)$, which we now define. Let $\mathrm{Sym}_A(\mathfrak{g}[-1])$ be the symmetric algebra over A of $\mathfrak{g}[-1]$ viewed as an A-dg-module.

Definition 2.2.0.5. *For a dg-Lie algebroid* $(\mathfrak{g},\rho)$ *over* A*, we will denote by* $\mathsf{CE}^*(\mathfrak{g},\rho)$ *the* A*-linear dual of the* A*-dg-module* $\mathrm{Sym}_A(\mathfrak{g}[-1])$*, endowed with the following structures:*

- *the grading* $\mathsf{CE}^*(\mathfrak{g},\rho)^{(n)} := \underline{\mathrm{Hom}}_A(\mathrm{Sym}^n_A(\mathfrak{g}[-1]),A)$, $n\geq 0$.
- *the graded commutative* A*-dg-algebra structure induced by being the* A*-dual of the graded cocommutative coalgebra* $\mathrm{Sym}_A(\mathfrak{g}[-1])$.
- *the internal differential*

$$d : (\mathsf{CE}^*(\mathfrak{g},\rho)^{(n)})^m \to (\mathsf{CE}^*(\mathfrak{g},\rho)^{(n)})^{m+1},\ \omega \longmapsto d_A\circ\omega-(-1)^m\omega\circ d_{\mathrm{Sym}^n_A(\mathfrak{g}[-1])}$$

i.e. the differential of the A*-dg module* $\underline{\mathrm{Hom}}_A(\mathrm{Sym}^n_A(\mathfrak{g}[-1]),A)$.
- *the mixed structure*

$$\epsilon^{(n)}_{\mathsf{CE}} : \mathsf{CE}^*(\mathfrak{g},\rho)^{(n)} \to (\mathsf{CE}^*(\mathfrak{g},\rho)^{(n+1)})[-1]$$

is given by (omitting Koszul signs)

$$\epsilon^{(n)}_{\mathsf{CE}}(\omega)(\xi_1,\dots,\xi_{n+1}) = \sum_i \rho(\xi_i)(\omega(\xi_1,\dots,\cancel{\xi_i},\dots,\xi_{n+1}))-\sum_{i<j}\omega([\xi_i,\xi_j],\xi_1,\dots,\cancel{\xi_i},\dots,\cancel{\xi_j},\dots,\xi_{n+1}).$$

Endowed with the above structures, $\mathsf{CE}^*(\mathfrak{g},\rho)$ *becomes a graded mixed cdga over* k.

Remark 2.2.0.6. Usually, $\mathsf{CE}^*(\mathfrak{g},\rho)$ is regarded as a cdga by taking the total differential, given by the sum of the internal differential and the mixed one. However, as we stressed out in several other points, it is quite important for our purposes to keep the two differentials separated and view it as a graded mixed cdga.

Since $\mathbf{DR}(-/k)$ is left adjoint to the weight 0 part functor (see Section 1.4), and

$$\mathsf{CE}^*(\mathfrak{g},\rho)^{(0)} = \underline{\mathrm{Hom}}_A(A,A),$$

the identity map $A\to A$ induces a morphism of graded mixed cdga's $\mathbf{DR}(A/k)\to\mathsf{CE}^*(\mathfrak{g},\rho)$. Therefore, if $\mathfrak{g}$ is a perfect A-dg-module, this yields the structure of a derived foliation on $\mathsf{CE}^*(\mathfrak{g},\rho)$ (Definition 2.1.1.1).

The following theorem expresses the precise comparison between perfect foliations and perfect dg-Lie algebroids.

Theorem 2.2.0.7. *Let A be a connective k-linear cdga of finite presentation over k. The ∞-functor sending a dg-Lie algebroid $(\mathfrak{g}, \rho)$ over A to its Chevalley complex $\mathsf{CE}^*(\mathfrak{g}, \rho)$ induces a equivalence*

$$\mathsf{CE}^* : (\mathbf{dgLieAlgds}_A^{\mathrm{perf}})^{op} \simeq \mathcal{F}ol^{\mathrm{perf}}(A/k),$$

where $\mathbf{dgLieAlgds}_A^{\mathrm{perf}}$ is the full sub-∞-category of dg-Lie algebroids which are perfect as an A-dg-module, and $\mathcal{F}ol^{\mathrm{perf}}(A/k)$ is the ∞-category of perfect derived foliations over A.

Proof. This is a consequence of [**Fu12**, Theorem 5.15] □

As an immediate consequence, we get the following corollary which relates perfect derived foliations over the point $\mathbf{Spec}\, k$ and dg-Lie algebras over k which are perfect as k-dg-modules. We note also that the above theorem extends to almost perfect derived foliations under the condition that A is furthermore cohomologically bounded. We can thus extract the most important case of application for us, which is the following corollary.

Corollary 2.2.0.8. *The Chevalley complex construction induces an equivalence of ∞-categories*

$$\mathbf{dgLie}_k^{dap} \simeq \mathcal{F}ol^{ap}(k/k)$$

between dual almost perfect dg-Lie algebras over k and almost perfect derived foliations over k.

In the previous corollary *dual almost perfect* means that the k-linear dual is almost perfect. As k is a field this condition is equivalent to the condition of being cohomologically bounded on the left and with finite dimensional individual cohomology spaces.

2.3. Formal structures

All along this section, we will assume that our base ring k is a field (of characteristic zero). We will be interested in derived foliations restricted to formal derived stacks, and their interpretations in terms of dg-Lie algebras over k. These results are not optimal and replacing k be a more general base is also possible. However, the general notion of formal derived stacks over general bases is out of the scope of this book, and we refer the interested reader to [**BMN25**] for recent and pretty optimal results in that direction.

2.3.1. Derived stacks and formal moduli problems. In this section we recall the fundamental results of [**Lur10**, **Lur11**] (see also [**Lur22b**, §IV] and [**Toë17**]), relating pointed derived formal stacks and dg-lie algebras. For this, we first introduce $\mathbf{dgart}_k$, the ∞-category of commutative, connective, Artinian local dg-algebras over k . By definition, this is the full sub-∞-category of $\mathbf{cdga}_k$ consisting of objects A satisfying the following conditions:

- $H^i(A) = 0$ for all $i > 0$
- $H^0(A)$ is a local Artinian k-algebra
- $H^*(A)$ is finite dimensional over k.

Note that the third condition implies in particular that $H^i(A)$ is finite dimensional for all i, but also that $H^i(A) = 0$ when $i << 0$. In a similar manner, we denote by $\mathbf{dgart}_k^*$ the ∞-catgeory of pointed objects in $\mathbf{dgart}_k$, or equivalently the ∞-category of augmented connective Artinian dg-algebras. Note that in particular, if $A \in \mathbf{dgart}_k^*$ the residue field of $H^0(A)$ is then automatically k itself.

Definition 2.3.1.1. *A* formal moduli problem *is an ∞-functor*

$$F : \mathbf{dgart}_k^* \to \mathbf{Top}$$

satisfying the following conditions.

(1) The space $F(k)$ is contractible.
(2) For any cartesian square in $\mathbf{dgart}_k^*$

$$\begin{tikzcd} B' \arrow[r] \arrow[d] & A' \arrow[d] \\ B \arrow[r] & A \end{tikzcd}$$

such that $H^0(A') \to H^0(A)$ is surjective, the square

$$\begin{tikzcd} F(B') \arrow[r] \arrow[d] & F(A') \arrow[d] \\ F(B) \arrow[r] & F(A) \end{tikzcd}$$

is cartesian in $\mathbf{Top}$*.*

Formal moduli problems over the field k form an ∞-category $\mathbf{FMP}_k$, which by definition is a full sub-∞-category of $Fun^\infty(\mathbf{dgart}_k^*, \mathbf{Top})$.

Formal moduli problems are related to pointed derived stacks by means of an adjunction, whose right adjoint is the formal completion functor. Let $\mathbf{dSt}_k^*$ be the ∞-category of pointed derived stacks. An object in $\mathbf{dSt}_k^*$ will be denoted as (X, x), where $X \in \mathbf{dSt}_k$ and $x : \mathbf{Spec}\, k \to X$ is the chosen base point. To any such object (X, x) we associated an ∞-functor

$$(X, x)^\sim : \mathbf{dgart}_k^* \to \mathbf{Top},$$

by sending $A \in \mathbf{dgart}_k^*$ to $Map_{\mathbf{dSt}_k^*}(\mathbf{Spec}\, A, (X, x))$. Here, $\mathbf{Spec}\, A$ is considered as an object in $\mathbf{dSt}_k^*$ by its augmentation $A \to k$. In general, $(X, x)^\sim$ is not a formal moduli problem (in general, it does not satisfy condition (2)). However, the natural inclusion

$$\mathbf{FMP}_k \hookrightarrow Fun^\infty(\mathbf{dgart}_k^*, \mathbf{Top})$$

is the right adjoint (see [**Lur10**, **Lur22b**, **Toë17**]) of an adjunction, and the image of $(X, x)^\sim$ by the left adjoint will be denoted by $(X, x)^\wedge \in \mathbf{FMP}_k$.

This construction defines an ∞-functor $(-)^\wedge : \mathbf{dSt}_k^* \to \mathbf{FMP}_k$ called the *formal completion* . It is easy to construction a left adjoint as follows. We start by the full embedding

$$\mathbf{Spec}\,:(\mathbf{dgart}_k^*)^{op} \hookrightarrow \mathbf{dSt}_k^*$$

and we left Kan extend it to get an ∞-functor

$$j : \mathbf{FMP}_k \hookrightarrow Fun^\infty(\mathbf{dgart}_k^*, \mathbf{Top}) \longrightarrow \mathbf{dSt}_k^*.$$

We thus have defined an adjunction of ∞-categories

$$j : \mathbf{FMP}_k \rightleftarrows \mathbf{dSt}_k^* : (-)^\wedge.$$

As a side comment, the left adjoint of the inclusion $\mathbf{FMP}_k \hookrightarrow Fun^\infty(\mathbf{dgart}_k^*, \mathbf{Top})$ is very non-explicit, and thus so is the ∞-functor $(-)^\wedge$ in general. However, this ∞-functor becomes very simple when applied to derived Artin stacks. Indeed, with the notations above, when (X,x) is a pointed derived Artin stack, the ∞-functor $(X,x)^\sim : \mathbf{dgart}_k^* \to \mathbf{Top}$ is in fact already a formal moduli problem, as this follows easily from the fact that derived Artin stack possesses an obstruction theory (see for instance [**TV08**, §1.4]). Therefore, $(X,x)^\sim \simeq (X,x)^\wedge$, and we thus have for any $A \in \mathbf{dgart}_k^*$, a cartesian square in $\mathbf{Top}$

$$\begin{tikzcd} (X,x)^\wedge(A) \arrow[r] \arrow[d] & X(A) \arrow[d] \\ \bullet \arrow[r, "x"'] & X(k). \end{tikzcd}$$

We can now combine the previous adjunction with the main equivalence $\mathbf{FMP}_k \simeq \mathbf{dgLie}_k$ of [**Lur10**] in order to obtain an adjunction between $\mathbf{dSt}_k^*$ and the ∞-category of dg-lie algebras. We thus have constructed an adjunction[1]

$$\mathcal{MC} : \mathbf{dgLie}_k \leftrightarrows \mathbf{dSt}_k^* : \ell_-.$$

The ∞-functor ℓ_- sends a pointed derived stack (X,x) to the dg-lie algebra $\ell_{(X,x)}$ controlling the formal moduli problem $(X,x)^\wedge$, and $\mathcal{MC}$ is its left adjoint. We have then the following important observation.

Propositon 2.3.1.2. *The ∞-functor*

$$\mathcal{MC} : \mathbf{dgLie}_k \longrightarrow \mathbf{dSt}_k^*$$

is fully faithful.

Proof. As a first reduction, we can start by noticing that $k \mapsto \mathbf{dgLie}_k$ and $k \mapsto \mathbf{dSt}_k^*$ are both stacks for the fpqc topology on k. Therefore, the ∞-functor $\mathcal{MC}$ being compatible with the change of base field k, we can assume that k is algebraically closed.

By construction, the ∞-functor $\mathcal{MC}$ is equivalent to the composition

$$j : \ \mathbf{FMP}_k \longrightarrow Fun^\infty(\mathbf{dgart}_k^*, \mathbf{Top}) \xrightarrow{\phi} \mathbf{dSt}_k^*.$$

[1]The functor $\mathcal{MC}$ is so named after L. Maurer and E. Cartan.

Here, the first ∞-functor is the canonical embedding and the second ∞-functor

$$\phi : Fun^{\infty}(\mathbf{dgart}_k^*, \mathbf{Top}) \to \mathbf{dSt}_k^*$$

is the composition of the left Kan extension

$$i_! : Fun^{\infty}(\mathbf{dgart}_k^*, \mathbf{Top}) \hookrightarrow */Fun^{\infty}(\mathbf{cdga}_k^{\leq 0}, \mathbf{Top})$$

along the inclusion $i : \mathbf{dgart}_k \hookrightarrow \mathbf{cdga}_k^{\leq 0}$, with the associated stack ∞-functor

$$*/Fun^{\infty}(\mathbf{cdga}_k^{\leq 0}, \mathbf{Top}) \to \mathbf{dSt}_k^*.$$

It is therefore enough to show that ϕ is fully faithful. The right adjoint to ϕ is the restriction ψ sending a pointed stack $F : \mathbf{cdga}_k^{\leq 0} \to \mathbf{Top}$ to its restriction to Artinian dg-algebras. However, as k is algebraically closed, any Artinian local dg-algebra is trivial for the étale topology, and thus for any prestack F, with associated stack F', the natural map $F \to F'$ induces equivalences $F(A) \simeq F'(A)$ for any Artinian local k-algebra A. As a consequence, for any $F \in Fun^{\infty}(\mathbf{dgart}_k^*, \mathbf{Top})$, the adjunction morphism $F \to \psi\phi(F)$, is simply equivalent to the adjunction morphims $F \to i^* i_!(F)$, which is an equivalence as the left Kan extension

$$i_! : Fun^{\infty}(\mathbf{dgart}_k^*, \mathbf{Top}) \hookrightarrow */Fun^{\infty}(\mathbf{cdga}_k^{\leq 0}, \mathbf{Top})$$

is fully faithful. □

We will mainly use the previous adjunction when restricted to derived Artin stacks. We note that for such a pointed derived Artin stack (X, x), the underlying complex of the dg-lie algebra $\ell_{(X,x)}$ is functorialy quasi-isomorphic to $\mathbb{T}_{X,x}[-1]$, the shifted tangent complex of X at x. This follows directly from a simple unfolding of the various definitions.

Finally, because of Proposition 2.3.1.2 we will often consider formal moduli problems as fully embedded in pointed derived stacks. In particular, we will often use the notation $(X, x)^\wedge$ to denote also the pointed derived stack $\mathcal{MC}((X, x)^\wedge)$ and call it the *formal completion of* X *at* x. This formal completion $(X, x)^\wedge$ can be in fact described in the usual manner on smooth atlases of X. Assuming first that $X = \mathbf{Spec}\, A$ is a pointed affine derived scheme, the point provides an augmentation $e : A \to k$, and we can define a pro-object in $\mathbf{dgart}_k^*$, called the formal completion of A along e, by

$$\widehat{A} := \text{''} \lim_{A \to A'} \text{''} A',$$

where A' runs in the ∞-category of morphism of augmented cdga morphism $A \to A'$, where A' is a local Artinian cdga. When A is an underived k-algebra of finite type, then it can be shown that this pro-object is equivalent to the usual formal completion $\text{''} \lim_n \text{''} A/m^n$ where $m \subset A$ is the maximal ideal corresponding to the augmentation e (see Proposition A.3.0.2). The pro-object $\widehat{A}$ pro-represents $(X, x)^\wedge$ is the sense that one has

$$(X, x)^\wedge \simeq \mathsf{Spf}(\underline{\widehat{A}}) = \operatorname{colim} \mathbf{Spec}\, A'.$$

In general, up to base changing to a finite extension k'/k, we can find a smooth morphism $p : U \to X$ of pointed derived stacks for which U is affine and such that x lies in the image of p. We can form the nerve $U_* \to X$ of p. We then have

$$(X,x)^\wedge \simeq \mathrm{colim}_n (U_n, x)^\wedge.$$

We can this way compute the formal completion of $(X,x)^\wedge$ inductively on the geometricity of X. We could also replace the nerve of p here by any pointed smooth hypercovering (locally around x) $U_* \to X$ where all U_i are affine, and get this way a formula for $(X,x)^\wedge$ as the colimit of formal completions $(U_n, x)^\wedge$ which can individually be described as the formal spectrum of a pro-Artinian cdga.

2.3.2. Derived foliations on formal moduli problems. In this part we show how derived foliations can be defined over formal moduli problems . For this we simply use the fully faithful ∞-functor defined in the previous section $j : \mathbf{FMP}_k \longrightarrow \mathbf{dSt}_k^*$.

Definition 2.3.2.1. *For a formal moduli problem $X \in \mathbf{FMP}_k$ we set*

$$\mathcal{F}ol(X/k) := \mathcal{F}ol(j(X)/k).$$

When the base field is clear we will simply use the notation $\mathcal{F}ol(X)$.

Note that $X \mapsto \mathcal{F}ol(X)$ naturally defines an ∞-functor $\mathcal{F}ol^{op} : \mathbf{FMP}_k \to \mathbf{Cat}_\infty$. Unfolding definitions and constructions we can give the following explicit formula

$$\mathcal{F}ol(X) \simeq \lim_{\mathbf{Spec}\, A \to X} \mathcal{F}ol(A),$$

where $\mathbf{Spec}\, A$ runs over the ∞-category of $A \in \mathbf{dgart}_k^*$ endowed with a morphism $\mathbf{Spec}\, A \to X$ (concretely the comma ∞-category $\mathbf{Spec}\,/X$ where $\mathbf{Spec} : \mathbf{dgart}_k^{op} \to \mathbf{FMP}_k$ is the Yoneda embedding).

Note that, for any pointed derived stack $(X,x) \in \mathbf{dSt}_k^*$, there exists a canonical ∞-functor

$$\mathcal{F}ol(X) \longrightarrow \mathcal{F}ol((X,x)^\wedge).$$

Indeed, by construction of the adjunction between j and $(-)^\wedge$, we have a natural adjunction morphism of derived stack $i_x : (X,x)^\wedge \to X$, and thus an induced pull-back ∞-functor $i_x^* : \mathcal{F}ol(X) \to \mathcal{F}ol((X,x)^\wedge)$.

Definition 2.3.2.2. *For a pointed derived stack (X,x) the ∞-functor $i_x^* : \mathcal{F}ol(X) \to \mathcal{F}ol((X,x)^\wedge)$ above is the* formalization at x, *or* formal completion at x. *It is denoted by $\mathcal{F} \mapsto \hat{\mathcal{F}}_x$.*

Proposition 2.3.2.3 below shows that derived foliations which are *almost perfect*, can be described explicitly on the formal completion of an affine derived scheme. Before explaining this, let A be a connective cdga with an augmentation $A \to k$, and assume that A is almost of finite presentation (see appendix §A.2). Wet set $X = \mathbf{Spec}\, A$ endowed with its natural point $x \in X(k)$ associated to the augmentation. We denote by $\underline{\hat{A}}_x$ (or simply by $\underline{\hat{A}}$) the formal completion of A at the augmentation $x : A \to k$. Recall (see appendix §A.3) that this is the pro-object in $\mathbf{dgart}_k^*$ given by " $\lim_{A \to A'} A'$, where A' runs in the ∞-category of all local Artinian cdga with an augmentation preserving morphism $A \to A'$. We let $X = \mathbf{Spec}\, A$ and $(X,x)^\wedge := \mathsf{Spf}(\underline{\hat{A}})$ be the corresponding derived affine schemes and formal completion.

We define the graded mixed cdga $\hat{\mathbf{DR}}(\underline{\hat{A}}/k)$, $\hat{\mathbf{DR}}(\underline{\hat{A}}/k)$ called the *formal de Rham algebra of* $\underline{\hat{A}}$, by the following limit

$$\hat{\mathbf{DR}}(\underline{\hat{A}}/k) := \lim_{A \to A'} \mathbf{DR}(A'/k).$$

By Corollary A.3.0.5 we know that, as graded cdga's we have

$$\hat{\mathbf{DR}}(\underline{\hat{A}}/k) \simeq Sym_{\hat{A}}(\hat{\mathbb{L}}_{A/k}[1]),$$

where $\hat{A} = \lim_{A \to A'} A'$ is the realization of the pro-object $\underline{\hat{A}}$, and $\hat{\mathbb{L}}_{A/k} = \lim_{A \to A'} \mathbb{L}_{A'/k}$ is the formal cotangent complex of $\underline{\hat{A}}$. Now, the limit ∞-functor provides a natural ∞-functor

$$\psi : \mathcal{F}ol((X,x)^\wedge) = \lim_{A \to A'} \mathcal{F}ol(Spec\, A') \longrightarrow (\hat{\mathbf{DR}}(\hat{A}/k) - \epsilon - \mathbf{cdga}_k^{gr})^{op},$$

from derived foliations on the formal completion of X at x to graded mixed commutative $\hat{\mathbf{DR}}(\hat{A}/k)$-algebras. Explicitly, the ∞-functor ψ sends an object $\{\mathcal{F}'_A\} \in \lim_{A \to A'} \mathcal{F}ol(Spec\, A')$ to $\lim_{A \to A'} \mathbf{DR}(\mathcal{F}'_A)$, as an algebra over $\lim_{A \to A'} \mathbf{DR}(A'/k)$.

Propositon 2.3.2.3. *Let $(X,x) = \mathbf{Spec}\, A$ be a pointed derived affine scheme almost of finite presentation over k. The ∞-functor*

$$\psi : \mathcal{F}ol((X,x)^\wedge) = \lim_{A \to A'} \mathcal{F}ol(Spec\, A') \longrightarrow (\hat{\mathbf{DR}}(\hat{A}/k) - \epsilon - \mathbf{cdga}_k^{gr})^{op}$$

constructed above, is fully faithful when restricted to almost perfect derived foliations. Its essential image consists of all graded mixed commutative $\hat{\mathbf{DR}}(\hat{A}/k)$-algebras whose underlying graded cdga'a are of the form $Sym_{\hat{A}}(E)$, with E an almost perfect $\hat{A}$-dg-module.

Proof. The limit ∞-functor

$$\lim_{A \to A'} (\mathbf{DR}(A'/k) - \epsilon - \mathbf{cdga}_k^{gr}) \longrightarrow \hat{\mathbf{DR}}(\hat{A}/k) - \epsilon - \mathbf{cdga}_k^{gr}$$

admits a left adjoint ϕ, which sends a graded mixed commutative $\hat{\mathbf{DR}}(\hat{A}/k)$-algbera B to the family of objects $\hat{\mathbf{DR}}(A'/k) \otimes_{\hat{\mathbf{DR}}(\hat{A}/k)} B$, together with their natural transition morphisms.

We denote by $\mathcal{C} \subset \hat{\mathbf{DR}}(\hat{A}/k) - \epsilon - \mathbf{cdga}^{gr}$ be the full sub-∞-category consisting of B which, as graded $\hat{\mathbf{DR}}(\hat{A}/k)$-cdga's, are of the form $Sym_{\hat{A}}(E)$ for some $E \in \mathbf{APerf}(\hat{A})$. We first observe that the essential image of ϕ is contained in $\mathcal{C}$. Let $\mathcal{F} \in \lim_{A \to A'} \mathcal{F}ol^{ap} Spec\, A')$ be a derived foliation on $(X,x)^\wedge$ which is assumed to be almost perfect It consists of a family of $\mathbf{DR}(\mathcal{F}_{A'})$ of graded mixed $\mathbf{DR}(A'/k)$-cdga, which, as graded $\mathbf{DR}(A'/k)$-algebras, are of the form $Sym_{A'}(\mathbb{L}_{\mathcal{F}_{A'}}[1])$ for an almost perfect A'-dg-module $\mathbb{L}_{\mathcal{F}_{A'}} \in \mathbf{APerf}(A')$. We set $\mathbb{L}_{\mathcal{F}} :=$

$\lim_{A\to A'} \mathbb{L}_{\mathcal{F}'_A}$, which is an $\hat{A}$-module, and let $\mathcal{N}^*_{\mathcal{F}}$ be the fiber of the natural morphism $\hat{\mathbb{L}}_A \otimes_{\hat{A}} A' \to \mathbb{L}_{\mathcal{F}}$.

We consider the natural commutative diagram of exact triangles of pro-$\underline{\hat{A}}$-dg-modules

$$\begin{tikzcd}
``\lim_{A\to A'}"(\hat{\mathbb{L}}_A \otimes_{\hat{A}} A') \arrow[r] \arrow[d] & ``\lim_{A\to A'}"(\mathbb{L}_{\mathcal{F}} \otimes_A A') \arrow[r] \arrow[d] & ``\lim_{A\to A'}"(\mathcal{N}^*_{\mathcal{F}} \otimes_A A')[1] \arrow[d] \\
``\lim_{A\to A'}"\mathbb{L}_{A'} \arrow[r] & ``\lim_{A\to A'}"\mathbb{L}_{\mathcal{F}_{A'}} \arrow[r] & ``\lim_{A\to A'}"\mathcal{N}^*_{\mathcal{F}_{A'}}[1].
\end{tikzcd}$$

The vertical left arrow is an equivalence of pro-objects thanks to theorem A.3.0.5. The right vertical arrow is also an equivalence (and even a levelwise equivalence) thanks to the fact that the natural ∞-functor

$$\lim_{A\to A'} \mathbf{APerf}(A') \longrightarrow \mathbf{APerf}(\hat{A})$$

is an equivalence of ∞-categories (see appendix A.4.0.1) and its inverse is given by $E \mapsto \lim_{A\to A'} E \otimes_{\hat{A}} A'$. Therefore, the natural morphism $``\lim_{A\to A'}"(\mathbb{L}_{\mathcal{F}} \otimes_{\hat{A}} A') \to ``\lim_{A\to A'}"\mathbb{L}_{\mathcal{F}_{A'}}$ is again an equivalence of pro-$\underline{\hat{A}}$-dg-modules. As a consequence, we see that for all p,

$$``\lim_{A\to A'}"(\wedge^p_{\hat{A}}(\mathbb{L}_{\mathcal{F}}) \otimes_{\hat{A}} A') \to ``\lim_{A\to A'}" \wedge^p_{A'} \mathbb{L}_{\mathcal{F}_{A'}}$$

is an equivalence of pro-objects. This implies in particular that $\psi(\mathcal{F})$ lies indeed in the sub-∞-category $\mathcal{C}$.

We thus have seen that the adjunction

$$\phi: \ \hat{\mathbf{DR}}(\hat{A}/k) - \epsilon - \mathbf{cdga}^{gr}_k \rightleftarrows \lim_{A\to A'} \mathbf{DR}(A'/k) - \epsilon - \mathbf{cdga}^{gr}_k \ : \psi$$

restricts to an adjunction

$$\phi: \ \mathcal{C} \rightleftarrows \lim_{A\to A'}(\mathcal{F}ol^{ap}(\mathbf{Spec}\, A'/k)) \ : \psi.$$

To finish the proof of the proposition we need to check that the unit and counit of this adjunction are equivalence. Let $B \in \mathcal{C}$, and $\mathbb{L} := B^{(1)}[1]$ be its weight one part. It is an object in $\mathbf{APerf}(\hat{A})$ by assumption, and thus if we denote by $\mathcal{N}^*_{\mathcal{F}}$ the fiber of $\widehat{\mathbb{L}_A} \to \mathbb{L}$, we have that $\mathcal{N}^*_{\mathcal{F}}$ is an almost perfect $\hat{A}$-dg-module. The adjunction morphism $B\psi(\phi(B))$, on the weight one part, is equivalent to the canonical morphism $\mathbb{L} \to \lim_{A\to A'} \mathbb{L} \otimes_{\hat{A}} A'$. This is an equivalence thanks to proposition A.4.0.1. This shows that ϕ is fully faithful. But ψ is also conservative. Indeed, if a morphism $f : \mathcal{F} \to \mathcal{F}'$ in $\lim_{A\to A'} \mathcal{F}ol^{ap}(Spec\, A')$ induces an equivalence after applying ϕ, then in particular the induced morphism on conormal complexes $\lim_{A\to A'} \mathcal{N}^*_{\mathcal{F}'_{A'}} \to \lim_{A\to A'} \mathcal{N}^*_{\mathcal{F}_{A'}}$ is an equivalence. The limit ∞-functor $\lim_{A\to A'} \mathbf{APerf}(A') \longrightarrow \mathbf{APerf}(\hat{A})$ is an equivalence, and is thus conservative, we have therefore that for each A' the induced morphism $\mathcal{N}^*_{\mathcal{F}'_{A'}} \to \mathcal{N}^*_{\mathcal{F}_{A'}}$ is an equivalence, and thus that $\mathcal{F}_{A'} \to \mathcal{F}'_{A'}$ is an equivalence of derived foliations on $\mathbf{Spec}\, A'$. This shows that ψ is conservative and thus the proof of the Proposition. $\square$

Finally, we have the formal version of Theorem 2.1.3.8. For this, recall that X^{art} is the ∞-site of pointed morphisms $\mathbf{Spec}\, A \to X$ with A a local Artinian connective cdga (see §A.6). It comes equipped with the canonical stack of cdga's $\mathcal{O}_{X^{art}}$, as well as its ∞-category of $\mathcal{O}_{X^{art}}$-dg-modules. We have the full sub-∞-category $\mathbf{QCoh}(X^{art}) \subset \mathcal{O}_{X^{art}} - \mathbf{dg}$ of quasi-coherent complexes, which can canonically be identified with $\mathbf{QCoh}((X,x)^\wedge)$. Any derived foliation $\mathcal{F} \in \mathcal{F}ol((X,x)^\wedge)$ possesses a big cotangent complex $\mathbb{L}^{big}_{\mathcal{F}/k} \in \mathcal{O}_{X^{art}} - \mathbf{dg}$, and a cotangent complex $\mathbb{L}_{F/k} \in \mathbf{QCoh}(X^{art})$. By definition, they are related by the formula $\mathbb{L}_{\mathcal{F}/k} = (\mathbb{L}^{big}_{\mathcal{F}/k})^{qcoh}$, where $(-)^{qcoh} : \mathcal{O}_{X^{art}} - \mathbf{dg} \longrightarrow \mathbf{QCoh}(X^{art})$ is the right adjoint to the natural inclusion ∞-functor.

Propositon 2.3.2.4. *Let (X,x) be a pointed derived stack locally almost of finite presentation and $(X,x)^\wedge$ its formal completion. Then, for any almost perfect derived foliation $\mathcal{F} \in \mathcal{F}ol^{ap}((X,x)^\wedge)$ such that $\mathcal{N}^*_{\mathcal{F}}$ is perfect, and all $i \geq 0$ the canonical morphism*

$$\wedge^i_{\mathcal{O}_{X^{art}}} \mathbb{L}_{\mathcal{F}} \longrightarrow (\wedge^i_{\mathcal{O}_{X^{art}}} \mathbb{L}^{big}_{\mathcal{F}})^{qcoh}$$

is a quasi-isomorphism.

Proof. This is exactly the same proof as that of Theorem 2.1.3.8, using Theorem A.6.0.1 instead of Theorem A.5.0.1. We leave the details to the reader. □

2.3.3. Formal integrability. We will now show that almost perfect derived foliations on the formal completion of a derived Artin stack locally almost of finite presentation can be described purely in algebraic terms, using dg-Lie algebras. This classification result also implies that these derived foliations are always formally integrable at each point.

Recall that the conormal complex of a derived foliation $\mathcal{F}$ is defined by the exact triangle

$$\mathcal{N}^*_{\mathcal{F}} \longrightarrow \mathbb{L}_X \longrightarrow \mathbb{L}_{\mathcal{F}}.$$

An important remark is that $\mathcal{N}^*_{\mathcal{F}}$ can be defined using either the cotangent quasi-coherent complexes or the big cotangent complexes. Indeed, the commutative square

$$\begin{tikzcd} \mathbb{L}_X \arrow[r] \arrow[d] & \mathbb{L}_{\mathcal{F}} \arrow[d] \\ \mathbb{L}^{big}_X \arrow[r] & \mathbb{L}^{big}_{\mathcal{F}} \end{tikzcd}$$

is always cartesian (see Lemma 2.1.3.9) so that the fibers of the horizontal arrows agree. As a result, $\mathcal{N}^*_{\mathcal{F}}$ always makes sense as an object in $\mathbf{QCoh}(X)$, even if X is merely a derived stack (i.e. not necessarily Artin). Another important fact is that conormal complexes of derived foliations are stable by pull-backs (as opposed to cotangent complexes themselves), and therefore almost perfect derived foliations form a full substack $\mathcal{F}ol^{ap} \subset \mathcal{F}ol$ of the stack of derived foliations. In particular, for any formal moduli problem $X \in \mathbf{FMP}$, we get a full sub-∞-category

$\mathcal{F}ol^{ap}(X) \subset \mathcal{F}ol(X)$, simply defined by $\mathcal{F}ol^{ap}(j(X))$ (Definition 2.3.2.1).

Let now (X, x) be a pointed derived Artin stack locally almost of finite presentation and $\mathcal{F} \in \mathcal{F}ol((X, x)^\wedge)$ a derived foliation on the formal completion of X at x. The natural base point $x : \mathbf{Spec}\, k \to (X, x)^\wedge$ provides $x^*(\mathcal{F}) \in \mathcal{F}ol(*)$. Moreover, the tautological derived foliation 0_X can be formally completed to $\hat{0}_{X_x} \in \mathcal{F}ol((X, x)^\wedge$.

LEMMA 2.3.3.1. *The object $\hat{0}_{X_x}$ is the initial object in $\mathcal{F}ol((X, x)^\wedge$.*

Proof. The formal completion construction $\mathcal{F} \mapsto \hat{\mathcal{F}}_x$ (Definition 2.3.2.2) being obtained by restriction, the conormal complex $\mathcal{N}_{\hat{F}_x}$ is simply the restriction of $\mathcal{N}_{\mathcal{F}}$. Applied to the case of $\mathcal{F} = 0_X$, we see that the conormal complex of $\hat{0}_{X_x}$ is identified with $\mathbb{L}_{(X,x)^\wedge}$, via the canonical morphism. Equivalently, $\mathbb{L}_{\hat{0}_{X,x}} \simeq 0$. We then finish the proof using the exact same argument as in the non-formal case (see example 2.1.5.1, by replacing the big site $\mathbf{dAff}/X$ with $\mathbf{dgart}^*_k/X$, and using Proposition 2.1.4.6 (and Proposition 2.3.2.4 instead of Theorem 2.1.3.8). $\square$

Lemma 2.3.3.1 provides a canonical morphism $x^*(0_X) \to x^*(\mathcal{F})$, of derived foliations over $\mathbf{Spec}\, k$. This defines an ∞-functor $\psi : \mathcal{F}ol((X, x)^\wedge) \to x^*(0_X)/\mathcal{F}ol(*)$, which can be restricted to perfect derived foliations

$$\psi : \mathcal{F}ol^{ap}((X, x)^\wedge) \to x^*(0_X)/\mathcal{F}ol^{ap}(*).$$

THEOREM 2.3.3.2. *With the above notations and assumptions, the ∞-functor*

$$\psi : \mathcal{F}ol^{ap}((X, x)^\wedge) \to x^*(0_X)/\mathcal{F}ol^{ap}(*)$$

is an equivalence of ∞-categories.

Before proving this theorem, we state its most important corollary. The target ∞-category of the equivalence of Theorem 2.3.3.2 can thus be identified, according to corollary 2.2.0.8, with the comma ∞-category of dual almost perfect dg-Lie algebras $t_X/\mathbf{dgLie}^{dap}_k$. Here t_X is the dg-Lie algebra corresponding to $x^*(0_X)$ according to the equivalence of 2.2.0.8. As this equivalence is functorial, it is easy to check that $x^*(0_X)$ is the dg-Lie algebra obtained by pulling back the dg-Lie algebroid $\mathbb{T}_X$ to the point x, which is nothing else that $\ell_{(X,x)}$, corresponding to the formal moduli problem $(X, x)^\wedge$. The main corollary of Theorem 2.3.3.2 thus reads as follows.

COROLLARY 2.3.3.3. *Let (X, x) be a pointed derived Artin stack locally almost of finite presentation. Then, the ∞-functor ψ combined with the equivalence of Corollary 2.2.0.8 induces an equivalence of ∞-categories*

$$\mathcal{F}ol^{ap}((X, x)^\wedge) \simeq \ell_{(X,x)}/\mathbf{dgLie}^{dap}_k,$$

between almost perfect derived foliations on the formal completion $(X, x)^\wedge$ and the comma ∞-category of dual almost perfect dg-Lie algebras under $\ell_{(X,x)}$. This equivalence will be denoted by

$$\mathcal{F} \mapsto \ell_{\mathcal{F}}.$$

The above corollary states formal integrability of derived foliations. Indeed, for any $\mathcal{F} \in \mathcal{F}ol^{ap}(X)$, its formal completion at a k-point $x \in X(k)$ provides by the equivalence of the corollary, a morphism of dg-Lie algebras $\ell_{(X,x)} \to \ell_{\hat{\mathcal{F}}_x}$. By applying the ∞-functor $\mathcal{MC}$: $\mathbf{dgLie}_k \to \mathbf{dSt}_k^*$ we get a morphism

$$u : \mathcal{MC}(\ell_{(X,x)}) \simeq (X,x)^\wedge \to \mathcal{MC}(\ell_{\hat{\mathcal{F}}_x}).$$

The pointed derived stack $\mathcal{MC}(\ell_{\hat{\mathcal{F}}_x})$ is, by definition, the *formal leaf space* of $\mathcal{F}$ at x. It is possible to show that the morphism u integrates $\hat{\mathcal{F}}_x$ in the sense that $\hat{F}_x \simeq u^*(0)$, but we will not do this here. We will however prove this in the next section, in the specific case where X is a smooth variety and $\mathcal{F}$ is rigid and quasi-smooth (see §2.3.4). For now, we gather some terminology related to Corollary 2.3.3.3.

Definition 2.3.3.4. *Let (X,x) be a pointed derived Artin stack locally almost of finite presentation and $\mathcal{F} \in \mathcal{F}ol^{ap}(X)$ be an almost perfect derived foliation.*

(1) The formal leaf space of $\mathcal{F}$ at x *is the formal moduli problem corresponding to the dg-Lie algebra $\ell_{\hat{\mathcal{F}}_x}$. It is denote by $\hat{\mathcal{M}}_x(\mathcal{F})$.*
(2) The formal leaf of $\mathcal{F}$ passing through x *is the formal moduli problem corresponding to the dg-Lie algebra which is the fiber of the canonical morphism $\ell_{(X,x)} \to \ell_{\hat{\mathcal{F}}_x}$. It is denoted by $\hat{\mathcal{L}}_x(\mathcal{F})$.*

We finish this section by proving Theorem 2.3.3.2.

Proof of Theorem 2.3.3.2. As a first reduction, by a standard Galois descent argument, we may assume that k is algebraically closed. We then proceed by induction on the geometricity of X. To start with, assume that X is affine, and thus of the form $X = \mathbf{Spec}\, A$ for A a connective cdga almost of finite presentation over k. We let $\underline{\hat{A}}$ be the formal completion of A at the point $x : \mathbf{Spec}\, A \to k$, and consider $\hat{\mathbf{DR}}(\hat{A}/k)$ the associated formal de Rham algebra. By Proposition 2.3.2.3 $\mathcal{F}ol^{ap}((X,x)^\wedge)$ is naturally equivalent to the (opposite) ∞-category of graded mixed $\hat{\mathbf{DR}}(\hat{A}/k)$-cdga's which are equivalent, as graded $\hat{\mathbf{DR}}(\hat{A}/k)$-cdga, to $Sym_{\hat{A}}(E)$ for E an almost perfect $\hat{A}$-dg-module. Tracing back the various equivalences of ∞-categories, we obtain a canonical equivalence of graded mixed cdga's

$$\mathbf{DR}(x^*(0_X)) \simeq \hat{A} \otimes_{\hat{\mathbf{DR}}(\hat{A}/k)} k,$$

and the ∞-functor of the theorem ψ is then induced by

$$\psi : \hat{\mathbf{DR}}(\hat{A}/k) - \epsilon - \mathbf{cdga}^{gr} \longrightarrow \epsilon - \mathbf{cdga}^{gr}/(A \otimes_{\hat{\mathbf{DR}}(\hat{A}/k)} k)$$

which sends $\hat{\mathbf{DR}}(\hat{A}/k) \to B$ to $B \otimes_{\hat{\mathbf{DR}}(\hat{A}/k)} k$ together with its natural augmentation to $\hat{A} \otimes_{\hat{\mathbf{DR}}(\hat{A}/k)} k$ induced by the natural morphism $B \to \hat{A}$. The ∞-functor ψ is thus subsumed by the following commutative diagrams of graded mixed cdga's with cocartesian squares

$$\begin{tikzcd} \hat{\mathbf{DR}}(\hat{A}/k) \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & \hat{A} \arrow[d] \\ k \arrow[r] & \psi(B) \arrow[r] & \hat{A} \otimes_{\hat{\mathbf{DR}}(\hat{A}/k)} k. \end{tikzcd}$$

We construct an ∞-functor ϕ in the other direction by using de Rham cohomology of the first kind explained in Section §A.8. For this, we first prove the following lemma.

LEMMA 2.3.3.5. *The canonical morphism $k \to \hat{\mathbf{DR}}(\hat{A}/k)$ induces an equivalence after taking realizations*

$$k \simeq |\hat{\mathbf{DR}}(\hat{A}/k)|.$$

Proof of the lemma. By definition $|\hat{\mathbf{DR}}(\hat{A}/k)|$ is the derived de Rham cohomology of the formal completion of $\mathbf{Spec}\, A$ at x, and the lemma simply states that derived de Rham cohomology is trivial on formal completion. By definition $\hat{\mathbf{DR}}(\hat{A}/k) \simeq \lim_{A \to A'} \mathbf{DR}(A'/k)$. As $|-|$ is a right adjoint it commutes with limits, and thus we are reduced to show that, for any local augmented dg-Artinian algebra A, the natural morphism $k \to \widehat{C}_{DR}(A/k)$ is a quasi-isomorphism. The lemma then follows from proposition A.8.0.3. $\square$

LEMMA 2.3.3.6. *There exists a canonical equivalence of graded mixed cdga's*

$$\hat{\mathbf{DR}}(\hat{A}/k) \simeq \widehat{C}^I_{DR}(\hat{A} \otimes_{\hat{\mathbf{DR}}(\hat{A}/k)} k/k).$$

Proof of the lemma. Let $D := \hat{\mathbf{DR}}(\hat{A}/k)$. By base change, we have a canonical equivalence of double graded mixed cdga's

$$\mathbf{DR}^{int}(\hat{A}/D) \otimes_D k \simeq \mathbf{DR}^{int}(\hat{A} \otimes_D k/k).$$

We therefore have a natural morphism of double graded mixed cdga's

$$\mathbf{DR}^{int}(\hat{A}/D) \to \mathbf{DR}^{int}(\hat{A}/D) \otimes_D k \simeq \mathbf{DR}^{int}(\hat{A} \otimes_D k/k),$$

where D is endowed with the trivial mixed structure as its second graded mixed structure. By applying the first realization functor we get a morphism of graded mixed cdga's

$$\widehat{C}^I_{DR}(\hat{A}/D) \to \widehat{C}^I_{DR}(\hat{A}/D) \widehat{\otimes}_{|D|_1} k \simeq \widehat{C}^I_{DR}(\hat{A} \otimes_D k/k),$$

where $\widehat{\otimes}$ is the completed tensor product of complete filtered complexes (see §1.3.4). Now, Lemma 2.3.3.5 precisely states that $|D|_1 \simeq k$, and thus the above morphism turns out to be an equivalence of graded mixed cdga's

$$\widehat{C}^I_{DR}(\hat{A}/D) \simeq \widehat{C}^I_{DR}(\hat{A} \otimes_D k/k).$$

To finish the proof of the lemma we apply proposition A.8.0.5 of Appendix A.8 showing the existence of a canonical equivalence of graded mixed cdga's $D \simeq \widehat{C}^I_{DR}(\hat{A}/D)$. $\square$

Using Lemma 2.3.3.6, we can define an ∞-functor

$$\phi : \epsilon - \mathbf{cdga}^{gr}/(\hat{A} \otimes_{\hat{\mathbf{DR}}(\hat{A}/k)} k) \longrightarrow \hat{\mathbf{DR}}(\hat{A}/k) - \epsilon - \mathbf{cdga}^{gr}$$

by sending an object $B' \to (\hat{A} \otimes_{\hat{\mathbf{DR}}(\hat{A}/k)} k)$ to the induced morphism in de Rham cohomology of the first kind

$$\widehat{C}^{I}_{DR}(\hat{A} \otimes_{\hat{\mathbf{DR}}(\hat{A}/k)} k/k) \longrightarrow \widehat{C}^{I}_{DR}(\hat{A} \otimes_{\hat{\mathbf{DR}}(\hat{A}/k)} k/B').$$

We claim that ψ and ϕ are inverse to each others when restricted to the appropriate full sub-∞-categories.

We start by showing that $\phi \circ \psi$ is equivalent to the identity. Let $\hat{\mathbf{DR}}(\hat{A}/k) \to B$ be an object in $\hat{\mathbf{DR}}(\hat{A}/k) - \epsilon - \mathbf{cdga}^{gr}$ corresponding to an almost perfect derived foliation on $(X,x)^{\wedge}$. By definition, the underlying graded cdga of B is of the form $Sym_{\hat{A}}(\mathbb{L}[1])$, for an almost perfect $\hat{A}$-dg-module $\mathbb{L}$. The object $\phi \circ \psi(B)$ is $\widehat{C}^{I}_{DR}(\hat{A} \otimes_D k/B \otimes_D k)$, where we keep using the notation $D := \hat{\mathbf{DR}}(\hat{A}/k)$. We have to produce an equivalence of graded mixed cdga's $u : B \simeq \widehat{C}^{I}_{DR}(\hat{A} \otimes_D k/B \otimes_D k)$, functorial in B. For this, we consider the commutative diagram with push-out squares in graded mixed cdga's

$$\begin{tikzcd}
\hat{\mathbf{DR}}(\hat{A}/k) \arrow[r] & \mathbf{DR}^{int}(\hat{A}/B) \arrow[r] & \mathbf{DR}^{int}(\hat{A} \otimes_D k/B \otimes_D k) \\
\hat{\mathbf{DR}}^{int}(B/k) \arrow[r] \arrow[u] & B \arrow[r] \arrow[u] & k \arrow[u]
\end{tikzcd}$$

where $\hat{\mathbf{DR}}^{int}(B/k)$ is by definition $\lim_{A \to A'} \mathbf{DR}^{int}(B \otimes_D \mathbf{DR}(A'/k)/k)$. By passing to the realizations along the first mixed structure, and applying proposition A.8.0.5, we obtain a natural morphism

$$B \simeq \widehat{C}^{I}_{DR}(\hat{A}/B) \to \widehat{C}^{I}_{DR}(\hat{A} \otimes_D k/B \otimes_D k). \tag{9}$$

It is then easy to check that this morphism is an equivalence, simply by considering the induced morphism on each graded pieces for the natural (complete) filtrations. Assume that B is of the form $Sym_{\hat{A}}(E)$ as a graded cdga, then in weight 1, the above morphism is the natural projection

$$E \longrightarrow E \otimes_{\hat{A}} |\hat{A} \otimes_D k|.$$

But, by considering the lemma 2.3.3.6 for the weight 0 pieces, we have $|\hat{A} \otimes_D k| \simeq \hat{A}$, and thus this morphism is an equivalence of complexes. The same argument shows that (9) is an equivalence also in arbitrary weights $i > 0$, being the canonical morphism

$$Sym^{i}_{\hat{A}}(E) \longrightarrow Sym^{i}_{\hat{A}}(E) \otimes_{\hat{A}} |\hat{A} \otimes_D k|.$$

In the other direction, to see that $\psi \circ \phi \simeq id$, we simply observe that, for any diagram of graded mixed cdga's $k \to B' \to \hat{A} \otimes_D k$, with B' of the form $Sym_k(L[1])$ as a graded cdga, the following square

$$\begin{tikzcd} D \arrow[r] \arrow[d] & \widehat{C}^I_{DR}(\hat{A} \otimes_D k/B') \arrow[d] \\ k \arrow[r] & B' \end{tikzcd}$$

is a push-out diagram of graded cdga's: this can be easily checked by considering the various cotangent complexes.

We have therefore finished the proof of the theorem for X *affine*, and it remains to extend the statement to the case of an arbitrary derived Artin stack X. For this we proceed on induction on the geometricity of X, as we have already seen that the theorem is true when X is affine, or more generally a disjoint union of affines. We fix a smooth atlas $U \to X$, where U is a disjoint union of of affine derived schemes locally of finite presentation. As k is algebraically closed we can lift the point $x : \mathbf{Spec}\, k \to X$ to a point $x_0 : \mathbf{Spec}\, k \to U$. We consider the nerve $U_* \to X$ of $U \to X$, which comes equipped with a k-point $x_* : \mathbf{Spec}\, k \to U_*$ compatible with the point x. The theorem is true for all $U_i = U \times_X \times \cdots \times_X U$ by our induction hypothesis. In order to deduce the result for X, we consider the following commutative square of ∞-categories

$$\begin{tikzcd} \mathcal{F}ol^{ap}((X,x)^\wedge) \arrow[r] \arrow[d] & \ell_{X,x}/\mathbf{dgLie}_k^{dap} \arrow[d] \\ \lim_i \mathcal{F}ol^{ap}((U_i,x_i)^\wedge) \arrow[r] & \lim_i \ell_{U_i,x_i}/\mathbf{dgLie}_k^{dad}. \end{tikzcd}$$

The bottom horizontal ∞-functor is an equivalence by our induction assumption. The vertical ∞-functor on the left is an equivalence, because $(U_*,u_*)^\wedge \to (X,x)^\wedge$ is a hypercovering of derived stacks. Finally, the induced augmented simplicial object $\ell_{U_*,u_*} \to \ell_{X,x}$ is a smooth hypercovering of dg-Lie algebras in the sense of [Lur11, Prop. 1.5.8], and thus induces an equivalence of dg-Lie algebras

$$\mathrm{colim}_i \ell_{U_*,u_*} \simeq \ell_{X,x}.$$

This implies that also the vertical ∞-functor on the right is an equivalence of ∞-categories. □

2.3.4. Transversely smooth and rigid derived foliations. In this section we prove a stronger formal integrability statement for quasi-smooth and rigid derived foliations, which is not only valid formally at each point but globally on X. This result provides an important connection between derived foliations and classical notions such as Gobdillon-Vey sequences that will be explored later on.

For this section we fix X a smooth DM-stack over k. The results of these sections can certainly be generalized to the more general setting, but we restrict to this case for simplicity.

DEFINITION 2.3.4.1. *A derived foliation $\mathcal{F} \in \mathcal{F}ol(X)$ is* transversely smooth and rigid *if its normal complex $\mathcal{N}_{\mathcal{F}}$ is a vector bundle.*

Note that because X is a smooth DM-stack then $\mathbb{L}_{\mathcal{F}}$ is quasi-isomorphic to the 2-term complex of vector bundles $\mathcal{N}^*_{\mathcal{F}} \to \Omega^1_X$, and that $\mathcal{F}$ is automatically quasi-smooth in the sense of the general Definition 2.1.4.5. The term "rigid" in Definition 2.3.4.1 refers here to the fact that $\Omega^1_X \to H^0(\mathbb{L}_{\mathcal{F}})$ must be an epimorphism of sheaves, so that $\mathcal{F}$ does not have any "infinitesimal automorphisms" or no "fixed points". As an example, the foliation induced by a group scheme G acting on X (see Example 2.1.5.6) is transversely smooth and rigid if and only if the stabilizers of the action are all discrete.

We start by constructing a canonical formal thickening $X \hookrightarrow \mathcal{J}_{\mathcal{F}}$ associated to a transversely smooth and rigid derived foliation $\mathcal{F}$. For this, we consider the push-out of graded mixed cdga's over X

$$\begin{tikzcd} \mathbf{DR}(X/k) \arrow[r] \arrow[d] & \mathbf{DR}(\mathcal{F}/k) \arrow[d] \\ \mathcal{O}_X \arrow[r] & B. \end{tikzcd}$$

The graded mixed cdga B is $\mathcal{O}_X$-linear via its canonical morphism $\mathcal{O}_X \to B$, and as a graded $\mathcal{O}_X$-linear cdga it is equivalent to $Sym_{\mathcal{O}_X}(\mathcal{N}^*_{\mathcal{F}}[2])$. Using the equivalence between graded mixed complexes and complete filtered complexes of Corollary 1.3.4.4, we associate to B a complete filtered $\mathcal{O}_X$-linear cdga $\mathcal{B}_{\mathcal{F}} := |B|^t$. The associated graded of B is canonical equivalent to $Sym_{\mathcal{O}_X}(\mathcal{N}^*_{\mathcal{F}})$. Moreover, as B comes equipped with an augmentation $B \to \mathcal{O}_X$, the filtered cdga $\mathcal{B}_{\mathcal{F}}$ is itself canonically augmented $\mathcal{B}_{\mathcal{F}} \to \mathcal{O}_X$.

Let us denote the filtration on $\mathcal{B}_{\mathcal{F}}$ by

$$\dots \longrightarrow F^i\mathcal{B}_{\mathcal{F}} \longrightarrow F^{i-1}\mathcal{B}_{\mathcal{F}} \longrightarrow \dots \qquad F^0\mathcal{B}_{\mathcal{F}} = \mathcal{B}_{\mathcal{F}},$$

and note that $F^i\mathcal{B}_{\mathcal{F}} = F^{i+1}\mathcal{B}_{\mathcal{F}} = \mathcal{B}_{\mathcal{F}}$ for all $i \geq 0$. We thus have a pro-objects in the ∞-category of quasi-coherent $\mathcal{O}_X$-linear cdga's

$$\hat{\mathcal{B}}_{\mathcal{F}} = \text{“}\lim_i\text{”}\, \mathcal{B}_{\mathcal{F}}/(F^i\mathcal{B}_{\mathcal{F}}).$$

This pro-object possesses a formal spectrum

$$\mathcal{J}_{\mathcal{F}} := \mathsf{Spf}(\hat{\mathcal{B}}_{\mathcal{F}}) = \mathrm{colim}_i\, \mathbf{Spec}\,(\mathcal{B}_{\mathcal{F}}/(F^i\mathcal{B}_{\mathcal{F}})) \to X,$$

which is an affine formal scheme over X. The formal scheme $\mathcal{J}_{\mathcal{F}}$ comes equipped with a canonical closed embedding $X \to \mathcal{J}_{\mathcal{F}}$ induced from the augmentation $\mathcal{B}_{\mathcal{F}} \to \mathcal{O}_X$, in such a way that X is a retract of $\mathcal{J}_{\mathcal{F}}$ because of the $\mathcal{O}_X$-linear structure on $\hat{\mathcal{B}}_{\mathcal{F}}$. Note also that $\mathcal{J}_{\mathcal{F}} \to X$ is locally on X of the form $X \times \hat{\mathbb{A}}^d$, where d is the codimension of $\mathcal{F}$ (i.e. the rank of the normal bundle $\mathcal{N}_{\mathcal{F}}$). Indeed, as $\mathcal{J}_{\mathcal{F}}$ is a relative formal spectrum, its construction is stable by base change, so we may assume that X is affine of the form $\mathbf{Spec}\, A$. In this case, the filtered A-cdga

$\mathcal{B}_{\mathcal{F}}$ is such that $Gr^*(\mathcal{B}_{\mathcal{F}}) \simeq Sym_A(\mathcal{N}^*_{\mathcal{F}})$, where $\mathcal{N}_{\mathcal{F}}*$ is of weight 1. Since $\mathcal{N}^*_{\mathcal{F}}$ is a vector bundle and X is affine, there are no obstructions to lift the canonical morphism of A-dg-modules

$$\mathcal{N}^*_{\mathcal{F}} \to \mathcal{B}_{\mathcal{F}}/F^2 \simeq A \oplus \mathcal{N}^*_{\mathcal{F}}$$

to a morphism of A-dg-modules $\mathcal{N}^*_{\mathcal{F}} \to \mathcal{B}_{\mathcal{F}}$. This extends by multiplicativity to a morphism of completed A-algebras $\hat{Sym}_A(\mathcal{N}^*_{\mathcal{F}}) \to \mathcal{B}_{\mathcal{F}}$ which is clearly an equivalence. This shows that $\mathcal{J}_{\mathcal{F}}$ is thus equivalent, as a formal affine scheme over X, to $\mathbb{V}(\hat{\mathcal{N}}_{\mathcal{F}}) \to X$, the formal completion at the zero section of the total space of the vector bundle $\mathcal{N}_{\mathcal{F}}$ over X. Note that however this is not a functorial description, and therefore the two formal schemes $\mathbb{V}(\hat{\mathcal{N}}_{\mathcal{F}})$ and $\mathcal{J}_{\mathcal{F}}$ are only locally (not globally) equivalent, for the étale topology on X, and the difference between these two objects is understood globally via the filtration on $\hat{\mathcal{B}}_{\mathcal{F}}$ interpolating between them.

So we have constructed, for any transversely smooth and rigid derived foliation $\mathcal{F}$ on X, a formal scheme $\mathcal{J}_{\mathcal{F}} \to X$ together with a section $j : X \to \mathcal{J}_{\mathcal{F}}$, which is moreover obviously functorial in the pair $(X, \mathcal{F})$.

Definition 2.3.4.2. *For a transversely smooth and rigid $\mathcal{F} \in \mathcal{F}ol(X)$, the relative formal scheme $\mathcal{J}_{\mathcal{F}} \to X$ constructed above is called the* (transversal) jet space of *$\mathcal{F}$.*

The importance of the transversal jet space lies in the following integrability result.

Propositon 2.3.4.3. *Let X be a smooth DM-stack, and $\mathcal{F} \in \mathcal{F}ol(X)$ transversely smooth and rigid with transverse jet space $\mathcal{J}_{\mathcal{F}} \to X$. There exists a canonical smooth, transversely smooth and rigid derived foliation $\mathcal{F}' \in \mathcal{F}ol(\mathcal{J}_{\mathcal{F}})$ such that $j^*(\mathcal{F}')$ is canonically equivalent to $\mathcal{F}$ in $\mathcal{F}ol(X)$.*

Proof. It is enough to prove the Proposition on $X = \mathbf{Spec}\, A$ in such a way that the construction of $\mathcal{F}'$, and the equivalence $j^*(\mathcal{F}') \simeq \mathcal{F}$, are both stable by base change along any étale morphism $Y \to X$ between smooth affine schemes. We are thus essentially reduced to the case where X is smooth and affine.

Let $\mathbf{DR}(\mathcal{F}/k)$ be the graded mixed $\mathbf{DR}(A/k)$-cdga corresponding to $\mathcal{F}$. We have an exact triangle of complexes $\mathcal{N}^*_{\mathcal{F}} \to \Omega^1_{A/k} \to \mathbb{L}_{\mathcal{F}}$. Equivalently, $\mathbb{L}_{\mathcal{F}}$ identifies canonically with the two terms complex $\mathcal{N}^*_{\mathcal{F}} \to \Omega^1_{A/k}$ where $\Omega^1_{A/k}$ sits in degree 0. The formal scheme $\mathcal{J}_{\mathcal{F}}$ is here the formal spectrum of the filtered A-algebra $\mathcal{B}_{\mathcal{F}} = |B|^t$, where B is the A-linear graded mixed cdga defined by $B := \mathbf{DR}(\mathcal{F}/k) \otimes_{\mathbf{DR}(A/k)} A$. Note that $\mathcal{B}_{\mathcal{F}}$ is non-canonically isomorphic to $\hat{Sym}_A(\mathcal{N}_{\mathcal{F}})$.

We let $\hat{\mathbf{DR}}(\mathcal{J}_{\mathcal{F}}/k)$ be defined by

$$\hat{\mathbf{DR}}(\mathcal{J}_{\mathcal{F}}/k) := \lim_n \mathbf{DR}((\mathcal{B}_{\mathcal{F}}/F^n)/k)$$

be the completed graded mixed cdga associated to the formal scheme $\mathcal{J}_{\mathcal{F}}$. We have the following statement concerning perfect derived foliations on $\mathcal{J}_{\mathcal{F}}$, whose proof is very much similar (actually simpler) than proposition 2.3.2.3, and is left to the reader as an exercise.

Lemma 2.3.4.4. *There is a natural equivalence between the following two ∞-categories:*

- *$\mathcal{F}ol^{ap}(\mathcal{J}_\mathcal{F})$, the ∞-category of perfect derived foliations on $\mathcal{J}_\mathcal{F}$*
- *the ∞-category of graded mixed $\hat{\mathbf{DR}}(\mathcal{J}_\mathcal{F}/k)$-cdga's D of the form $Sym_{\mathcal{B}_\mathcal{F}}(V[1])$ as graded cdga where V is an almost perfect $\mathcal{B}_\mathcal{F}$-dg-module.*

To achieve the proof proposition 2.3.4.3, it is enough to construct a smooth derived foliation $\mathcal{F}' \in \mathcal{F}ol(\mathcal{J}_\mathcal{F})$ together with an equivalence $j^*(\mathcal{F}') \simeq \mathcal{F}$, which is functorial under étale base change on X. Indeed, as $\mathcal{F}$ is rigid, such an $\mathcal{F}'$ must automatically be rigid as well: the conormal complex $\mathcal{N}^*_{\mathcal{F}'}$ being perfect on $\mathcal{J}_\mathcal{F}$ is such that $j^*(\mathcal{N}^*_{\mathcal{F}'}) \simeq \mathcal{N}^*_\mathcal{F}$ is vector bundle, and thus $\mathcal{N}^*_{\mathcal{F}'}$ must already be a vector bundle on the formal thickening $\mathcal{J}_\mathcal{F}$. As a result, such a smooth $\mathcal{F}'$ with $j^*(\mathcal{F}') \simeq \mathcal{F}$ must be transversely smooth, and thus is automatically rigid because the short exact sequence of bundles $0 \to \mathcal{N}^*_{\mathcal{F}'} \to \Omega^1_{\mathcal{J}_\mathcal{F}} \to \mathbb{L}_{\mathcal{F}'} \to 0$ locally splits.

In order to construct $\mathcal{F}'$, we use our notion of de Rham cohomology of the first kind explained in §A.8. We recall the push-out square of graded mixed cdga's

$$\begin{tikzcd} \mathbf{DR}(X/k) \arrow[r] \arrow[d] & \mathbf{DR}(\mathcal{F}/k) \arrow[d] \\ \mathcal{O}_X \arrow[r] & B, \end{tikzcd}$$

and we consider $\widehat{C}^I_{DR}(B/\mathbf{DR}(\mathcal{F}/k))$, the de Rham cohomology of the first kind. A direct computation of cotangent complexes shows that, as a graded algebra, $\widehat{C}^I_{DR}(B/\mathbf{DR}(\mathcal{F}/k))$ is of the form $Sym_{\mathcal{B}_\mathcal{F}}(\Omega^1_X \otimes_A \mathcal{B}_\mathcal{F}[1])$. Therefore, by the equivalence of ∞-categories describing $\mathcal{F}ol(\mathcal{J}_\mathcal{F})$, $\widehat{C}^I_{DR}(B/\mathbf{DR}(\mathcal{F}/k))$ defines a perfect derived foliation $\mathcal{F}' \in \mathcal{F}ol(\mathcal{J}_\mathcal{F}/k)$. Moreover, the cotangent complex of $\mathcal{F}'$ is $p^*(\Omega^1_{X/k})$, where $p : \mathcal{J}_\mathcal{F} \to X$ is the natural projection, and thus $\mathcal{F}'$ is smooth.

We consider now the canonical morphism $\widehat{C}^I_{DR}(B/\mathbf{DR}(\mathcal{F}/k)) \to \widehat{C}^I_{DR}(A/\mathbf{DR}(\mathcal{F}/k))$, via the augmentation $B \to A$. By Proposition A.8.0.5, we know that the right hand side is naturally equivalent to $\mathbf{DR}(\mathcal{F}/k)$, and we thus have a morphism of graded mixed cdga's $\widehat{C}^I_{DR}(B/\mathbf{DR}(\mathcal{F}/k)) \to \mathbf{DR}(\mathcal{F}/k)$, making the following diagram commutative

$$\begin{tikzcd} \widehat{C}^I_{DR}(B/\mathbf{DR}(\mathcal{F}/k)) \arrow[r] & \mathbf{DR}(\mathcal{F}/k) \\ \hat{\mathbf{DR}}(\mathcal{J}_\mathcal{F}/k) \arrow[r] \arrow[u] & \mathbf{DR}(A/k). \arrow[u] \end{tikzcd}$$

This produces a morphism

$$\widehat{C}^I_{DR}(B/\mathbf{DR}(\mathcal{F}/k)) \otimes_{\hat{\mathbf{DR}}(\mathcal{J}_\mathcal{F}/k)} \mathbf{DR}(A/k) \longrightarrow \mathbf{DR}(\mathcal{F}/k)$$

which is easily seen to be an equivalence by computing the cotangent complexes. By construction, this is the required equivalence $j^*(\mathcal{F}') \simeq \mathcal{F}$. $\square$

Remark 2.3.4.5. *(1)* An important point of proposition 2.3.4.3 is that the association $\mathcal{F} \mapsto (\mathcal{J}_\mathcal{F}, \mathcal{F}')$ is canonical. The foliation $\mathcal{F}'$ will be called the *canonical spread* of the derived foliation $\mathcal{F}$. We will see below that the pair $(\mathcal{J}_{\mathcal{F}}, \mathcal{F}')$ is closely related to the classical notion of Godbillon-Vey sequences (see for instance [**Cas02**] and [**Mal77**, (3.4)] for classical references).

(2) In the special case where, in the statement of the proposition 2.3.4.3 $\mathcal{F}$ is *integrable*, the objects $\mathcal{J}_{\mathcal{F}}$ and $\mathcal{F}'$ can be described explicitly as follows. Let $f : X \to Y$ be a moprhism of smooth separated schemes such that $f^*(0_Y) \simeq \mathcal{F}$. Then $\mathcal{J}_{\mathcal{F}}$ is identified with the formal completion of $X \times_k Y$ along the graph of f, and $\mathcal{F}'$ is $q^*(0_Y)$ where $q : X \times_k Y \to Y$ is the second projection.

(3) Being at the same time smooth and transverally smooth, the derived foliation $\mathcal{F}'$ is a genuine foliation on $\mathcal{J}_{\mathcal{F}}$, given by an integrable sub-bundle $\mathbb{T}_{\mathcal{F}'} \subset \mathbb{T}_{\mathcal{J}_{\mathcal{F}}}$ of the tangent sheaf.

One important corollary of proposition 2.3.4.3 is the following formal integrability result, that could also be obtained from the more general theorems 2.3.3.2 and 2.3.3.3.

Corollary 2.3.4.6. *Let X be a smooth DM-stack and $x : \mathbf{Spec}\, k \to X$ a global point. Let $\mathcal{F} \in \mathcal{F}ol(X/k)$ be a transversely smooth and rigid derived foliation on X. There exists a morphism of formal schemes $f : (X,x)^\wedge \to \hat{\mathbb{A}}^d$, such that $\hat{F}_x \simeq f^*(0)$ in $\mathcal{F}ol((X,x)^\wedge)$. In other words, $\mathcal{F}$ is formally integrable at each point on X.*

Proof of the Corollary. The statement is local at x, so we can assume that X is affine, and moreover that the normal bundle $\mathcal{N}_{\mathcal{F}} \simeq \mathcal{O}_X^d$ is trivial, as well as $\Omega^1_{X/k}$. We use the canonical spread $j : X \hookrightarrow X \times \hat{\mathbb{A}}^d$, with $\mathcal{F}' \in \mathcal{F}ol(X \times \hat{\mathbb{A}}^d)$, $j^*(\mathcal{F}') \simeq \mathcal{F}$. We formally complete at x to get

$$\hat{j} : (X,x)^\wedge \longrightarrow (X \times \mathbb{A}^d, (x,0))^\wedge.$$

The formal completion at $(x,0)$ defines a foliation $\hat{\mathcal{F}'} \in \mathcal{F}ol((X \times \mathbb{A}^d, (x,0))^\wedge)$, which is here given by a family of formal vector fields $\nu_1, \ldots, \nu_n$ on the formal scheme $(X \times \mathbb{A}^d, (x,0))^\wedge \simeq \hat{\mathbb{A}}^{n+d}$, where $n = Dim_k X$, satisfying the classical Frobenius condition: each $[\nu_i, \nu_j]$ can be written as a linear combination of ν_k's. The formal Frobenius lemma (see e.g. [**BZ09**]) then tells us that there is a morphism of formal schemes $p : (X \times \mathbb{A}^d, (x,0))^\wedge \to \hat{\mathbb{A}}^d$, such that $\hat{\mathcal{F}'} \simeq p^*(0)$. We can thus take $f = p \circ \hat{j}$. □

2.4. Existence of derived enhancements

In this section we study the relations between the notion of derived foliation on a derived scheme X, and the classical notion of differential ideal on the truncation $\tau_0(X)$. We show that any derived foliation underlies a classical differential ideal. However, not every differential ideal arises this way, and in the case where X is a smooth variety, we provide a criterion for the existence of a derived foliation enhancing a given classical differential ideal, based on the classical notion of Gobdillon-Vey sequences (see for instance [**Cas02**, **Mal77**]).

2.4.1. Derived foliations and differential ideals. Let X be a derived scheme locally almost of finite presentation over k, and $\mathcal{F} \in \mathcal{F}ol(X/k)$ be a derived foliation on X. We assume that $\mathcal{F}$ is connective in the sense of Definition 2.1.1.1, i.e. $H^i(\mathbb{L}_{\mathcal{F}/k}) \simeq 0$ for $i > 0$. We consider the corresponding sheaf of graded mixed $\mathbf{DR}_{X/k}$-cdga $\mathbf{DR}_{\mathcal{F}}$. We have the canonical morphism of quasi-coherent complexes

$$a : \mathbb{L}_{X/k} \to \mathbb{L}_{\mathcal{F}/k}$$

from which we form the kernel of the induced morphism on H^0

$$K_{\mathcal{F}} := Ker(H^0(\mathbb{L}_{X/k}) \to H^0(\mathbb{L}_{\mathcal{F}/k})).$$

The sheaf $K_{\mathcal{F}}$ is a coherent subsheaf of $H^0(\mathbb{L}_{X/k}) \simeq \Omega^1_{X_0/k}$, where $X_0 = \tau_0(X)$ is the underived scheme obtained from X by truncation (see Appendix A.2).

LEMMA 2.4.1.1. *The subsheaf $K_{\mathcal{F}} \subset \Omega^1_{X_0/k}$ is a coherent differential ideal:*

$$dR(D_{\mathcal{F}}) \subset Im(D_{\mathcal{F}} \otimes \Omega^1_{X_0/k} \to \Omega^2_{X_0/k}).$$

Proof. Since the graded mixed structure on $\mathbf{DR}_{\mathcal{F}} \simeq Sym_{\mathcal{O}_X}(\mathbb{L}_{\mathcal{F}/k}[1])$ is compatible with the de Rham differential on $Sym_{\mathcal{O}_X}(\mathbb{L}_{X/k})$, we have a commutative diagram of sheaves of k-modules on X

$$\begin{tikzcd} H^0(\mathbb{L}_{X_0}/k) \arrow[r, "dR"] \arrow[d, "a"'] & H^0(\wedge^2_{\mathcal{O}_X}\mathbb{L}_{X_0}/k) \arrow[d, "\wedge^2 a"] \\ H^0(\mathbb{L}_{\mathcal{F}/k}) \arrow[r, "\epsilon"'] & H^0(\wedge^2_{\mathcal{O}_X}\mathbb{L}_{\mathcal{F}/k}). \end{tikzcd}$$

Because $\mathbb{L}_{X_0/k}$ and $\mathbb{L}_{\mathcal{F}/k}$ are both connective, the above diagram is isomorphic to

$$\begin{tikzcd} \Omega^1_{X_0/k} \arrow[r, "dR"] \arrow[d, "a"'] & \Omega^2_{X_0/k} \arrow[d, "\wedge^2 a"] \\ H^0(\mathbb{L}_{\mathcal{F}/k}) \arrow[r, "\epsilon"'] & \wedge^2_{\mathcal{O}_X} H^0(\mathbb{L}_{\mathcal{F}/k}). \end{tikzcd}$$

The lemma thus follows by considering the induced morphism on the kernels of the vertical morphisms. □

DEFINITION 2.4.1.2. *(1) With the notations and conditions above, the differential ideal $K_{\mathcal{F}} \subset \Omega^1_{X_0/k}$ is called the* classical truncation *of the derived foliation $\mathcal{F} \in \mathcal{F}ol(X/k)$.*
(2) For a coherent differential ideal $K \subset \Omega^1_{X_0/k}$, a derived enhancement of *D is a connective derived foliations $\mathcal{F} \in \mathcal{F}ol(X/k)$ such that $K = K_{\mathcal{F}}$.*

Note that, if $\mathcal{F} \to \mathcal{F}'$ is a morphism of connective derived foliations, then we have a natural inclusion of differential ideals $K'_{\mathcal{F}} \subset K_{\mathcal{F}} \subset \Omega^1_{X_0/k}$. Therefore, if we denote by $IDiff(X_0)$ the category of coherent differential ideals on X_0 (with morphisms given by inclusions), the classical truncation is an ∞-functor

$$K_{-} : \mathcal{F}ol^c(X/k)^{op} \longrightarrow IDiff(X_0),$$

from connective derived foliations on X to differential ideals on X_0. As we will now explain, there are no reasons to expect that the above ∞-functor is essentially surjective, even in the case where X is a nice smooth scheme over k. Indeed, if $K \subset \Omega^1_{X/k}$ is a coherent differential ideal, we can certainly form a graded mixed cdga by considering $Sym^{st}_{\mathcal{O}_X}(E[1])$, where E is the coherent sheaf $\Omega^1_{X/k}/K$, and where the mixed structure is defined by the unique possible extension of the de Rham differential along the quotient map $\mathbf{DR}(X/k) \to Sym^{st}_{\mathcal{O}_X}(E[1])$.

It is important to note that here Sym^{st}, as opposed to notation Sym, denotes the strict or naive symmetric algebra, defined in the tensor abelian category of coherent sheaves on X. This makes a difference as E is in general not a flat $\mathcal{O}_X$-module, and thus the canonical morphism $Sym_{\mathcal{O}_X}(E[1]) \to Sym^{st}_{\mathcal{O}_X}(E[1])$, from the derived Sym-algeba to the naive Sym-algebra is not a quasi-isomorphism. In particular, the graded mixed cdga $Sym^{st}_{\mathcal{O}_X}(E[1])$ does not satisfy the condition appearing in Definition 2.1.1.1, and thus does not define a derived foliation on X. In order to promote $Sym^{st}_{\mathcal{O}_X}(E[1])$ into a derived foliation, we would have to lift the de Rham differential along the natural morphism $Sym_{\mathcal{O}_X}(E[1]) \to Sym^{st}_{\mathcal{O}_X}(E[1])$. We will see in the next section that there are obstructions to the existence of such a lift in general, and that these obstructions can be expressed explicitly in simple terms.

The main reason for expecting that the classical truncation K_{-} is not essentially surjective comes from integrability results for derived foliations, such as Theorem 2.3.3.3 or proposition 2.3.4.3, which indicate that derived foliations are always formally integrable at each point. There are however example of differential ideals, or even analytic germs of differential ideals, which are not formally integrable (see remark 2.4.1.3 (1) below). Of course, the notion of derived enhancement of Definition 2.4.1.2 is extremely general as we do not restrict the type of derived foliation $\mathcal{F}$, so theorem 2.3.3.3 or proposition 2.3.4.3 are not strictly speaking counterexamples to the existence of derived enhancements, and possibly the questions of their existence is complicated and not well posed. However, when restricting the type of derived foliations allowed, the question of existence of derived enhancements can be treated in great detail. This is what we will do in the next section for *transversely smooth and rigid* derived foliations.

Remark 2.4.1.3.

- Examples of non formally integrable differential ideals already exist in dimension 2. Indeed, for a germ ω of holomorphic 1-differential with an isolated zero at $0 \in \mathbb{C}^2$ and satisfying $d\omega \wedge \omega = 0$, formal integrability and holomorphic integrability are equivalent conditions by the main result of [**Mal76**]. Moreover, it is well known that holomorphic integrability is restricted by topological obstructions, see for instance [**MM80**, Theorem B]. This provides examples of (analytic germs of) differential ideals that do not admit derived enhancements.

- It is worth mentioning that the truncation functor K_- of Definition 2.4.1.2 can also be understood in terms of a natural t-structure on graded mixed complexes. Indeed, we can introduce a t-structure on graded mixed complexes by declaring that $E \in \epsilon - \mathbf{dg}_k^{gr}$ is *connective* if for each $n \in \mathbb{Z}$, the part $E^{(n)}$ of weight n is n-connective: $H^i(E^{(n)}) = 0$ for $i < -n$. This t-structure is compatible with the symmetric monoidal structure, and thus t-truncations functors extends to commutative algebra objects. A connective derived foliation $\mathcal{F} \in \mathcal{F}ol^c(X/k)$ is such that $\mathbf{DR}(\mathcal{F})$ is a connective graded mixed cdga, and its 0-truncation for the above mentioned t-structure is precisely $Sym^{st}_{\mathcal{O}_X}(E[1])$ where $E = \Omega^1_{X_0/k}/K_{\mathcal{F}}$.

2.4.2. Derived enhancements and Godbillon-Vey sequences. Let $X = \mathbf{Spec}\, A$ be a smooth affine scheme over k, and $K \subset \Omega^1_{X/k}$ be a differential ideal and we assume that K is a free an A-module (but we do not assume that it is a sub-bunld eof $\Omega^1_{X/k}$). We chose a set of generators $(\omega_1, \dots, \omega_d)$ of K as an A-module, which is a basis. We recall from [**Mal77**, (3.4)] that a *Godbillon-Vey (*or simply *GV) sequence of forms adapted to* $(\omega_1, \dots, \omega_d)$ consists of a sequence of elements $\omega_{i,\alpha} \in \Omega^1_{X/k}$, where $\alpha \in \mathbb{N}^p$, satisfying the equations

$$dR(\omega_{i,\alpha}) = \sum_{j,\beta} \binom{\alpha}{\beta} \omega_{j,\beta} \wedge \omega_{i,\alpha-\beta+e_j} \qquad \omega_{i,0} = \omega_i \tag{10}$$

where $e_j = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^p$ is the j-th element of the standard basis. The GV sequences can also be interpreted as follows.

LEMMA 2.4.2.1. *With the notations above, there is a bijection between the following sets.*

- *The set of GV sequences adapted to* $(\omega_1, \dots, \omega_d)$.
- *Families of differential forms* $(\widetilde{\omega_1}, \dots, \widetilde{\omega_d})$ *on the formal scheme* $X \times \widehat{\mathbb{A}}^d$ *with the following conditions satisfied.*
  1. $j^*(\widetilde{\omega_i}) = \omega_i$, *where* $j : X \hookrightarrow X \times \widehat{\mathbb{A}}^d$ *is the embedding at* 0.
  2. *For each* i, *the form* $\widetilde{\omega_i}$ *can be written as*
  $$\widetilde{\omega_i} = dR(t_i) + \sum_k a_k . p^*(\lambda_k)$$
  with $\lambda_k \in \Omega^1_{X/k}$, $p : X \times \widehat{\mathbb{A}}^d \to X$ *being the projection onto the first factor,* $(t_1, \dots, t_p)$ *the standard coordinates on* $\widehat{\mathbb{A}}^d$, *and* $a_k \in A[[t_1, \dots, t_p]]$ *formal functions on* $X \times \widehat{\mathbb{A}}^d$.
  3. *The forms* $(\widetilde{\omega_1}, \dots, \widetilde{\omega_d})$ *define a differential ideal in* $\widehat{\Omega}^1_{X \times \widehat{\mathbb{A}}^d}$.

Proof. This is almost straightforward: to a GV sequence $\omega_{i,\alpha}$ we associate the forms

$$\widetilde{\omega_i} := dR(t_i) + \sum_{i,\alpha} p^*(\omega_{i,\alpha}) . \frac{t^\alpha}{\alpha!}$$

We refer to [Mal77, Prop. 3.2] for more details. □

It is important to note that because of condition (2), the forms $(\widetilde{\omega_1},\ldots,\widetilde{\omega_d})$ are automatically linearly independent everywhere, and thus the differential ideal they generate is a free subbundle of rank p in $\widehat{\Omega}^1_{X\times\widehat{\mathbb{A}}^d}$. Therefore, they define a smooth, transversely smooth and rigid foliation (i.e. a classical foliation in the usual sense) on the formal scheme $X\times\widehat{\mathbb{A}}^d$ exactly as in proposition 2.3.4.3. In fact, proposition 2.3.4.3 can be used in order to prove the following result, which expresses that GV sequences and transversely smooth and rigid derived derived enhancements are essentially the same thing.

Propositon 2.4.2.2. *Let X be a smooth scheme over k and $K\subset\Omega^1_{X/k}$ be a coherent differential ideal such that K is a vector bundle. The following two conditions are equivalent:*

(1) Locally for the Zariski topology on X, there exists a Godbillon-Vey sequence (adapated to some choice of generators $(\omega_1,\ldots,\omega_p)$ of K).
(2) Locally for the Zariski topology on X there exists a transversely smooth and rigid derived foliation $\mathcal{F}\in\mathcal{F}ol(X/k)$ such that $K=K_{\mathcal{F}}$.

Proof. We have already seen that (1) ⇒ (2). Indeed, a GV sequence for $K\in IDiff(X)$ defines a smooth, transversely smooth and rigid derived foliation $\mathcal{F}'$ on $X\times\widehat{\mathbb{A}}^d$. We let $\mathcal{F}:=j^*(\mathcal{F}')\in\mathcal{F}ol(X/k)$. As pull-backs of derived foliations preserve conormal complexes, we see that $\mathcal{F}$ is transversely smooth and rigid. Moreover, by construction, the differential ideal $K_{\mathcal{F}}$ is generated by $(\omega_1,\ldots,\omega_d)$ and thus coincides with K.

Conversely, suppose that K is of the form $K_{\mathcal{F}}$. We apply proposition 2.3.2.3 to $\mathcal{F}$. As the statement is local, we can even assume that the normal bundle of $\mathcal{F}$ is trivialized, and the proposition gives us $\mathcal{F}'\in\mathcal{F}ol(X\times\widehat{\mathbb{A}}^d/k)$ such that $j^*(\mathcal{F}')=\mathcal{F}$. Moreover, by observing the proof of proposition 2.3.2.3, it is easy to see that $\mathcal{F}'$ is given by a differential ideal $(\widetilde{\omega_1},\ldots,\widetilde{\omega_d})\subset\widehat{\Omega}^1_{X\times\widehat{\mathbb{A}}^d}$, where each $\widetilde{\omega}_i$ is of the form

$$\widetilde{\omega}_i=\sum_j f_{i,j}.dR(t_j)+\sum_k a_k.p^*(\lambda_k)$$

with $f_{i,j}\in A[[t]]=A[[t_1,\ldots,t_p]]$ a formal function such that the the $p\times p$ matrix $M:=(f_{i,j})$ is invertible. Indeed, it is easy to see by construction that the cotangent complex of $\mathcal{F}'$ is a vector bundle of rank $dimX$, and that the anchor map

$$\widehat{\Omega}^1_{X\times\widehat{\mathbb{A}}^d}\longrightarrow\mathbb{L}_{\mathcal{F}'},$$

precomposed with $p^*(\Omega^1_{X/k})\to\widehat{\Omega}^1_{X\times\widehat{\mathbb{A}}^d}$ induces an isomorphism $p^*(\Omega^1_{X/k})\simeq\mathbb{L}_{\mathcal{F}'}$. As

$$\widehat{\Omega}^1_{X\times\widehat{\mathbb{A}}^d}\simeq p^*(\Omega^1_{X/k})\bigoplus\oplus_i A[[t]].dR(t_i),$$

we see that the kernel of the anchor map, which is the subsheaf generated by the $\widetilde{\omega}_i$'s, becomes isomorphic to $\oplus_i A[[t]].dR(t_i)$ via the projection $\widehat{\Omega}^1_{X\times\widehat{\mathbb{A}}^d}\to\oplus_i A[[t]].dR(t_i)$. We can thus change the basis $(\widetilde{\omega_1},\dots,\widetilde{\omega_d})$ by $M^{-1}(\widetilde{\omega_1},\dots,\widetilde{\omega_d})$, which defines the same foliation $\mathcal{F}'$, and thus simply assume that $f_{i,j}=\delta_{i,j}$. By Lemma 2.4.2.1, the family $(\widetilde{\omega_1},\dots,\widetilde{\omega_d})$ defines a GV sequence for the differential ideal K. □

Proposition 2.4.2.2 has an important consequence concerning existence of derived enhancement under appropriate codimension conditions. The question of uniqueness of a derived enhancement is more complicated and will be studied later using analytic methods (see §5.4).

COROLLARY 2.4.2.3. *Let $X=\mathbf{Spec}\,A$ be a smooth affine scheme over k and $K\subset\Omega^1_{X/k}$ be a differential ideal. We assume that the coherent sheaf $\Omega^1_{X/k}/K$ is a vector bundle on a dense open $U\subset X$, whose closed complement Z is of codimension at least 3 in X. Then, locally for the Zariski topology on X, K is of the form $K=K_{\mathcal{F}}$ for a transversely smooth and rigid derived foliation $\mathcal{F}\in\mathcal{F}ol(X/k)$.*

Proof of the Corollary. We first notice that K must be a vector bundle on U, and thus on X by Hartogs. Since the statement is local for the Zariski topology on X, we can assume the existence of an isomorphism of coherent sheaves $\mathcal{O}_X^d\simeq K$. We consider the induced injective morphism $\mathcal{O}_X^d\hookrightarrow\Omega^1_{X/k}$. This defines a complex of vector bundles concentrated in cohomological degress $[-1,0]$ that we denote by $\mathbb{L}$. We then use the following well-known vanishing lemma.

LEMMA 2.4.2.4. *Under the conditions above, for all $n\geq 0$ we have $H^i(\wedge^{n+2}_{\mathcal{O}_X}(\mathbb{L}))\simeq 0$ for all $i\leq -n$.*

Proof of the lemma. The complex $\wedge^{n+2}_{\mathcal{O}_X}(\mathbb{L})$ is a perfect complex which is a direct factor of $\mathbb{L}^{\otimes n+2}$, which is itself perfect and represented by the genuine complex of vector bundles $(\mathcal{O}_X^d\to\Omega^1_{X/k})^{\otimes n+2}$. This is a complex of vector bundles sitting in cohomological degrees $[-n-2,0]$. We form the dual perfect complex $\mathbb{L}^\vee$ which is a bounded complex of vector bundles concentrated in degrees $[0,n+2]$. Moreover, on the open U the complex $\mathbb{L}$ is equivalent to a vector bundle, and this all the cohomology sheaves $H_i:=\underline{H}^i(\mathbb{L}^\vee)$ vanishes on U as soon as $i>0$. In particular, by Grothendieck duality the ext-sheaves $\underline{Ext}^i_{\mathcal{O}_X}(H_i,\mathcal{O}_X)$ are zero for $i=0$, $i=1$ and $i=2$. This clearly implies, by postnikov induction on $\mathbb{L}^\vee$, that $\mathbb{L}\simeq\mathbb{R}\underline{Hom}_{\mathcal{O}_X}(\mathbb{L}^\vee,\mathcal{O}_X)$ has cohomology sheaves concentrated in cohomological degrees $[-n,0]$ as wanted. □

Coming back to the proof of the corollary, by Proposition 2.4.2.2 it is enough to show the existence of a GV sequence $\omega_{i,\alpha}$ adapted to the basis $(\omega_1,\dots,\omega_d)$ of K coming from our choice of isomorphism $\mathcal{O}_X^d\simeq K$. We construct the $\omega_{i,\alpha}$ by induction on the multi-index α. Suppose that we have already constructed $\omega_{i,\alpha}$ for all $|\alpha|\leq n$, such that equations (10) are satisfied for all $|\alpha|<n$. For all i, and a multi-index α with $|\alpha|=n$, consider the equation (10) in the following form

$$\sum_j \omega_{i,0}\wedge\omega_{i,\alpha+e_j}=dR(\omega_{i,\alpha})-\sum_{j,\beta\geq 1}\binom{\alpha}{\beta}\omega_{j,\beta}\wedge\omega_{i,\alpha-\beta+e_j}, \tag{11}$$

where $\omega_{i,\alpha+e_j}$ are to be determined. The lemma 2.4.2.4 tells us that the complex $\wedge^{n+2}_{\mathcal{O}_X}(\mathbb{L})$ is exact in degree $[n-2,n]$, and thus, in particular, that the sequence

$$Sym^{n+1}(\mathcal{O}_X^d)\otimes\Omega^1_{X/k}\to Sym^n(\mathcal{O}_X^d)\otimes\Omega^2_{X/k}\to Sym^{n-1}(\mathcal{O}_X^d)\otimes\Omega^3_{X/k}$$

is exact. The right hand side of (11) can be considered as cocycle in $Sym^n(\mathcal{O}_X^d)\otimes\Omega^2_{X/k}$, because equations (10) are satisfied for $|\alpha|<n$. The condition that this cocycle is in fact a coboundary precisely says that there exists forms $\omega_{i,\alpha+e_j}$ such that the above equation (11) is satisfied. This finishes the proof of the Corollary. □

We finish by emphasising a important consequence of lemma 2.4.2.4 that will be used later on. Let $\mathcal{F}$ be a derived foliation which is rigid and quasi-smooth, and $K_{\mathcal{F}}$ the corresponding ideal. We can compare the derived de Rham cohomology of $\mathcal{F}$ with the *naive de Rham cohomology* of $K_{\mathcal{F}}$ by means of the natural projection

$$p:\mathbf{DR}_{\mathcal{F}}\to\mathbf{DR}^{st}_{\mathcal{F}}=Sym^{st}_{\mathcal{O}_X}(\Omega^1_{\mathcal{F}}[1]).$$

Here, $\Omega^1_{\mathcal{F}}=H^0(\mathbb{L}_{\mathcal{F}})$ is the coherent sheaf of 1-forms along $\mathcal{F}$, which is quasi-isomorphic to $\mathbb{L}_{\mathcal{F}}$ via the natural truncation map $\mathbb{L}_{\mathcal{F}}\to H^0(\mathbb{L}_{\mathcal{F}})\simeq\Omega^1_X/\mathcal{N}^*_{\mathcal{F}}$. The graded mixed cdga $Sym^{st}_{\mathcal{O}_X}(\Omega^1_{\mathcal{F}}[1])$ is the naive or *strict* symmetric product algebra over the coherent sheaf $\Omega_{\mathcal{F}}[1]$, where the mixed structure is induced by the de Rham differential $\mathcal{O}_X\to\Omega^1_{\mathcal{F}}$. The morphism p is also the natural projection of the connective object $\mathbf{DR}_{\mathcal{F}}$ to its 0-th truncation for the Beilinson's t-structure on graded mixed objects (see 1.2.3.4). By passing to the non-Tate realization we have an induced morphism of the corresponding sheaves of de Rham cohomologies

$$|\mathbf{DR}_{\mathcal{F}}|\simeq\widehat{C}^*_{DR}(\mathcal{F})^u\to|\mathbf{DR}^{st}_{\mathcal{F}}|=:C^*_{DR,naive}(\mathcal{F}).$$

Passing on hyper cohomology groups over X we get an induced morphism

$$\widehat{H}^i_{DR}(\mathcal{F})\to H^i_{DR,naive}(\mathcal{F})$$

relating the Hodge completed derived de Rham cohomology with naive de Rham cohomology of $\mathcal{F}$.

Corollary 2.4.2.5. *With the notations above, assume that $\mathbb{L}_{\mathcal{F}}$ is a vector bundle outside of a closed subset $Z\in X$ of codimension d. Then, the induced morphism*

$$\widehat{H}^i_{DR}(\mathcal{F})\to H^i_{DR,naive}(\mathcal{F})$$

is an isomorphism for all $i<d-1$.

Proof of the corollary. This is a direct application of the lemma 2.4.2.4 and the compatibility of the morphism $|\mathbf{DR}_{\mathcal{F}}|\simeq\widehat{C}^*_{DR}(\mathcal{F})^u\to|\mathbf{DR}^{st}_{\mathcal{F}}|=:C^*_{DR,naive}(\mathcal{F})$ with the Hodge filtrations on both sides. □

2.5. Direct images

Let $f : X \to Y$ be a morphism of derived stacks. The pull-back ∞-functor $f^* : \mathbf{dSt}/Y \to \mathbf{dSt}/X$, sending $(Z \to Y)$ to $(X \times_Y Z \to X)$, commutes with arbitrary colimits. In particular, it admits a right adjoint

$$f_* : \mathbf{dSt}/X \to \mathbf{dSt}/Y,$$

called the *direct image*, or *Weil restriction* along f.

For any $Z \to X$, the adjunction morphism is a commutative diagram of derived stacks

$$\begin{tikzcd} X \times_Y f_*(Z) \arrow[r] \arrow[dr] & Z \arrow[d] \\ & X. \end{tikzcd}$$

The adjunction counit $X \times_Y f_*(Z) \to Z$ will be denoted by ev and will be called the *evaluation morphism*. The following theorem shows that, under certain properness conditions, derived foliations can be push-forward. This provides a very powerful tool to construct new derived foliations from existing ones.

THEOREM 2.5.0.1. *Assume that $f : X \to Y$ is a morphism of derived Artin stacks (locally of finite presentation) which is quasi-smooth and representable by a proper derived algebraic space. Let $Z \to X$ be a derived Artin stack locally of finite presentation over X. Then, there exists an ∞-functor*

$$f_* : \mathcal{F}ol^p(Z/X) \to \mathcal{F}ol^p(f_*(Z)/Y)$$

from perfect derived foliations on Z relative to X to perfect derived foliations on $f_*(Z)$ relative to Y (see Example 2.1.5.3). Moreover, the cotangent complex of $f_*(\mathcal{F})$ is given by

$$\mathbb{L}_{f_*(\mathcal{F})} \simeq q_+(ev^*(\mathbb{L}_{\mathcal{F}}))$$

where $q : X \times_Y f_*(Z) \to f_*(Z)$ is the second projection and $q_+ : \mathbf{Perf}(X \times_Y f_*(Z)) \to \mathbf{Perf}(f_*(Z))$ is defined on perfect complexes by $q_+(E) := (q_*(E^\vee))^\vee$.

The above theorem is proven in [**Alf25**] and its proof will not be covered in this book. In the next two subsections we will simply explain how the ∞-functor f_* is constructed, the reader will find details in [**Alf25**].

2.5.1. Derived linear stacks. For an affine derived scheme $X = \mathbf{Spec}\, A$, and an A-dg-module E, we can form the linear derived stack associated to E, denoted by $\mathbb{V}(E)$, as follows. We consider the ∞-functor

$$\mathbb{V}(E) : \mathbf{cdga}_A^{\leq 0} \to \mathbf{Top}$$

sending a commutative connective A-linear cdga B to $\mathbf{Map}_{\mathbf{dg}_A}(E, B)$, the mapping space of A-dg-modules from E to B. Note that, by adjunction, $\mathbf{Map}_{\mathbf{dg}_A}(E, B) \simeq \mathbf{Map}_{\mathbf{dg}_B}(E \otimes_A^L B, B)$. The group scheme $\mathbb{G}_m$ acts on $\mathbb{V}(E)$ by acting on B by its weight 1 action. More concretely, $\mathbb{G}_m(B)$ can be identified with the simplicial set of self autoequivalences of B as a B-dg-modules.

This is, by definition, the simplicial sub-monoid of $\mathbf{Map}_{\mathbf{dg}_B}(B,B)$ of endomorphisms of B as a B-dg-modules, and thus it acts naturally on B as an object in the ∞-category $\mathbf{dg}_B$.

The linear derived stack associated to E, $\mathbb{V}(E)$, will be considered as an object in $\mathbf{dSt}_X^{\mathbb{G}_m}$, the ∞-category of derived stacks over X endowed with a $\mathbb{G}_m$-action. These local constructions are obviously compatible with base change on X, so they can be glued to obtain, for any derived stack X, an ∞-functor

$$\mathbb{V} : \mathbf{QCoh}(X)^{op} \longrightarrow \mathbf{dSt}_X^{\mathbb{G}_m}.$$

It is shown in [**Mon21**] that the ∞-functor $\mathbb{V}$ has a left adjoint sending an $\mathbb{G}_m$-equivariant derived stack $Y \to X$ to its sheaf of functions of weight 1. Moreover, when restricted to bounded above quasi-coherent complexes, the counit of this adjunction is an equivalence, so that the induced ∞-functor

$$\mathbb{V} : \mathbf{QCoh}^{<0}(X)^{op} \longrightarrow \mathbf{dSt}_X^{\mathbb{G}_m}$$

is a full embedding. An object in the image of $\mathbb{V}$ will be called a *derived linear stack over* X. We will mainly use this notion for the more specific case of *perfect* complexes on X.

On a final note, when $E \in \mathbf{Perf}(X)$ is a perfect complex on X, then the linear derived stack $\mathbb{V}(E) \to X$ is a relative derived Artin stack which is strongly finitely presented (see [**TV07**, Prop. 2.23]). Such $\mathbb{G}_m$-equivariant derived stacks will be called *perfect derived linear stacks over* X. They form an ∞-category naturally equivalent to $\mathbf{Perf}(X)^{op}$ by means of the construction $E \mapsto \mathbb{V}(E)$. Note that $\mathbb{V}(E)$ is a derived Artin stack locally of finite presentation over k if X is so. Moreover, the projection $\mathbb{V}(E) \to X$ is smooth if and only if E is of positive Tor amplitude, and is affine if and only if E is connective.

2.5.2. Derived foliations as equivariant derived linear stacks. The importance of derived linear stacks for us is the fact that they can be used in order to provide a geometric interpretation for derived foliations. This geometric interpretation has its own interests, and we will use it for instance in order to construct the Hodge filtration for general derived foliations. It will also be useful for the notion of derived foliations in non-zero characteristic discussed in Chapter 7.

We recall the group stack $\mathcal{H}_0$, defined as the semi-direct product of $B\mathbb{G}_a$ and $\mathbb{G}_m$ for the $\mathbb{G}_m$-action of weight -1 on $\mathbb{G}_a$ (see §1.5). For a derived affine scheme $X = \mathbf{Spec}\, A$, we have its *graded loop space* $\mathcal{L}^{gr}(X/k)$, defined as the derived mapping stack

$$\mathcal{L}^{gr}(X/k) := \underline{\mathbf{Map}}(B\mathbb{G}_a, X) \in \mathbf{dSt}_k.$$

The group $\mathcal{H}_0$ acts on $B\mathbb{G}_a$, where the $\mathbb{G}_m$ component acts by weight -1 and $B\mathbb{G}_a$ acts by translation on itself. As a result, $\mathcal{H}_0$ acts on $\mathcal{L}^{gr}(X/k)$, and we will thus consider the graded loop stack as a $\mathcal{H}_0$-equivariant derived stack. Note also that as X is affine, so is $\mathcal{L}^{gr}(X/k)$, and it is explicitly given by

$$\mathcal{L}^{gr}(X/k) \simeq \mathbf{Spec}\,(Sym_A(\mathbb{L}_{A/k}[1])) \simeq \mathbb{V}(\mathbb{L}_{A/k}[1]),$$

and as a result we have a natural equivalence of graded cdga's
$$\mathcal{O}(\mathcal{L}^{gr}(X/k)) \simeq \mathbf{DR}(X/k).$$
Moreover, as shown in [**MRT22**], the $\mathcal{H}_0$-action on $Sym_A(\mathbb{L}_{A/k}[1]) = \mathbf{DR}(X/k)$ corresponds, by means of the above identification and our corollary 1.5.0.3, to its usual graded mixed structure given by the de Rham differential.

When X is a general derived stack, we define the completed graded loop stack by imposing descent
$$\widehat{\mathcal{L}}^{gr}(X/k) := \mathrm{colim}_{\mathbf{Spec}\,A\to X}\mathcal{L}^{gr}(\mathbf{Spec}\,A/k) \in \mathbf{dSt}_k^{\mathcal{H}_0}.$$
Note that $\widehat{\mathcal{L}}^{gr}(X/k)$ can not always be identified with the derived mapping stack $\underline{\mathbf{Map}}(B\mathbb{G}_a, X)$. There exists a natural descent morphism $\underline{\mathbf{Map}}(B\mathbb{G}_a, X) \to \widehat{\mathcal{L}}^{gr}(X/k)$, but it is not an equivalence in general (it is so when X is a derived scheme, or more generally a derived Deligne-Mumford stack). Finally, for an arbitrary morphism of derived stacks $X \to Y$, we can define the relative graded loop stack
$$\widehat{\mathcal{L}}^{gr}(X/Y) := \widehat{\mathcal{L}}^{gr}(X/k) \times_{\widehat{\mathcal{L}}^{gr}(Y/k)} Y$$
which is now an $\mathcal{H}_0$-equivariant derived stack *over* Y.

We are now ready to define a notion of *geometric derived foliations*, based on linear stacks and graded loop stacks. This definition is shown in [**Alf25**] to be equivalent to Definition 2.1.3.3. For any derived stack X, we consider its derived graded loop stack $\widehat{\mathcal{L}}^{gr}(X/k)$ endowed with its natural $\mathcal{H}_0$-action. It comes equipped with a projection $\pi : \widehat{\mathcal{L}}^{gr}(X/k) \to X$ by evaluating at the natural base point of $B\mathbb{G}_a$. We note here that even though π is not $\mathcal{H}_0$-equivariant (for the trivial action of $\mathcal{H}_0$ on X), it is still endowed with a canonical $\mathbb{G}_m$-equivariant structure, given by considering the induced action via the canonical section $\mathbb{G}_m \to \mathcal{H}_0$.

Definition 2.5.2.1. *With the notations above, a* geometric derived foliation $\mathcal{F}$ on X (relative to k) *consists of the datum of a $\mathcal{H}_0$-equivariant derived stack $\mathbb{V}(\mathcal{F})$ over $\widehat{\mathcal{L}}^{gr}(X/k)$, such the following condition is satisfied: for any derived affine scheme S and morphism $S \to X$, the $\mathbb{G}_m$-equivariant morphism*
$$\mathbb{V}(\mathcal{F}) \times_{\widehat{\mathcal{L}}^{gr}(X/k)} \widehat{\mathcal{L}}^{gr}(S/k) \to S$$
makes $\mathbb{V}(\mathcal{F})$ into a perfect derived linear stack over S.

By definition, geometric derived foliations $\mathcal{F}$ on X form a full sub-∞-category of $\mathbf{dSt}^{\mathcal{H}_0}_{/\widehat{\mathcal{L}}^{gr}(X/k)}$, the ∞-category of $\mathcal{H}_0$-equivariant derived stacks over $\widehat{\mathcal{L}}^{gr}(X/k)$.

As a first comment concerning the above definition, it is important to note that $\mathbb{V}(\mathcal{F})$ is not a derived linear stack globally over X, in the exact same manner as the big cotangent complex of a derived foliation is not a quasi-coherent complex (see Definition 2.1.3.4). This is expressed by the strange looking condition in Definition 2.5.2.1. When X is an affine derived scheme, taking $S = X$, this condition simply states that $\mathbb{V}(\mathcal{F})$ is a derived linear stack over X. The same holds, more generally, whenever X is a derived DM stack, since $\mathcal{L}^{gr}(-/k)$ has étale descent. It is shown in [**Alf25**] that the above notion of geometric derived foliations is equivalent to our original notion of derived foliations. The equivalence between the two notions passes through the interpretation of graded mixed dg-modules as quasi-coherent complexes with $\mathcal{H}_0$-action (see §1.5). We will not go into too many details here, and we will limit ourselves to mention some basic facts. As a first comment, we can always restrict to the case where X is an affine derived scheme, and then proceed by gluing for general derived Artin stacks. For a derived foliation $\mathcal{F}$, on an affine derived scheme X, the corresponding geometric derived foliation will be given by $\mathbb{V}(\mathbb{L}_{\mathcal{F}/k}[1])$, the derived linear stack associated to the shifted tangent $\mathbb{L}_{\mathcal{F}/k}[1]$. One of the result of [**Alf25**] states that the graded mixed structure on $\mathbf{DR}(\mathcal{F}/k)$ induces a $\mathcal{H}_0$-action on $\mathbb{V}(\mathbb{L}_{\mathcal{F}/k}[1])$ which makes it into a geometric derived foliation in the sense of Definition 2.5.2.1. Conversely, if $\mathcal{F}$ is a geometric derived foliation on X, the corresponding derived foliation can be obtained by considering the push-forward

$$\pi_*(\mathcal{O}) \in \mathbf{QCoh}(X \times B\mathbb{G}_m),$$

of the structure sheaf along the projection $\mathbb{V}(\mathcal{F}) \to X$. This quasi-coherent sheaf is endowed with a commutative multiplication, as well as a graded mixed structure coming from the action of $\mathcal{H}_0$ (see Corollary 1.5.0.3), and is therefore a graded mixed cdga. The projection $\mathbb{V}(\mathcal{F}) \to \widehat{\mathcal{L}}^{gr}(X/k)$ induces a structure morphism $\mathbf{DR}_{X/k} \to \pi_*(\mathcal{O})$, which makes $\pi_*(\mathcal{O})$ into a derived foliation over X (relative to k).

Definition 2.5.2.1 will be reconsidered (with an important variation) in Chapter 7 in order to define derived foliations in non-zero characteristic.

CHAPTER 3

De Rham cohomology of derived foliations

In this third chapter we study the foliated de Rham cohomology of derived foliations. For this we introduce the notion of crystals along derived foliations. Crystals must be thought of as quasi-coherent complexes endowed with partial flat connections along the foliation, and are natural coefficients for foliated de Rham cohomology. In the third section of the chapter we construct the Hodge filtration associated to a derived foliation $\mathcal{F} \in \mathcal{F}ol(X/k)$, which is a filtration on the de Rham cohomology of X. Its associated graded are identified with the foliated de Rham cohomology with coefficients in exterior powers of the conormal complex. Finally, the conormal complex of a derived foliation possesses very rich structures, and defines a Lie algebra objects inside the ∞-category of crystals. This can be used to study the transversal geometry of a derived foliations, such as transversal jets of functions, but also algebraic invariants related to the classical notion of holonomy.

All along this section k is a fixed $\mathbb{Q}$-algebra.

3.1. Crystals along derived foliations

Let $\mathcal{F} \in \mathcal{F}ol(X/k)$ be a derived foliation on a derived stack X over k, $\mathbf{DR}_{\mathcal{F}}$ the corresponding sheaf of graded mixed cdga's on X, and

$$\mathbf{DR}_{\mathcal{F}} - \epsilon - \mathbf{dg}_X^{gr} := \lim_{\mathbf{Spec}\, A \to X} \mathbf{DR}_{\mathcal{F}}(A) - \epsilon - \mathbf{dg}_k^{gr}$$

the ∞-category of sheaves of graded mixed $\mathbf{DR}_{\mathcal{F}}$-dg-modules on X (see Definition 2.1.3.5 and corollary 2.1.3.6). By definition of derived foliations, the weight 0 part of $\mathbf{DR}_{\mathcal{F}}^{(0)}$ comes equipped with a canonical identification to $\mathcal{O}_X$. Therefore, for any sheaf E of graded mixed $\mathbf{DR}_{\mathcal{F}}$-dg-modules, the weight n part $E^{(n)}$ is endowed with a natural $\mathcal{O}_X$-dg-module structure.

Definition 3.1.0.1. *With the notations above, a* quasi-coherent crystal over $\mathcal{F}$ *is a graded mixed $\mathbf{DR}_{\mathcal{F}}$-dg-module E satisfying the following two conditions.*

- *The weight 0 dg-module $E^{(0)}$, considered as a $\mathbf{DR}_{\mathcal{F}}^{(0)} = \mathcal{O}_X$-dg-module, is quasi-coherent over X.*
- *For any $n \in \mathbb{Z}$, the natural morphism*

$$E(0) \otimes_{\mathcal{O}_X} \mathbf{DR}_{\mathcal{F}}^{(n)} \longrightarrow E^{(n)}$$

is a quasi-isomorphisms of $\mathcal{O}_X$-dg-modules.

The ∞-category of quasi-coherent crystals over $\mathcal{F}$ is the full sub-∞-category $\mathbf{QCoh}(\mathcal{F})$ of $\mathbf{DR}_{\mathcal{F}} - \epsilon - \mathbf{dg}_X^{gr}$ consisting of quasi-coherent crystals.

Remark 3.1.0.2. It should be noted that quasi-coherent crystals on X are not quasi-coherent sheaves on X in the sense that, except for $n = 0$, the weight n pieces $E^{(n)}$ are $\mathcal{O}_X$-dg-modules which are not quasi-coherent over $\mathcal{O}_X$. However, a quasi-coherent crystal is always quasi-coherent over the sheaf of graded mixed cdga's $\mathbf{DR}_{F/k}$.

Various operations on crystals will be studied in Chapter 4, but we will mention here the existence of pull-backs. The ∞-category $\mathbf{QCoh}(\mathcal{F})$ is contravariantly functorial in the pair $(X, \mathcal{F})$ in the following sense. Suppose that we have two pairs $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ consisting of derived schemes endowed with derived foliations. A morphism $f : (X, \mathcal{F}) \longrightarrow (Y, \mathcal{G})$ will be a pair (g, u) consisting of

- a morphism $g : X \to Y$ of derived schemes,
- a morphism $u : \mathcal{F} \to g^*(\mathcal{G})$ of derived foliations over X (i.e. a morphism $\mathbf{DR}_{g^*\mathcal{G}} :=\to \mathbf{DR}_{\mathcal{F}}$ of graded mixed cdga's over X).

Associated to such a morphism $f = (g, u)$ there is a pull-back ∞-functor

$$f^! : \mathbf{QCoh}(\mathcal{G}) \longrightarrow \mathbf{QCoh}(\mathcal{F})$$

constructed as follows. By definition of pull-backs of derived foliations, we have an equivalence of graded mixed cdga's on X

$$\mathbf{DR}_{g^*(\mathcal{G})} \simeq \mathbf{DR}_X \otimes_{g^{-1}(\mathbf{DR}_Y)} g^{-1}(\mathbf{DR}_{\mathcal{G}}).$$

The morphism u thus corresponds to a morphism of sheaves of graded mixed cdgas over X under $\mathbf{DR}_X$

$$u : \mathbf{DR}_X \otimes_{g^{-1}(\mathbf{DR}_Y)} g^{-1}(\mathbf{DR}_{\mathcal{G}}) \longrightarrow \mathbf{DR}_{\mathcal{F}}$$

or equivalently, to a morphism of sheaves of graded mixed cdga's under $g^{-1}(\mathbf{DR}_Y)$

$$u : g^{-1}(\mathbf{DR}_{\mathcal{G}}) \longrightarrow \mathbf{DR}_{\mathcal{F}}.$$

The pull-back ∞-functor $f^!$ on quasi-coherent crystals is thus simply defined by the following formula (for E a graded mixed $\mathbf{DR}_Y$-module)

$$f^!(E) := g^{-1}(E) \otimes_{g^{-1}(\mathbf{DR}_{\mathcal{G}})} \mathbf{DR}_{\mathcal{F}}.$$

Clearly, the rule $((X, \mathcal{F}) \mapsto \mathbf{QCoh}(\mathcal{F}), f \mapsto f^!)$ can be promoted to an ∞-functor $\mathbf{QCoh}^!$ from the ∞-category of pairs $(X, \mathcal{F})$ to the ∞-category of ∞-categories. Note also that $\mathbf{QCoh}(\mathcal{F})$ comes equipped with a natural symmetric monoidal structure (induced by the tensor product of graded mixed $\mathbf{DR}_{\mathcal{F}}$-modules), and that the pull-back $f^!$ has a natural symmetric monoidal structure as well.

Moreover, $f^!$ is compatible with the pull-back of quasi-coherent sheaves on derived stacks in the following sense. By definition of $\mathbf{QCoh}(\mathcal{F})$, we have a forgetful ∞-functor

$$\mathbf{QCoh}(\mathcal{F}) \longrightarrow \mathbf{QCoh}(X)$$

which sends a graded mixed $\mathbf{DR}_{\mathcal{F}}$-module E to its weight zero part $E^{(0)} \in \mathbf{QCoh}(X)$. For a morphism $f = (g,u) : (X,\mathcal{F}) \longrightarrow (Y,\mathcal{G})$ as above,the compatibility of $f^!$ with the pull-back of quasi-coherent sheaves is expressed by the commutativity of the following square

$$\begin{tikzcd} \mathbf{QCoh}(\mathcal{G}) \arrow[r, "f^!"] \arrow[d] & \mathbf{QCoh}(\mathcal{F}) \arrow[d] \\ \mathbf{QCoh}(X) \arrow[r, "g^*"'] & \mathbf{QCoh}(Y), \end{tikzcd}$$

as this can be easily seen using the explicit formula $f^!(E) := g^{-1}(E) \otimes_{g^{-1}(\mathbf{DR}_{\mathcal{G}})} \mathbf{DR}_{\mathcal{F}}$ and the condition (2) of Definition 3.1.0.1.

The following definition fixes some terminology.

Definition 3.1.0.3. *Let $X \in \mathbf{dSt}_k$ and $\mathcal{F} \in \mathcal{F}ol(X/k)$.*

(1) For a quasi-coherent crystal $E \in \mathbf{QCoh}(\mathcal{F})$, the underlying quasi-coherent complex of *E is $E^{(0)}$.*
(2) For $E' \in \mathbf{QCoh}(X)$, a crystal structure on E' along $\mathcal{F}$ *consists of $E \in \mathbf{QCoh}(X)$ together with an equivalence $E^{(0)} \simeq E'$.*

We warn the reader that we will often refer to objects in $\mathbf{QCoh}(\mathcal{F})$ by their underlying objects in $\mathbf{QCoh}(X)$. The typical example is the unit crystal $\mathcal{O}_X$, that is the structure sheaf endowed with its canonical crystal structure. Strictly speaking this object should be written $\mathbf{DR}_{\mathcal{F}}$, as it corresponds to $\mathbf{DR}_{\mathcal{F}}$ as a module over itself. However, we will keep referring to this object with the symbol $\mathcal{O}_X$, to keep in mind that crystals are somehow $\mathcal{D}$-module structures (see Chapter 4 for a precise statement).

3.1.1. Examples. We conclude this section by listing some basic examples of the notion quasi-coherent crystals.

Crystals over the trivial foliation. Assume that X is a derived Artin stack, so the initial derived foliation exists (see Example 2.1.5.1). Now, for $\mathcal{F} = 0_X$ the initial foliation, the ∞-category $\mathbf{QCoh}(\mathcal{F})$ clearly is equivalent to $\mathbf{QCoh}(X)$, the ∞-category of quasi-coherent complexes over X. This equivalence is realized by sending $E \in \mathbf{QCoh}(\mathcal{F})$ to its weight 0 part $E(0) \in \mathbf{QCoh}(X)$. It can be promoted to an equivalence of symmetric monoidal ∞-categories.

Crystals over the tautological foliation. Assume that $\mathcal{F} = *_X$ is the final foliation. If X is a smooth variety, then there is a canonical equivalence of ∞-categories

$$\mathbf{QCoh}(*_X) \simeq \mathcal{D}_X - \mathbf{dg},$$

between quasi-coherent crystals along $*_X$ and quasi-coherent complexes of left $\mathcal{D}_X$-modules. This equivalence will be reviewed and generalized in the next Section 3.2. This equivalence is again compatible with the natural symmetric monoidal structures involved. When X is not smooth anymore, and for instance is a derived scheme, we believe that our notion of quasi-coherent crystals on $*_X$ coincides with the notion of derived $\mathcal{D}_X$-modules of [**Ber21**].

Crystals over the Dolbeault foliation. Let $\mathcal{F} = *_{Dol}$ be the Dolbeault foliation on a smooth variety X, defined by $\mathbf{DR}(\mathcal{F}) := Sym_{\mathcal{O}_X}(\Omega^1_X[1]))$ where the graded cdga $Sym_{\mathcal{O}_X}(\Omega^1_X[1]))$ is endowed with the *zero* mixed structure. Then $\mathbf{QCoh}(*_{Dol})$ is naturally equivalent to the derived ∞-category of complexes of quasi-coherent Higgs sheaves on X in the sense of [**Sim96**]. Dolbeault foliations and the tautological foliations can be combined together into a *Hodge foliation* denoted by $*_{Hod}$. It is a derived foliation on $X \times \mathbb{A}^1$ interpolating between $*_{DR}$ (fiber at 1) and $*_{Dol}$ (fiber at 0). This can be constructed using the deformation to the normal bundle of our appendix A.7 and will be reviewed in our next section concerning the Hodge filtration (see §3.3).

Crystals over integrable foliations. Let $f : X \longrightarrow Y$ be a morphism of derived schemes and $\mathcal{F} := f^*(0_Y)$ be the corresponding derived foliation (see example 2.1.5.3). Then $\mathbf{QCoh}(\mathcal{F})$ is, by definition, the ∞-category of *relative $\mathcal{D}$-modules on X/Y*. When f is a smooth morphism between smooth varieties these relative $\mathcal{D}$-modules can be written as complexes of modules over $\mathcal{D}_{X/Y}$, the sheaf (of algebras) of relative differential operators along the fibers of f. When f is no more supposed to be smooth, we will see that $\mathcal{D}_{X/Y}$ only exists as a sheaf of *dg-algebras* on X (see Definition 4.1.1.2).

Representations of Lie algebras as crystals. Let $X = \mathbf{Spec}\, k$ and $\mathfrak{g}$ be a dual almost perfect dg-Lie algebra over a field k. By theorem 2.2.0.7, L determines a derived foliation $\mathcal{F}_\mathfrak{g} \in \mathcal{F}ol(*)$, for which the graded mixed cdga is the Chevalley-Eilenberg complex $\mathbf{DR}(\mathcal{F}) := \mathsf{CE}^*(\mathfrak{g})$. There exists a natural ∞-functor

$$\phi : \mathfrak{g} - \mathbf{dg}_k \longrightarrow \mathbf{DR}(\mathcal{F}) - \epsilon - \mathbf{dg}^{gr},$$

from $\mathfrak{g}$-dg-modules to the ∞-category of graded mixed $\mathbf{DR}(\mathcal{F})$-dg-modules, given by the Chevalley-Eilenberg complex for dg-modules. As for theorem 2.2.0.7, it can be shown that the ∞-functor ϕ is an equivalence of ∞-categories. This example of course can be generalized to the case of dg-Lie algebroids.

3.2. Foliated de Rham cohomology

The notion of quasi-coherent crystals along derived foliations can be used in order to define a general notion of *foliated de Rham cohomology with coefficients* . Let $X \in \mathbf{dSt}_k$ be any derived stack and $\mathcal{F} \in \mathcal{F}ol(X/k)$ be a derived foliation over X. The symmetric monoidal ∞-category $\mathbf{QCoh}(\mathcal{F})$ sits inside $\mathbf{DR}_\mathcal{F} - \epsilon - \mathbf{dg}^{gr}$. We have a global section ∞-functor

$$\Gamma : \mathbf{DR}_\mathcal{F} - \epsilon - \mathbf{dg}^{gr} \longrightarrow \epsilon - \mathbf{dg}^{gr}_k,$$

which is a lax monoidal ∞-functor to graded mixed complexes. It can be composed with our Tate realization functor of §1.3.2, in order to get a lax monoidal global section ∞-functor

$$\Gamma : \mathbf{DR}_{\mathcal{F}} - \epsilon - \mathbf{dg}^{gr} \longrightarrow \widehat{\mathbf{dg}}_k^{fil}$$

to complete filtered complexes.

Definition 3.2.0.1. *The above lax monoidal ∞-functor Γ, when restricted to $\mathbf{QCoh}(\mathcal{F})$ is denoted by*

$$\widehat{C}_{DR}(\mathcal{F}, -) : \mathbf{QCoh}(\mathcal{F}) \longrightarrow \widehat{\mathbf{dg}}_k^{fil}.$$

For any object $E \in \mathbf{QCoh}(\mathcal{F})$, the complex $\widehat{C}_{DR}(\mathcal{F}, E)$ is called the foliated de Rham cohomology along $\mathcal{F}$ with coefficients in E*.*

By construction, $\widehat{C}_{DR}(\mathcal{F}, E)$ is a complete filtered complex, which is functorial in E, but also in $\mathcal{F}$ and X. The lax monoidal structure provides natural morphisms of complete filtered complexes

$$\widehat{C}_{DR}(\mathcal{F}, E) \widehat{\otimes} \widehat{C}_{DR}(\mathcal{F}, E') \to \widehat{C}_{DR}(\mathcal{F}, E \otimes E').$$

In particular, if E is any (commutative) algebra object (e.g the unit crystal $\mathcal{O}_X$), we get a complete filtered (commutative) dg-algebra of cohomology $\widehat{C}_{DR}(\mathcal{F}, E)$. Typically, the de Rham cohomology of the unit crystal along $\mathcal{F}$ provides a complete filtered cdga $\widehat{C}_{DR}(\mathcal{F}, \mathcal{O}_X)$, where $\mathcal{O}_X$ is the structure sheaf endowed with its canonical crystal structure along $\mathcal{F}$.

Coming back to the basic examples of crystals given in §3.1.1, we can for each of them relate the foliated de Rham cohomology with more classical notions.

Initial and final foliations. We have $\mathbf{QCoh}(0_X) \simeq \mathbf{QCoh}(X)$ and in this case foliated de Rham cohomology simply coincides with quasi-coherent cohomologies of X with coefficients in quasi-coherent complexes. For the final foliation $*_X$ on a smooth variety, $\mathbf{QCoh}(*_X)$ identifies with the ∞-category of quasi-coherent $\mathcal{D}$-modules, and foliated de Rham cohomology simply is the usual de Rham cohomology of X relative to $\mathbf{Spec}\, k$.

Dolbeault foliation. In this case $\mathbf{QCoh}((*_{Dol})$ is identified with quasi-coherent Higgs complexes over X, and foliated de Rham cohomology coincide here with the *Hodge completed derived Higgs cohomology*. This is defined the exact same manner as the Hodge complete derived de Rham cohomology.

Lie algebras representations. For a derived foliation $\mathcal{F}$ on $\mathbf{Spec}\, k$ determines by a dual almost perfect dg-Lie algebra $\mathfrak{g}$, $\mathbf{QCoh}(\mathcal{F})$ identifies with the ∞-category of $\mathfrak{g}$-dg-modules and foliated de Rham cohomology is equivalent to the usual Lie algebra cohomology defined via the Chevalley complex.

3.3. The Hodge filtration

In this section we show that a derived foliation $\mathcal{F}$ on a derived stack X induces a canonical filtration on the completed de Rham complex $\widehat{C}_{DR}(X/k)$, which is moreover multiplicative and functorial in X. We call it the *Hodge filtration associated to* $\mathcal{F}$, as the usual Hodge filtration on $\widehat{C}_{DR}(X/k)$ will be recovered for the special case where $\mathcal{F} = 0_X$ is the initial derived foliation. When the derived foliation $\mathcal{F}$ is integrable by a morphism $f : X \to Y$, the associated Hodge filtration is intimately related to the Gauss-Manin connection with respect to f.

As usual, we start by the case of an affine derived scheme $X = \mathbf{Spec}\, A$, and obtain the globalization by usual gluing. Let $\mathcal{F} \in \mathcal{F}ol(X/k)$ be a derived foliation on X, which we assume to be almost perfect. We consider the corresponding $\mathcal{H}_0$-equivariant derived linear stack $p : \mathbb{V}(\mathcal{F}) \to \mathcal{L}^{gr}(X/k)$ (see §2.5.2), an we perform the deformation to the normal bundle for the morphism p, as recalled in Appendix A.7. This provides a morphism of $\mathcal{H}_0$-equivariant derived stacks over $\mathcal{A} = [\mathbb{A}^1/\mathbb{G}_m]$

$$j : \mathbb{V}(\mathcal{F}) \times \mathcal{A} \to Def(p).$$

Over the open point $[\mathbb{G}_m/\mathbb{G}_m]$, the morphism j recovers the original morphism p, whereas over the closed point we get the zero section of the shifted relative tangent stack $T(p)[1]$, or equivalently $\mathbb{V}(\mathbb{L}_p[-1])$, the derived linear stack associated to the shifted cotangent complex of the morphism p. More generally, $Def(p) \to \mathcal{A}$ is itself a relative derived linear stack, associated to $\mathbb{L}_{X/k}[1] \to \mathbb{L}_{\mathcal{F}}[1]$, considered as a one step filtered quasi-coherent complexes on X, and thus as a quasi-coherent complexes on $\mathcal{A}$ (see the end of §A.7). The cotangent complex $\mathbb{L}_p$ can furthermore be described thanks to the following lemma.

LEMMA 3.3.0.1. *Let $X = \mathbf{Spec}\, A$ be a derived affine k-scheme, E be a bounded above quasi-coherent complex on X and $\pi : \mathbb{V}(E) \to X$ the induced derived linear stack. Then, we have a functorial (in E and X) equivalence in $\mathbf{QCoh}(\mathbb{V}(E))$*

$$\mathbb{L}_\pi \simeq \pi^*(E).$$

Proof of the lemma. For any derived linear stack $\pi : \mathbb{V}(E) \to X$ the relative cotangent complex $\mathbb{L}_\pi$, by definition, satisfies the following property: for any $A \to B$ of connective cdga's, andy $E \to B$ morphism of A-dg-modules corresponding to $u : \mathbf{Spec}\, B \to \mathbb{V}(E)$, and any connective $M \in \mathbf{dg}_B$, we have functorial (in B and M) equivalences

$$\mathbf{Map}_{\mathbf{dg}_B}(u^*(\mathbb{L}_\pi), M) \simeq \mathbf{Map}_{\mathbf{dg}_{A/B}}(E, B \oplus M) \simeq \mathbf{Map}_{\mathbf{dg}_A}(E, M).$$

This shows that $u^*(\mathbb{L}_\pi) \simeq B \otimes_A E$ functorially in B, and thus that $\mathbb{L}_\pi \simeq \pi^*(E)$. This identification is furthermore functorial in E in an obvious manner. $\square$

By passing to $\mathcal{H}_0$-equivariant global functions on $Def(p)$, we get this way a cdga endowed with a complete filtration. The underlying object of the filtration is given by $\mathcal{H}_0$-equivariant functions on $\mathcal{L}^{gr}(X/k)$, i.e. by the completed de Rham complex $\widehat{C}_{DR}(X/k)$. The associated graded for this filtration is given by $\mathcal{H}_0$-equivariant functions on $\mathbb{V}(\mathbb{L}_p[-1])$, and can be written as foliated de Rham cohomology with coefficient in some crystals as follows.

The graded cdga of graded functions on $\mathbb{V}(\mathbb{L}_p[-1])$, thanks to Lemma 3.3.0.1 is given, as a graded $\mathbf{DR}(\mathcal{F})$-cdga, by

$$Sym_{\mathbf{DR}(\mathcal{F})}(\mathbb{L}_p[-1]) \simeq Sym_A(\mathcal{N}^*_{\mathcal{F}}[1]) \otimes_A \mathbf{DR}(\mathcal{F}),$$

where $\mathcal{N}^*_{\mathcal{F}}$ is the conormal complex of $\mathcal{F}$ defined as usual as being the fiber of $\mathbb{L}_{X/k} \to \mathbb{L}_{\mathcal{F}/k}$. We need to be slightly more precise here. The cdga $Sym_{\mathbf{DR}(\mathcal{F})}(\mathbb{L}_p[-1]) \simeq Sym_A(\mathcal{N}^*_{\mathcal{F}}[1]) \otimes_A \mathbf{DR}(\mathcal{F})$ has an action of $\mathcal{H}_0$, its natural graded mixed structure, and a second action of $\mathbb{G}_m$, as $\mathbb{V}(\mathbb{L}_p[-1])$ lives over $B\mathbb{G}_m$. These two actions combined in an action of $\mathbb{G}_m \times \mathcal{H}_0$, and defines a graded objects inside the ∞-category of graded mixed $\mathbf{DR}(\mathcal{F})$-dg-modules. As such we have a decomposition of graded mixed $\mathbf{DR}(\mathcal{F})$-dg-modules

$$Sym_{\mathbf{DR}(\mathcal{F})}(\mathbb{L}_p[-1]) \simeq \bigoplus_{i\geq 0}((\wedge^i \mathcal{N}^*_{\mathcal{F}}) \otimes_A \mathbf{DR}(\mathcal{F}))[i],$$

but for which $(\wedge^i \mathcal{N}^*_{\mathcal{F}})$ is of weight i. In other words, the graded mixed structure on each piece $(\wedge^i \mathcal{N}^*_{\mathcal{F}}) \otimes_A \mathbf{DR}(\mathcal{F})$ is not quite a crystal in the sense of Definition 3.1.0.1 because it is not of weight 0. Rather, the graded mixed $\mathbf{DR}(\mathcal{F})$-dg-modules $(\wedge^i \mathcal{N}^*_{\mathcal{F}}) \otimes_A \mathbf{DR}(\mathcal{F})$ are such that their weight shifts ("Tate twists")

$$((\wedge^i \mathcal{N}^*_{\mathcal{F}}) \otimes_A \mathbf{DR}(\mathcal{F}))(-i)$$

belongs to $\mathbf{QCoh}(\mathcal{F})$.

Therefore, the $\mathcal{H}_0$-action on each piece $((\wedge^i \mathcal{N}^*_{\mathcal{F}}) \otimes_A \mathbf{DR}(\mathcal{F}))(-i)$ induces a canonical crystal structure on the quasi-coherent complexes $\wedge^i \mathcal{N}^*_{\mathcal{F}}$. The associated graded of the filtration on $\widehat{C}_{DR}(X/k)$ can thus be written as

$$|Sym_{\mathbf{DR}(\mathcal{F})}(\mathbb{L}_p[-1])|^t \simeq \bigoplus_{i\geq 0} |(\wedge^i \mathcal{N}^*_{\mathcal{F}}) \otimes_A \mathbf{DR}(\mathcal{F})|^t \simeq \oplus_{\geq 0} \widehat{C}_{DR}(\mathcal{F}, \wedge^i \mathcal{N}^*_{\mathcal{F}})[-i],$$

where the extra shifts $[-i]$ comes from the fact that $(\wedge^i \mathcal{N}^*_{\mathcal{F}}) \otimes_A \mathbf{DR}(\mathcal{F})$ is of weight i rather than weight 0 and thus its Tate realization is shifted by $-2i$ according its definition (see §1.3.2). We gather the above construction into the following proposition.

Propositon 3.3.0.2. *For a perfect derived foliation $\mathcal{F}$ on a derived Artin stack X.*

(1) The wedge powers of the cotangent complex $\wedge^i_{\mathcal{O}_X} \mathcal{N}^*_{\mathcal{F}}$ admit canonical crystal structure along $\mathcal{F}$.

(2) There exists a canonical (descending) filtration $F^*_{\mathcal{F}}$ on the (complete filtered) cdga $\widehat{C}_{DR}(X/k)$ of completed de Rham cohomology on X. It possesses moreover the following properties.

(a) The filtration $F^*_{\mathcal{F}}$ is functorial in X and $\mathcal{F}$.

(b) The associated graded, as a graded cdga, is given by for $i \leq 0$

$$Gr^i_{F^*_{\mathcal{F}}}(\widehat{C}_{DR}(X/k)) \simeq \widehat{C}_{DR}(\mathcal{F}, \wedge^{-i} \mathcal{N}^*_{\mathcal{F}})[i].$$

In the above proposition, we have two extreme cases: when $\mathcal{F} = 0_X$ is the initial derived foliation the filtration $F^*_{\mathcal{F}}$ is the usual Hodge filtration, and when $\mathcal{F} = *_X$ is the final foliation then the filtration $F^*_{\mathcal{F}}$ is the trivial filtration.

Interesting examples of the Hodge filtration already appear for derived foliations over the point $* = \mathbf{Spec}\, k$ for a field k. By corollary 2.2.0.8, we know that an almost perfect derived foliation $\mathcal{F}$ on $\mathbf{Spec}\, k$ (relative to k) is given by a dg-Lie algebra $\mathfrak{g}^\vee$ which is dual almost perfect over k. The ∞-category of quasi-coherent crystals on $\mathcal{F}$ is equivalent to the ∞-category of $\mathfrak{g}$ dg-modules, and de Rham cohomology is given by the Lie algebra cohomology. The completed de Rham complex of X reduces here to k. The conormal complex $\mathcal{N}^*_{\mathcal{F}}$ naturally identifies with $\mathfrak{g}[-1]$, and its crystal structure is given by the coadjoint action. The Hodge filtration is then a filtration on k whose associated graded are given by $C^*(\mathfrak{g}, Sym^i\mathfrak{g}^\vee)[-2i]$, the Lie algebra cohomology with coefficients in the wedge powers of the coadjoint representation. One way to see this filtration in more geometric terms, is to consider the formal derived stack $\mathcal{MC}(\mathfrak{g}) \in \mathbf{dSt}^*_k$ associated to $\mathfrak{g}$, as explained in §2.3.1. By corollary A.8.0.3, we have $\widehat{C}_{DR}(\mathcal{MC}(\mathfrak{g})) \simeq k$, and the usual Hodge filtration on the completed de Rham cohomology provides the Hodge filtration for the derived foliation $\mathcal{F}$.

We finish this part by mentioning another construction of a Hodge filtration, which we will call the *longitudinal Hodge filtration* in order to distinguish it from what we have seen in Proposition 3.3.0.2. For any graded mixed cdga D, we define a new graded mixed cdga D_{Dol} as follows. The underlying graded cdga of D_{Dol} will be $D \otimes_k k[T]$ where the new variable T has weight 0, and the new mixed structure will be $k[T]$-linear and given by $T.\epsilon$, where ϵ is the mixed structure on D. This new graded mixed structure lives in $\epsilon - \mathbf{cdga}^{gr}_{k[T]}$ and is equipped with a canonical compatible $\mathbb{G}_m$-action for which $T^n.D \subset D \otimes_k k[T]$ is of weight n. Finally, the weight 0-part of D_{Dol} is $D^{(0)}[T]$.

We now apply this to a derived foliation $\mathcal{F}$ on a derived affine k-scheme $X = \mathbf{Spec}\, A$ and its corresponding graded mixed cdga $\mathbf{DR}(\mathcal{F}/k)$. The graded mixed cdga $\mathbf{DR}(\mathcal{F}/k)_{Dol}$ is, as a graded cdga, equivalent to $Sym_{A[T]}((\mathbb{L}_{\mathcal{F}} \otimes_A A[T])[1])$, and thus defines a new derived foliation on $X \times \mathbb{A}^1$. Moreover, the $\mathbb{G}_m$-action defines a descent data as a derived foliation on $X \times \mathcal{A}$. This new derived foliation will be denoted by $\mathcal{F}_{Hod}$.

Definition 3.3.0.3. *The* Dolbeault foliation associated to $\mathcal{F}$ *is* $\mathcal{F}_{Hod} \in \mathcal{F}ol(X \times \mathcal{A})$ *defined above.*

Note that the pull-back of $\mathcal{F}_{Hod}$ at $X = [X \times \mathbb{G}_m/\mathbb{G}_m] \hookrightarrow X \times \mathcal{A}$ recovers the original $\mathcal{F}$. On ther other hand, the pull-back at the origin $X \to X \times \mathcal{A}$ is given by the graded mixed cdga $Sym_A(\mathbb{L}_{\mathcal{F}}[1])$ with the zero mixed structure, with the induced morphism $\mathbf{DR}(X/k) \to Sym_A(\mathbb{L}_{\mathcal{F}}[1])$ being the factorization $\mathbf{DR}(X/k) \to \mathcal{O}_X \to Sym_A(\mathbb{L}_{\mathcal{F}}[1])$. This derived foliation will be denoted by $\mathcal{F}_{Dol}$ and is called the Dolbeault foliation associated to $\mathcal{F}$. When $\mathcal{F} = *_{DR}$, the Dolbeault foliation $\mathcal{F}_{Hod}$ is naturally equivalent to $*_{Dol}$ described in our previous section.

Note that $(*_{DR})_{Hod}$, which we will denote by $*_{Hod}$, provides an interpolation between $*_{DR}$ and $*_{Dol}$.

Finally, on the level of foliated cohomology, $\widehat{C}_{DR}(\mathcal{F}_{Hod}, \mathcal{O})$ now has an induced filtration, called the *longitudinal Hodge filtration*, whose associated graded in weight i is $C^*(X, \wedge^i \mathbb{L}_{\mathcal{F}})[-i]$. We gather the previous constructions in the following proposition.

Propositon 3.3.0.4. *For any derived foliation $\mathcal{F}$ on a derived stack X, we can associated functorially $\mathcal{F}_{Hod} \in \mathcal{F}ol(X \times \mathcal{A})$. The pull-back of $\mathcal{F}_{Hod}$ on $X = [X \times \mathbb{G}_m/\mathbb{G}_m] \hookrightarrow X \times \mathcal{A}$ canonically identifies with $\mathcal{F}$, whereas the pull-back of $\mathcal{F}_{Hod}$ at the origin $X \to X \times \mathcal{A}$ is $\mathcal{F}_{Dol}$.*

The foliated de Rham cohomology of F_{Hod} relative to $\mathcal{A}$ defines a (descending) filtration on $\widehat{C}_{DR}(\mathcal{F}, \mathcal{O})$ whose associated graded in weight $i \leq 0$ is $C^*(X, \wedge^{-i} \mathbb{L}_{\mathcal{F}})[i]$.

As a final comment, the two Hodge filtrations given by propositions 3.3.0.2 and 3.3.0.4 are different but are produced using similar constructions based on the exact triangle

$$\mathcal{N}^*_{\mathcal{F}} \to \mathbb{L}_X \to \mathbb{L}_{\mathcal{F}}$$

relating the cotangent complexes and the conormal complex. The filtration of 3.3.0.2 is closely related by the natural $\mathbb{A}^1$-homotopy $\lambda.b$, where $b : \mathbb{L}_{\mathcal{F}} \to \mathcal{N}^*_{\mathcal{F}}[1]$ is the boundary map. Therefore, at the linear level, this filtration is simply induced by the one step filtration $\mathcal{N}^*_{\mathcal{F}} \to \mathbb{L}_X$ on $\mathbb{L}_X$. On the other side the filtration 3.3.0.4 is related to the $\mathbb{A}^1$-homotopy $\lambda.a$ where now $a : \mathbb{L}_X \to \mathbb{L}_{\mathcal{F}}$ is the anchor map. Therefore at the linear level it is given by the one step filtration $\mathbb{L}_{\mathcal{F}}[-1] \to \mathcal{N}^*_{\mathcal{F}}$ on the conormal complex. These filtrations are both of independent interest, depending if one wants to study foliated de Rham cohomology, or rather absolute de Rham cohomology. The Hodge filtration will be used later in the study of characteristic classes associated to foliations (see §7.4).

3.4. Transversal geometry

In the previous section we have seen that conormal complex $\mathcal{N}^*_{\mathcal{F}}$ of a perfect derived foliation $\mathcal{F}$ on a derived Artin stack X carries a canonical crystal structure along $\mathcal{F}$ and thus lifts to an object in $\mathbf{QCoh}(\mathcal{F})$. This object reflects the transversal geometry of $\mathcal{F}$, as when $\mathcal{F}$ is integrable by a morphism $f : X \to Y$ the conormal complex $\mathcal{N}^*_{\mathcal{F}}$ is given by $f^*(\mathbb{L}_Y)$, the pull-back of the cotangent complex of Y, equipped with its natural crystal structure relative to f (see 3.1). In this section we will show moreover that the object $\mathcal{N}_{\mathcal{F}}[-1]$ comes equipped with a natural Lie bracket, encoded in a graded mixed structure on $Sym_{\mathcal{O}_X}(\mathcal{N}^*_{\mathcal{F}}[2])$. More importantly this Lie bracket is compatible with the crystal structure and makes $\mathcal{N}_{\mathcal{F}}[-1]$ into a Lie algebra object in $\mathbf{QCoh}(\mathcal{F})$. The Koszul dual to this Lie object will be by definition the sheaf of transversal jets, which should be understood as the sheaf of formal functions in a transversal direction along $\mathcal{F}$.

To start with let us assume as usual that $X = \mathbf{Spec}\, A$ is a derived affine scheme of finite presentation, and that $\mathcal{F}$ is a perfect derived foliation on X (relative to k). We consider the morphism of graded mixed cdga's $\mathbf{DR}(X) \to \mathbf{DR}(\mathcal{F})$ and its internal de Rham algebra $\mathbf{DR}^{gr}(\mathbf{DR}(\mathcal{F})/\mathbf{DR}(X))$ as introduced in our Appendix A.8. This is a double graded mixed cdga whose underlying bi-graded cdga is given by

$$\mathbf{DR}^{gr}(\mathbf{DR}(\mathcal{F})/\mathbf{DR}(X)) \simeq Sym_{\mathbf{DR}(\mathcal{F})}(\mathbf{DR}(\mathcal{F}) \otimes_A \mathcal{N}^*_{\mathcal{F}}[2]) \simeq \mathbf{DR}(\mathcal{F}) \otimes_A Sym_A(\mathcal{N}^*_{\mathcal{F}}[2]).$$

This bi-graded cdga is endowed with two different graded mixed structure. This first one is induced by the mixed structure on $\mathbf{DR}(X) \to \mathbf{DR}(F)$, whereas the second one is the de Rham differential in $\mathbf{DR}^{gr}(\mathbf{DR}(\mathcal{F})/\mathbf{DR}(X))$. When considered as a graded mixed structure for the former, $\mathbf{DR}(\mathcal{F}) \otimes_A Sym_A(\mathcal{N}^*_{\mathcal{F}}[2])$ defines a commutative algebra object inside $\mathbf{QCoh}(\mathcal{F})$, the symmetric monoidal ∞-category of quasi-coherent crystals along $\mathcal{F}$. The underlying quasi-coherent complex is here $Sym_A(\mathcal{N}^*_{\mathcal{F}}[2])$. The second mixed structure tells us that moreover this commutative algebra object in $\mathbf{QCoh}(\mathcal{F})$ is endowed with an internal graded mixed structure. We can therefore consider its de Rham cohomology of the second kind as in Definition A.8.0.2, and define

$$\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}}} := \widehat{C}^{II}_{DR}(\mathbf{DR}(\mathcal{F})/\mathbf{DR}(X)) = |\mathbf{DR}^{gr}(\mathbf{DR}(\mathcal{F})/\mathbf{DR}(X))|_2$$

which is a complete filtered commutative algebra object in $\mathbf{QCoh}(\mathcal{F})$.

When X is a more general derived Artin stack locally of finite presentation, and $\mathcal{F}$ is a perfect derived foliation on X, we obtain (by gluing the above constructions over all affines over X) a globally defined object $\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}}}$ which is a complete filtered commutative algebra object in $\mathbf{QCoh}(\mathcal{F})$.

Definition 3.4.0.1. *The sheaf of* transversal formal functions *along $\mathcal{F}$ is $\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}}}$ defined above. It is a complete filtered commutative algebra object in $\mathbf{QCoh}(\mathcal{F})$.*

By construction, the underlying object of $\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}}}$ is a sheaf of complete filtered $\mathcal{O}_X$-linear cdga's on X whose associated graded is equivalent to $Sym_{\mathcal{O}_X}(\mathcal{N}^*_{\mathcal{F}})$ where $\mathcal{N}^*_{\mathcal{F}}$ is of weight -1 (remind from (4) before proposition 1.3.2.2 that the associated graded of the realization $|E|$ of a graded mixed complexes is of the form $\oplus_p E^{-p}[-2p]$).

To conclude this part, we mention here that the object $\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}}}$ can be used in order to define a very general notion of algebraic holonomy of $\mathcal{F}$, by using the Tannakian formalism. This algebraic holonomy will be compared to the more classical notion of holonomy in the analytic setting later in §6.4. Note however that the definition below makes sense over any general bases (including non-zero characteristics bases, see §7). It provides a notion of algebraic holonomy defined over smaller fields, for instance number fields, if $\mathcal{F}$ is itself defined over smaller fields.

Let $\mathcal{F}$ be a perfect derived foliation over a derived Artin stack X locally of finite presentation over k. Let K be a k-field and $x \in X(K)$ be a global K-point. We can consider the fiber functor

$$\omega_x : \mathbf{Perf}(\mathcal{F}) \longrightarrow \mathbf{Perf}(K),$$

sending a perfect crystal along $\mathcal{F}$ to its fiber at x. We let $< \mathcal{N}^*_{\mathcal{F}} >^{\otimes} \subset \mathbf{Perf}(\mathcal{F})$ be the smallest thick rigid tensor triangulated ∞-category containing the object $\mathcal{N}^*_{\mathcal{F}}$, or in other words the thick triangulated sub-∞-category generated by all tensor powers of $\mathcal{N}^*_{\mathcal{F}}$ and its duals. We restrict the fiber functor to a symmetric monoidal ∞-functor

$$\omega_x : < \mathcal{N}^*_{\mathcal{F}} >^{\otimes} \to \mathbf{Perf}(K).$$

As explained in [**Toe**], we can form the stack of k-linear tensor automorphisms $aut^{\otimes}(\omega_x)$ of ω_x. This is the (underived) group stack

$$aut^{\otimes}(\omega_x) : \mathbf{Aff}_k^{op} \longrightarrow Gp(\mathbb{T}),$$

sending a (genuine) commutative k-algebra A to the group-like simplicial monoid of self-equivalences of $A \otimes_k \omega_x(-)$ considered as an object in the ∞-category

$$Fun_k^{\infty,\otimes}(< \mathcal{N}_{\mathcal{F}}^* >^{\otimes}, \mathbf{Perf}(A)),$$

of k-linear symmetric monoidal ∞-functors from $< \mathcal{N}_{\mathcal{F}}^* >^{\otimes}$ to $\mathbf{Perf}(A)$. Its classifying stack $Baut^{\otimes}(\omega_x)$ can be shown to be a schematic homotopy type over K in the sense of [**Toe06**]. For the sake of completeness we include a proof.

LEMMA 3.4.0.2. *The (underived) stack $Baut^{\otimes}(\omega_x)$ above is a schematic homotopy type over the field K.*

Proof of the lemma. By Definition [**Toe06**, 3.1.2] we have to prove the following two assertions.

(1) The stack $aut^{\otimes}(\omega_x)$ is affine.
(2) The stack $Baut^{\otimes}(\omega_x)$ is P-local.

As $aut^{\otimes}(\omega_x)$ coincide with $end^{\otimes}(\omega_x)$ (by the natural inclusion), because ω_x is a symmetric monoidal ∞-functor between rigid symmetric monoidal ∞-categories. But $end^{\otimes}(\omega_x)$ is always equivalent to a certain limit of stacks of the form $\underline{End}_K(E)$ for E a perfect complex over K, where $\underline{End}_K(E)$ is the stack of endomorphisms of E. Clearly this last stack is linear and thus affine, and explicitly given by the spectrum of $Sym_K(E \otimes_K E^{\vee})$. As affine stacks are stable by limits the first assertion follows. Moreover, each stack $\underline{End}_K(E)$ is P-local in the sense of [**Toe06**, Def. 3.1.1], as it classifies endomorphisms of E. In other words, for any other K-stack F, we have

$$\mathbf{Map}(F, \underline{End}_K(E)) \simeq \mathbf{Map}_{\mathbf{Perf}(F)}(p^*(E), p^*(E))$$

for $p : F \to \mathbf{Spec}\, K$ the canonical projection. As P-local stacks are stable by limits this finishes the proof of the lemma. ☐

By the lemma 3.4.0.2, we can apply [**Toe06**, Thm. 3.2.9], and show that the sheaves $\pi_i(Baut^{\otimes}(\omega_x)) \simeq \pi_{i-1}(aut^{\otimes}(\omega_x))$ are all representable by affine group schemes over K, which are furthermore products of additive groups when $i > 1$.

DEFINITION 3.4.0.3. *The* algebraic holonomy type *of $\mathcal{F}$ at x is the schematic homotopy type $Baut^{\otimes}(\omega_x)$ defined above, and is denoted by $\widehat{Hol}_{\mathcal{F}}^{alg}(x)$. Its homotopy groups are called the* abstract algebraic holonomy groups *of $\mathcal{F}$ at x .*

REMARK 3.4.0.4. *(1)* As a first comment, we have used the expression *abstract algebraic holonomy groups* in order to emphasis the existence of a more classical/topological version of algebraic holonomy in the analytic setting, which will be discussed in §6.2.

For π_1 a more concrete definition of algebraic holonomy will be also discussed below in Definition 3.4.0.5.

(2) The above definition depends on the choice of a global base point x in $X(K)$. There exists a base-point free version by considering instead the whole stack of k-linear symmetric monoidal ∞-functors $<\mathcal{N}_{\mathcal{F}}^*>\longrightarrow \mathbf{Perf}$, to the stack of perfect complexes. This leads to a stack that deserves the name of *foliated algebraic holonomy type of* $\mathcal{F}$. The sheaf of connected components of this algebraic homotopy type is an algebraic version of an hypothetical *space of leaves* and can be very degenerate in general. A better behaved leaf space will be introduced in the analytic setting in §6.2

(3) Definition 3.4.0.3 can also be extended by considering instead the tensor automorphisms of the fiber functor defined now on the whole ∞-category $\mathbf{Perf}(\mathcal{F})$ rather than just on $<\mathcal{N}_{\mathcal{F}}^*>$. This leads to a similar schematic homotopy type which is called the *algebraic homotopy (or monodromy) type of* $\mathcal{F}$ *at* x*, denoted by* $\widehat{Mon}_{\mathcal{F}}^{alg}(x)$. By restriction, there is a canonical projection morphism from the algebraic monodromy to the algebraic holonomy, which corresponds to some form of a maximal effective quotient, in the same spirit as the maximal effective quotient of a DM stack discussed in 6.2. When X is a smooth variety over a field k, and $\mathcal{F} = *_X$ is the totaulogical foliation, this algebraic holonomy is the differential-Galois homotopy types, and its fundamental group is the usual (affine part) of the differential Galois group of X. In this case the algebraic holonomy is easily seen to be trivial as $\mathcal{N}_{\mathcal{F}}^* = 0$. Finally, in the other extreme case where $\mathcal{F} = 0_X$, the algebraic holonomy is trivial because of the Tannakian reconstruction theorem (see for instance [**BHL17**,**Iwa18**]).

We want to discuss a more concrete realization of the abstract holonomy groups defined above, in particular for $\pi_1(\widehat{Hol}_{\mathcal{F}}^{alg}(x))$, in order to relate it to the more usual notion of holonomy as the action of fundamental groups of leaves on transversals. For this, we first notice that $\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}}}$ can be considered as a complete filtered algebra objects in the ∞-category $Pro(<\mathcal{N}_{\mathcal{F}}^*>)$ of pro-objects. Indeed, as the filtration on $\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}}}$ is concentrated in non-negative degrees, we can extra from it a pro-object

$$\text{"}\lim_{i\leq 0}\text{"}\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}}}/F^i.$$

Each individual piece $\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}}}/F^i$ is here the realization of the truncated internal de Rham algebra $\mathbf{DR}^{gr,\leq i}(\mathbf{DR}(\mathcal{F})/\mathbf{DR}(X))$, which is itself considered as a graded mixed commutative algebra object in $\mathbf{QCoh}(\mathcal{F})$ as explained earlier in this section. Each $\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}}}/F^i$ is endowed with its natural filtration (as being a realization of a graded mixed object) whose associated graded is $Sym_{\mathcal{O}_X}^{\leq i}(\mathcal{N}_{\mathcal{F}}^*)$. This implies that indeed each piece $\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}}}/F^i$ is a commutative algebra object in $<\mathcal{N}_{\mathcal{F}}^*>$, and thus defines $\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}}} = \text{"}\lim_{i\leq 0}\text{"}\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}}}/F^i$ as a pro-object in commutative algebras in $<\mathcal{N}_{\mathcal{F}}^*>$.

The symmetric monoidal ∞-functor ω_x induces an ∞-functor on the corresponding ∞-categories of commutative algebra objects, and we can thus consider the action of the $aut^{\otimes}(\omega_x)$

on $\omega_x(\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}}})$ by automorphisms of complete cdga's. We get this way a natural morphism of group stacks

$$aut^{\otimes}(\omega_x) \longrightarrow aut_{\mathbf{cdga}^{fil}}(\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}},x})$$

where $\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}},x}$ is the fiber of $\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}}}$ at x, and $aut_{\mathbf{cdga}^{fil}}$ denotes the automorphism group stack of self-equivalences of filtered cdga's. Note that $\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}},x}$ can be morally identified here with the cdga of formal functions at x along in the transversal direction. This defines an action of $aut^{\otimes}(\omega_x)$ on the formal transversal at x, which is closer to the usual notion of a holonomy action.

To go a tiny bit further, we assume now $\mathcal{F}$ is transversely smooth, or in other words that $\mathcal{N}^*_F$ is a vector bundle on X. the filtered cdga $\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}},x}$ is thus a discrete filtered cdga, as its associated graded of is $Sym_K(\mathcal{N}^*_{\mathcal{F},x})$, the symmetric algebra over the fiber $\mathcal{N}^*_{\mathcal{F},x}$ of $\mathcal{N}^*_F$ at x. In this case, the group stack $aut_{\mathbf{cdga}^{fil}}(\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}},x})$ is a genuine sheaf of groups, which is moreover representable by an affine group scheme over K which is abstractly isomorphic to the group scheme $\widehat{G}^d$ of (origin preserving) automorphisms of a formal affine space $\widehat{\mathbb{A}}^d$ where d is the rank of $\mathcal{N}^*_{\mathcal{F}}$. The morphism $aut^{\otimes}(\omega_x) \longrightarrow aut_{\mathbf{cdga}^{fil}}(\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}},x})$ thus factors as

$$\pi_1(\widehat{Hol}^{alg}_{\mathcal{F}}(x))) = \pi_0(aut^{\otimes}(\omega_x)) \longrightarrow aut_{\mathbf{cdga}^{fil}}(\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}},x}) \simeq \widehat{G}^d.$$

Definition 3.4.0.5. *With the notations and convention above, if $\mathcal{F}$ is a transversely smooth derived foliation on X, and $x \in X(K)$, the* algebraic holonomy group at x *is the image of the morphism of affine group schemes over K*

$$\pi_1(\widehat{Hol}^{alg}_{\mathcal{F}}(x))) \longrightarrow aut_{\mathbf{cdga}^{fil}}(\widehat{\mathcal{O}}_{\mathcal{N}_{\mathcal{F}},x}).$$

It is denoted by $Hol^{alg}_{\mathcal{F}}(x)$.

Later in this book, the topological version of holonomy will be introduced in the complex analytic setting. We will show that $Hol^{alg}_{\mathcal{F}}(x)$ always contains the topological holonomy as a sub-group. When X and $\mathcal{F}$ are defined over smaller fields, for instance number fields, this can provide interesting informations about the topological holonomy.

CHAPTER 4

Operations on crystals

In this chapter we study furthermore crystals and their functorial properties. We present in particular a new description of crystals, as dg-modules over a certain sheaf of differential operators which is closer to the usual notion of $\mathcal{D}$-modules. We study the standard functorialities of crystals, pull-backs and push-forwards. We use the $\mathcal{D}$-module approach in order to introduce the notion of good filtration on crystals and use it to define characteristic cycles. The Grothendieck-Riemann-Roch theorem then states that characteristic cycles are compatible with push-forward along proper and quasi-smooth morphisms.

All along this section k is a fixed $\mathbb{Q}$-algebra.

4.1. Sheaves of differential operators

In this Section we associate to a derived foliation $\mathcal{F}$ on a derived DM-stack X, a sheaf of filtered dg-algebras $\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}}$ on X, called the sheaf of *differential operators along* $\mathcal{F}$, and prove that dg-modules over the underlying dg-algebra $\mathcal{D}_{\mathcal{F}} := (\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}})^u$ (i.e. $\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}}$ with filtration forgotten) corresponds to quasi-coherent crystals along $\mathcal{F}$.

4.1.1. Sheaf of differential operators. Let $X = \mathbf{Spec}\, A$ be a derived affine k-scheme, and $\mathcal{F}$ be a derived foliation on X. We have a corresponding graded mixed cdga $\mathbf{DR}(\mathcal{F})$, and a canonical augmentation of graded mixed cdga's $\mathbf{DR}(\mathcal{F}) \to A$. This makes A into a graded mixed $\mathbf{DR}(\mathcal{F})$-dg-module.

Remark 4.1.1.1. The graded mixed $\mathbf{DR}(\mathcal{F})$-dg-module A is not a crystal along $\mathcal{F}$, as it is obviously not free as a graded $\mathbf{DR}(\mathcal{F})$-module (unless X is étale over $\mathbf{Spec}\, k$). It should not be confused with the canonical crystal structure on the structure sheaf $\mathcal{O}_X$ in the sense of Definition 3.1.0.3, which corresponds to $\mathbf{DR}(\mathcal{F})$ as a graded mixed dg-module over itself. In this section we will use at several places non-free graded mixed modules, which thus do not correspond to crystals along $\mathcal{F}$.

We consider $\mathbb{R}\underline{\mathcal{H}om}_{\mathbf{DR}(\mathcal{F})}(A, A)$, the internal Hom object of endomorphisms of $A \in \mathbf{DR}(\mathcal{F}) - \epsilon - \mathbf{dg}^{gr}_X$. This is a graded mixed associative algebra in the ∞-category $\mathbf{DR}(\mathcal{F}) - \epsilon - \mathbf{dg}^{gr}_k$ of graded mixed $\mathbf{DR}(\mathcal{F})$-dg-modules

$$\mathbb{R}\underline{\mathcal{H}om}_{\mathbf{DR}(\mathcal{F})}(A, A) \in \mathbf{Alg}(\epsilon - \mathbf{dg}^{gr}_k). \tag{12}$$

Its underlying graded dg-algebra, obtained by forgetting the mixed structure, is explicitly given by

$$\mathbb{R}\underline{\mathcal{H}om}_{\mathbf{DR}(\mathcal{F})}(A,A)^{\notin} \simeq \mathbb{R}\underline{\mathcal{H}om}_{Sym_A(\mathbb{L}_{\mathcal{F}}[1])}(A,A) \simeq \bigoplus_{i\geq 0}(Sym_A^i(\mathbb{L}_{\mathcal{F}}))^{\vee}[-2i],$$

where $(-)^{\vee}$ denotes the $\mathcal{O}_X$-linear dual. When $\mathcal{F}$ is a perfect derived foliation, $\mathbb{L}_{\mathcal{F}}$ is perfect and thus duality commutes with symmetric powers, and in this case we can write furthermore

$$\mathbb{R}\underline{\mathcal{H}om}_{\mathbf{DR}(\mathcal{F})}(A,A)^{\notin} \simeq Sym_A(\mathbb{T}_{\mathcal{F}}[-2])$$

where the tangent complex $\mathbb{T}_{\mathcal{F}} = \mathbb{L}_{\mathcal{F}}^{\vee}$ is here of weight -1. Note that, in general, the mixed structure induced on the right hand side is non-trivial. For instance, it induces a morphism in $\mathbf{dg}_k$ from the piece of weight -2 to the (shift by -1 of the) piece of weight -1 $Sym_A^2(\mathbb{T}_{\mathcal{F}})[-4] \to \mathbb{T}_{\mathcal{F}}[-3]$, which defines a extension class

$$Sym_A^2(\mathbb{T}_{\mathcal{F}}) \to \mathbb{T}_{\mathcal{F}}[1].$$

The corresponding triangle

$$\mathbb{T}_{\mathcal{F}} \longrightarrow \mathcal{D}_{\mathcal{F}}^{[1,2]} \longrightarrow Sym_A^2(\mathbb{T}_{\mathcal{F}})$$

which defines the complex of $\mathcal{D}_{\mathcal{F}}^{[1,2]}$ differential operators of degree bigger than 1 less than 2 along $\mathcal{F}$.

Definition 4.1.1.2. *The* filtered ring of differential operators of $\mathcal{F}$ *is defined to be the realization*

$$\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}} := |\mathbb{R}\underline{\mathcal{H}om}_{\mathbf{DR}(\mathcal{F})}(A,A)|^{t} \in \mathbf{Alg}(\mathbf{dg}_k^{fil}).$$

The ring of differential operators along $\mathcal{F}$ *is the underlying E_1-algebra over X obtained by forgetting the filtration and is denoted by*

$$\mathcal{D}_{\mathcal{F}} := (\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}})^{u} = colim_{i\geq 0} F^i\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}} \in \mathbf{Alg}(\mathbf{dg}_k).$$

Remark 4.1.1.3. The filtered dg-algebra $\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}}$ depends on the base ring k, and should rather be denoted by $\mathcal{D}_{\mathcal{F}/k}^{\mathrm{fil}}$. We will keep the base ring implicit and continue to use the notation $\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}}$ unless necessary.

By construction, $\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}}$ is k-linear associative dg-algebras. As usual, we set

$$\mathcal{D}_{\mathcal{F}}^{\leq i} := F^i\mathcal{D}_{\mathcal{F}},$$

and call $\mathcal{D}_{F}^{\leq i}$ the *ring of differential operators along $\mathcal{F}$ of order $\leq i$.* Using the explicit description of the ∞-functor $|-|^t$ (see §1.3.2 (4)) it is straightforward to verify that, when $\mathcal{F}$ is perfect the associated graded $\mathrm{Gr}(\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}})$ is naturally equivalent to $Sym_{\mathcal{O}_X}(\mathbb{T}_{\mathcal{F}})$, where $\mathbb{T}_{\mathcal{F}}$ has weight 1. When $\mathcal{F}$ is not perfect a more complicated formula holds

$$\mathrm{Gr}(\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}}) \simeq \bigoplus_{i\geq 0}(Sym_A^i(\mathbb{L}_{\mathcal{F}}))^{\vee}[-2i].$$

Suppose that $\mathcal{F}$ is a smooth derived foliation, and X is a non-derived scheme (e.g. a smooth variety), $\mathbb{T}_{\mathcal{F}}$ is a vector bundle on X, and thus $Gr^*(\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}})$ is automatically cohomologically concentrated degree 0. As it is a complete filterted object, this implies that $\mathcal{D}_{\mathcal{F}}^{\leq i}$ is also in degree 0 for all i. Therefore, in this case, $\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}}$ is a genuine filtered k-algebra. For more general X, $\mathcal{D}_{\mathcal{F}}$ is a dg-algebra that might have non-trivial cohomologies in an infinite number of degrees. As $\mathcal{F}$ is assumed to be perfect, $\mathbb{T}_{\mathcal{F}}$ is a perfect A-module, as well as its symmetric powers $Sym_A^i(\mathbb{T}_{\mathcal{F}})$. As a result each $\mathcal{D}_{\mathcal{F}}^{\leq i}$ is a perfect A-module. It is moreover concentrated in non-negative degrees when X is a non-derived scheme. When $\mathcal{F}$ and X are both quasi-smooth (i.e. $\mathbb{L}_{\mathcal{F}}$ and $\mathbb{L}_X$ are perfect of tor-amplitude $[-1,0]$), the dg-algebra $\mathcal{D}_F$ is moreover cohomologically bounded. This can be checked, for instance, using the exact triangles

$$\mathcal{D}_{\mathcal{F}}^{\leq i} \longrightarrow \mathcal{D}_{\mathcal{F}}^{\leq i+1} \longrightarrow Sym_{\mathcal{O}_X}^{i+1}(\mathbb{T}_{\mathcal{F}})$$

and induction on i. More is true: if X is a non-derived scheme and $\mathbb{T}_{\mathcal{F}}$ can be represented by a two term complex of vector bundles $V \to W$, then $\mathcal{D}_{\mathcal{F}}$ is cohomologically concentrated in degrees $[0, rk(W)]$.

Definition 4.1.1.2 can be generalized to non-affine objects by sheafification. However, we have to restrict to Deligne-Mumford stacks, as the construction of $\mathcal{D}_{\mathcal{F}}$ is not stable by pull-backs unless we restrict to étale morphisms. It is possible to define a non-quasi-coherent sheaf of dg-algebras over general derived Artin stacks, but we do not find it particularly helpful so will not discuss this here.

Let $f : X = \mathbf{Spec}\, A \to Y = \mathbf{Spec}\, B$ be a formally étale morphism between derived affine schemes over k, and $\mathcal{F} \in \mathcal{F}ol^p(Y/k)$ be a perfect derived foliation on Y. We contemplate the commutative diagram of graded mixed cdga's

$$\begin{tikzcd}
\mathbf{DR}(B) \arrow[r] \arrow[d] & \mathbf{DR}(A) \arrow[d] \\
\mathbf{DR}(\mathcal{F}) \arrow[r] \arrow[d] & \mathbf{DR}(f^*(\mathcal{F})) \arrow[d] \\
B \arrow[r] & A
\end{tikzcd}$$

and realize that the top square is cartesian by Definition of $f^*(\mathcal{F})$, and the bottom square is cartesian to because f is formally étale. As a result, the exterior square is cartesian. Therefore, the base change ∞-functor on graded mixed dg-modules

$$\mathbf{DR}(f^*(\mathcal{F})) \otimes_{\mathbf{DR}(\mathcal{F})} - : \mathbf{DR}(\mathcal{F}) - \epsilon - \mathbf{dg}_k^{gr} \to \mathbf{DR}(f^*(\mathcal{F})) - \epsilon - \mathbf{dg}_k^{gr}$$

sends B to A via the canonical morphism $\mathbf{DR}(f^*(\mathcal{F})) \otimes_{\mathbf{DR}(\mathcal{F})} B \to A$. This implies that the formation of $\mathcal{D}_{\mathcal{F}}$ is functorial in f and in particular it can be sheafified over the small étale site of any derived Deligne-Mumford stack.

For any derived Deligne-Mumford stack X over k and $\mathcal{F} \in \mathcal{F}ol(X)$, we first have a sheaf of graded mixed associative algebras $\mathbb{R}\underline{\mathcal{H}om}_{\mathbf{DR}_{\mathcal{F},et}}(\mathcal{O}_X, \mathcal{O}_X)$, and its realization

$$\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}} := |\mathbb{R}\underline{\mathcal{H}om}_{\mathbf{DR}_{\mathcal{F},et}}(\mathcal{O}_X, \mathcal{O}_X)| \in \mathbf{Alg}(\epsilon - \mathbf{dg}_k(X)),$$

where $\epsilon - \mathbf{dg}_k(X)$ is the ∞-category of sheaves of graded mixed k-modules over $X_{\text{ét}}$ the small étale site of X.

Definition 4.1.1.4. *For a derived DM-stack X over k, the* filtered sheaf of differential operators of *a derived foliation $\mathcal{F} \in \mathcal{F}ol(X/k)$ is*

$$\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}} := |\mathbb{R}\underline{\mathcal{H}om}_{\mathbf{DR}_{\mathcal{F},et}}(\mathcal{O}_X, \mathcal{O}_X)|^t \in \mathbf{Alg}(\mathbf{dg}_k^{fil}(X))$$

defined above. The sheaf of differential operators along $\mathcal{F}$ *is the underlying E_1-algebra over X obtained by forgetting the filtration and is denoted by*

$$\mathcal{D}_{\mathcal{F}} := (\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}})^u = colim_{i \geq 0} F^i \mathcal{D}_{\mathcal{F}}^{\mathrm{fil}} \in \mathbf{Alg}(\mathbf{dg}_k(X)).$$

When X is a smooth DM-stack over k and $\mathcal{F} = *_X$ is the tautological foliation, $\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}}$ is a sheaf of filtered associative k-algebras whose associated graded in $Sym_{\mathcal{O}_X}(\mathbb{T}_{X/k})$. The following result identifies it with the usual ring of differential operators on X relative to k.

Propositon 4.1.1.5. *Let X be a smooth Deligne-Mumford stack over* $\mathbf{Spec}\, k$. *There is a natural isomorphism of sheaf of filtered k-algebras on $X_{\acute{e}t}$*

$$Diff_X \simeq \mathcal{D}_{*_X}^{\mathrm{fil}}$$

where $Diff_X$ is the usual sheaf of rings of differential operators (relative to k) filtered by order of differential operators.

Proof. We use corollary 1.3.4.4 in order to compute $|\mathbb{R}\underline{\mathcal{H}om}_{\mathbf{DR}_{\mathcal{F},et}}(\mathcal{O}_X, \mathcal{O}_X)|^t$. It can be identified as the internal derived endomorphism objects inside $\widehat{C}_{DR}(-) - \widehat{\mathbf{dg}}_k^{fil}(X)$, the ∞-category of sheaves of complete filtered complexes of $\widehat{C}_{DR}(-)$-modules on $X_{\text{ét}}$

$$|\mathbb{R}\underline{\mathcal{H}om}_{\mathbf{DR}_{\mathcal{F},et}}(\mathcal{O}_X, \mathcal{O}_X)|^t \simeq \mathbb{R}\underline{\mathcal{H}om}_{\widehat{C}_{DR}(-) - \widehat{\mathbf{dg}}_k^{fil}(X)}(\mathcal{O}_X, \mathcal{O}_X).$$

Here $\widehat{C}_{DR}(-)$ is the sheaf of filtered dg-algebras $(U \to X) \mapsto \widehat{C}_{DR}(U, \mathcal{O}_U)$, defined on the small étale site of X and given by the Hodge-completed derived de Rham cohomology of Definition 1.4.2.1. As X is smooth this is nothing else than the genuine de Rham complex $(\Omega_{X/k}^*, dR)$, with $\Omega_{X/k}^i$ in cohomological degrees i, and endowed with its Hodge filtration $\Omega_{X/k}^{\geq p} \subset \Omega_{X/k}^*$. The statement of the proposition is then that the usual ring of differential operators $Diff_X$ appears as

$$Diff_X \simeq \mathbb{R}\underline{\mathcal{H}om}_{\widehat{C}_{DR}(-) - \widehat{\mathbf{dg}}_k^{fil}(X)}(\mathcal{O}_X, \mathcal{O}_X),$$

which is a consequence of the classical Koszul duality between the ring of differential operators and the de Rham complex. One manner to see this is to use the standard resolution of $\mathcal{O}_X$ by locally free module over $Diff_X$ given by the Spencer complex $Diff_X \otimes_{\mathcal{O}_X} \wedge^* \mathbb{T}_{X/k}$

$$0 \longrightarrow Diff_X \otimes_{\mathcal{O}_X} \omega_{X/k} \longrightarrow Diff_X \otimes_{\mathcal{O}_X} \wedge^{d-1} \mathbb{T}_{X/k} \dots \longrightarrow Diff_X \otimes_{\mathcal{O}_X} \mathbb{T}_{X/k} \longrightarrow Diff_X \longrightarrow \mathcal{O}_X$$

where d is the rank of $\mathbb{T}_{X/k}$ (we can assume X connected). This resolution implies the existence of a natural quasi-isomorphism of sheaves of filtered dg-algebras

$$\mathbb{R}\underline{\mathcal{H}om}_{Diff_X - \widehat{\mathbf{dg}}_k^{fil}(X)}(\mathcal{O}_X, \mathcal{O}_X) \simeq \widehat{C}_{DR}(-).$$

By adjunction, this produces a canonical morphism of sheaves of complete filtered dg-algebras on X

$$\phi : Diff_X \longrightarrow \mathbb{R}\underline{\mathcal{H}om}_{\widehat{C}_{DR}(-) - \widehat{\mathbf{dg}}_k^{fil}(X)}(\mathcal{O}_X, \mathcal{O}_X).$$

This last morphism can then be checked to be an equivalence by looking at the associated graded, which reduces to the standard equivalence of graded rings

$$Gr^*(\phi) : Sym_{\mathcal{O}_X}(\mathbb{T}_{X/k}) \simeq \mathbb{R}\underline{\mathcal{H}om}_{\Omega^*_{X/k} - \mathbf{dg}_k^{gr}(X)}(\mathcal{O}_X, \mathcal{O}_X).$$

□

Here are some basic examples of Rings of differential operators.

EXAMPLE 4.1.1.6.

(1) When $\mathcal{F}$ is the *zero foliation* (i.e. $\mathbb{L}_{\mathcal{F}} = 0$) then $\mathcal{D}_{\mathcal{F}} = \mathcal{O}_X$ endowed with the trivial filtration.

(2) When $\mathcal{F}$ is the *tautological foliation* on a smooth variety X (i.e. $\mathbf{DR}(\mathcal{F}) = \mathbf{DR}(X)$), then $\mathcal{D}_{\mathcal{F}} = \mathcal{D}_X$ is the usual ring of differential operators endowed with its usual filtration by the order of operators. This is the proposition 6.1.0.2 above.

(3) When $\mathcal{F}$ is the *Dolbeault foliation* on a smooth variety, that is $\mathbf{DR}(\mathcal{F}) = Sym_{\mathcal{O}_X}(\Omega^1_X[1])$ with trivial mixed structure, then $\mathcal{D}_{\mathcal{F}} = Sym_{\mathcal{O}_X}(\mathbb{T}_X)$ with the split filtration. More generally, when $\mathbf{DR}(\mathcal{F}) = Sym_{\mathcal{O}_X}(\mathbb{L}_{\mathcal{F}}[1])$ with trivial mixed structure (abelian derived foliation), then $\mathcal{D}_{\mathcal{F}} \simeq Sym_{\mathcal{O}_X}(\mathbb{T}_{\mathcal{F}})$.

(4) If the foliation $\mathcal{F}$ is *smooth*, i.e it is a Lie algebroid (see Theorem 2.2.0.7 or [**TV23a**, Section 1.3.1.]), induced by a smooth groupoid G acting on X, then $\mathcal{D}_{\mathcal{F}}$ is the ring of distributions on G, i.e. the $\mathcal{O}_X$-linear dual of formal functions of G, endowed with the convolution product. This coincides with the universal enveloping algebra of the Lie algebroid.

(5) When $\mathcal{F}$ is *globally integrable* by a flat and generically smooth morphism of smooth varieties $f : X \longrightarrow Y$, $\mathcal{D}_{\mathcal{F}}$ is called the dg-algebra of *relative differential operators*. The reason for this name comes from the fact that $H^0(\mathcal{D}_{\mathcal{F}})$ is indeed a subring of $\mathcal{D}_X$ consisting of differential operators stabilizing the fibers of f.

REMARK 4.1.1.7. *(1)* It is interesting to note the following basic example. Let $X = \mathrm{Spec}(k[u])$ with u in degree -2, and $\mathcal{F} = *_X$ be the tautological (i.e. final) derived foliation on X. Then $\mathcal{D}_{\mathcal{F}}$ is here the Weyl dg-algebra over one generator u in degree

-2. In other words, it is the dg-algebra freely generated by two cocycle u and $\frac{\partial}{\partial u}$, respectively in degrees -2 and 2, with the usual commutation relation $[u, \frac{\partial}{\partial u}] = -1$. Note that $\mathcal{D}_{\mathcal{F}}$ is *not* Morita equivalent to k, as opposed to the case when $\deg u$ is odd (see [**Ber21**, Proof of Cor. 4.3.13]). The reader will find more details about $\mathcal{D}_{*_X}$ in [**Ber21**], even for more general X than the ones considered here.

(2) The previous example shows that our notion of differential operators, and thus of crystals (see below) differs from the notion of $\mathcal{D}$-modules use in [**GR14**]. For more precise comparison we refer to [**Buc**].

For X a fixed derived DM-stack, the construction $\mathcal{F} \mapsto \mathcal{D}_{\mathcal{F}}$ defines an ∞-functor $\mathcal{F}ol(X) \to \mathbf{Alg}(\mathbf{dg}^{fil}(X))$: if $\mathcal{F} \to \mathcal{F}'$ is a morphism in $\mathcal{F}ol(X)$, given by a morphism $\varphi : \mathbf{DR}'_{\mathcal{F}} \to \mathbf{DR}_{\mathcal{F}}$ of sheaves of graded mixed cdga's over X, φ induces a morphism on internal endomorphism objects $\mathbb{R}\underline{\mathcal{H}om}_{\mathbf{DR}_{\mathcal{F}}}(\mathcal{O}_X, \mathcal{O}_X) \to \mathbb{R}\underline{\mathcal{H}om}_{\mathbf{DR}'_{\mathcal{F}}}(\mathcal{O}_X, \mathcal{O}_X)$, and thus a morphism $\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}} \to \mathcal{D}^{\mathrm{fil}}_{\mathcal{F}'}$ in $\mathbf{Alg}(\mathbf{dg}^{fil}(X))$. In particular, the maps $0_X \to \mathcal{F} \to *_X$, from the initial and to the final foliations, provide maps of filtered dg-algebras over X

$$\mathcal{O}_X \longrightarrow \mathcal{D}^{\mathrm{fil}}_{\mathcal{F}} \longrightarrow \mathcal{D}^{\mathrm{fil}}_{X},$$

where $\mathcal{O}_X$ has the trivial filtration, and $\mathcal{D}^{\mathrm{fil}}_X$ is by definition the ring of differential operators on X with its natural filtration by order of differential operators. Moreover, the first morphism exhibits $\mathcal{O}_X$ as $F^0\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}}$. By induction on the filtration it is easy to see, when $\mathcal{F}$ is assumed to be perfect, that $\mathcal{D}_{\mathcal{F}}$ is a quasi-coherent complex on X, for the left $\mathcal{O}_X$-module structure given by the morphism above $\mathcal{O}_X \to \mathcal{D}_{\mathcal{F}}$

4.1.2. Quasi coherent $\mathcal{F}$-crystals and $\mathcal{D}_{\mathcal{F}}$-modules. As recalled in §1.3.4.4, the realization functor on X provides an equivalence of symmetric monoidal ∞-categories

$$|-|^t : \epsilon - \mathbf{dg}^{gr}_k \simeq \widehat{\mathbf{dg}}^{fil}_k,$$

where the right hand side is endowed with the complete tensor product of complete filtered complexes. By passing to sheaves on the small étale sites, we have a symmetric monoidal equivalence

$$|-|^t : \epsilon - \mathbf{dg}^{gr}_k(X) \simeq \widehat{\mathbf{dg}}^{fil}_k(X),$$

for any derived DM-stack X over k.

Let us fix a derived DM-stack X and $\mathcal{F} \in \mathcal{F}ol(X)$ a derived foliation on X. We will construct an ∞-functor

$$\theta_X : \mathbf{QCoh}(\mathcal{F}) \longrightarrow \mathcal{D}_{\mathcal{F}} - \mathbf{dg},$$

from quasi-coherent crystals along $\mathcal{F}$ to sheaves of (left) $\mathcal{D}_{\mathcal{F}}$-dg-modules on X_{et}. We start, as in formula (12) above, by letting

$$S_{\mathcal{F}} := \mathbb{R}\underline{\mathcal{H}om}_{\mathbf{DR}_{\mathcal{F},et}}(\mathcal{O}_X, \mathcal{O}_X) \in \mathbf{Alg}(\epsilon - \mathbf{dg}^{gr}(X)),$$

the internal Hom object of endomorphisms of $\mathcal{O}_X$ as a sheaf of graded mixed $\mathbf{DR}_{\mathcal{F},et}$-modules over X_{et}.

The object $\mathcal{O}_X$ can thus be considered as a graded mixed bi-module with its right action by $\mathbf{DR}_{\mathcal{F},et}$ and left action by $S_{\mathcal{F}}$. Therefore it can be used in order to produce the following ∞-functor (between categories of left modules)

$$\mathcal{O}_X \otimes_{\mathbf{DR}_{\mathcal{F},et}} - : \mathbf{DR}_{\mathcal{F},et} - \epsilon - \mathbf{dg}^{gr} \longrightarrow S_{\mathcal{F}} - \epsilon - \mathbf{dg}^{gr}.$$

Note that this ∞-functor obviously sends $\mathbf{DR}_{\mathcal{F},et}$-dg-modules on X which are quasi-coherent on X and graded free on weight zero, i.e. quasi-coherent crystals along $\mathcal{F}$, to graded mixed $S_{\mathcal{F}}$-dg-modules whose underlying graded modules are pure of weight 0 and $\mathcal{O}_X$-quasi-coherent. We thus get an induced ∞-functor

$$\theta'_X : \mathbf{QCoh}(\mathcal{F}) \longrightarrow S_{\mathcal{F}} - \epsilon - \mathbf{dg}^{gr}_{qcoh,w=0},$$

where $S_{\mathcal{F}} - \epsilon - \mathbf{dg}^{gr}_{qcoh,w=0} \subset S_{\mathcal{F}} - \epsilon - \mathbf{dg}^{gr}$ is the full sub-∞-category of objects whose underlying graded $S_{\mathcal{F}}$-module are pure of weight 0 and quasi-coherent over $\mathcal{O}_X$.

We now compose the previous ∞-functor with the Tate realization $|-|^t$ in order to get an ∞-functor to dg-modules over $\mathcal{D}_{\mathcal{F}}$

$$\theta_X : \ \mathbf{QCoh}(\mathcal{F}) \xrightarrow{\theta'_X} S_{\mathcal{F}} - \epsilon - \mathbf{dg}^{gr}_{qcoh,w=0} \xrightarrow{|-|^t} \mathcal{D}_{\mathcal{F}} - \mathbf{dg}.$$

THEOREM 4.1.2.1. *Let X be a derived DM-stack over k and $\mathcal{F} \in \mathcal{F}ol^p(X)$ a perfect derived foliation. The ∞-functor $\theta_X : \mathbf{QCoh}(\mathcal{F}) \to \mathcal{D}_{\mathcal{F}} - \mathbf{dg}$ defined above is fully faithful. Its essential image, $\mathcal{D}_{\mathcal{F}} - \mathbf{dg}^{qcoh}_X$, consists of all $\mathcal{D}_{\mathcal{F}}$-modules over X which are quasi-coherent as $\mathcal{O}_X$-modules*

$$\theta_X : \mathbf{QCoh}(\mathcal{F}) \simeq \mathcal{D}_{\mathcal{F}} - \mathbf{dg}^{qcoh}_X.$$

Proof. The key lemma to prove the theorem is the following one, that is a well known version of Koszul duality in the graded context.

LEMMA 4.1.2.2. *Let A be a cdga and M a perfect A-dg-module. We let $C := Sym_A(E)$. Then, the following holds.*

(1) There exists a natural identification of graded dg-algebras

$$\mathbb{R}\underline{\mathcal{H}om}_C(A, A) \simeq Sym_A(E^\vee[-1]) =: D.$$

(2) Using the identification above makes A into a bi-dg-module over (C, D), and the induced ∞-functor on ∞-categories of graded modules

$$A \otimes_C - : C - \mathbf{dg}^{gr} \longrightarrow D - \mathbf{dg}^{gr}$$

induces an equivalence from the full sub-∞-category of graded C-modules which are graded free over their part of weight 0, and the graded D-modules which are pure of weight 0 as graded modules over k (i.e. comes from graded A-dg-modules via the augmentation $D \to A$).

Proof of the lemma. This is pretty standard. The ∞-functor $A \otimes_C -$ graded modules of the form $C \otimes_A M(0)$ to $M(0)$ with the trivial D-module structure induced by the augmentation

$D \to A$. Conversely, the right adjoint to $A \otimes_C -$ is $\mathbb{R}\underline{\mathcal{H}om}_D(A,-)$. We first prove that for any A-dg-module $M(0)$, considered as graded pure of weight 0, the adjunction morphism

$$C \otimes_A M(0) \to \mathbb{R}\underline{\mathcal{H}om}_D(A, M(0))$$

is an equivalence. For this, we use the explicit cofibrant replacement $QA \twoheadrightarrow A$ in $D-\mathbf{dg}^{gr}$ given explicitly by $QA := Sym_A(E^\vee \oplus E^\vee[-1])$ where the differential on $E^\vee \oplus E^\vee[-1]$ is the usual differential making it contractible $\begin{pmatrix} d & 0 \\ id & -d \end{pmatrix}$. This implies that $\mathbb{R}\underline{\mathcal{H}om}_D(A, M(0))$ is equivalent to $\mathbb{R}\underline{\mathcal{H}om}_A(Sym_A(E^\vee), M(0))$, the graded hom complexes over A of morphisms from $Sym_A(E^\vee)$ to $M(0)$. But, as E is perfect, and as these are computed in the graded modules ∞-categories, we have

$$\mathbb{R}\underline{\mathcal{H}om}_A(Sym_A(E^\vee), M(0)) \simeq Sym(E) \otimes_A M(0)$$

as required. This shows that the adjunction $(A \otimes -, \mathbb{R}\underline{\mathcal{H}om}_D(A,-))$ restricts to an equivalence on the ∞-categories stated in the lemma. $\square$

We now prove the theorem, and we prove first that the ∞-functor

$$\theta'_X : \mathbf{QCoh}(\mathcal{F}) \longrightarrow S(\mathcal{F}) - \epsilon - \mathbf{dg}^{gr}_{qcoh,w=0}$$

is an equivalence of ∞-categories, and then identify the essential image of θ_X.

For this, we consider the ∞-functor that forgets the mixed structures and only retain the graded structures. We have obtain this way a commutative diagram of ∞-categories

$$\begin{tikzcd} \mathbf{DR}_{\mathcal{F},et} - \epsilon - \mathbf{dg}^{gr} \arrow[r, "\theta'_X"] \arrow[d] & S_{\mathcal{F}} - \epsilon - \mathbf{dg}^{gr} \arrow[d] \\ \mathbf{DR}_{\mathcal{F},et} - \mathbf{dg}^{gr} \arrow[r] & S_{\mathcal{F}} - \mathbf{dg}^{gr} \end{tikzcd}$$

where the vertical ∞-functors consists of forgetting the mixed structures. The horizontal bottom ∞-funcor is the one of the lemma $\mathcal{O}_X \otimes_{\mathbf{DR}_{\mathcal{F},et}} -$. Moreover, the above square is right adjointable in the sense of see [**Lur22a**, Def. 4.7.4.13], and the right adjoints are given by $\mathbb{R}\underline{\mathcal{H}om}_{S_{\mathcal{F}}}(\mathcal{O}_X, -)$. The lemma 4.1.2.2 implies that θ'_X is an equivalence of ∞-categories, as the unit of conunit of the involved adjunction can be checked on the level of the underlying graded objects without mixed structures.

To finish the proof of the theorem, we now consider the realization

$$|-|^t : S_{\mathcal{F}} - \epsilon - \mathbf{dg}^{gr}_X \longrightarrow \mathcal{D}^{\mathrm{fil}}_{\mathcal{F}} - \mathbf{dg}^{fil},$$

and observe that, by definition, $\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}} := |S_{\mathcal{F}}|^t$. We know that this ∞-functor is fully faithful, and its image consists of complete filtered $\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}}$-modules over X. We restrict this to $S_{\mathcal{F}}-\epsilon-\mathbf{dg}^{gr}_{qcoh,w=0}$, the full sub-$\infty$-category of graded mixed module which are quasi-coherent and of weight 0. Its image by $|-|^t$ is then easily checked to consist of all filtered $\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}}$-modules E satisfying the following two conditions

(1) The filtration on E is tautological: $F^i(E) = E$ if $i \geq 0$ and 0 if $i < 0$.
(2) The underlying $\mathcal{O}_X$-module of E is quasi-coherent.

Now, these two conditions define a full sub-∞-category of $\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}} - \mathbf{dg}^{fil}$ which is equivalent, via the underlying object ∞-functor $(-)^u : \mathbf{dg}^{fil}_k(X) \to \mathbf{dg}_k(X)$, to the ∞-category $\mathcal{D}_{\mathcal{F}} - \mathbf{dg}_{qcoh}$, of *unfiltered* $\mathcal{D}_{\mathcal{F}}$-modules that are $\mathcal{O}_X$-quasi-coherent. ☐

The following definition/notation will be used throughout the rest of this chapter.

Definition 4.1.2.3. Let $\mathcal{F} \in \mathcal{F}ol(X)$ be a perfect derived foliation over a derived scheme X. We denote by $\mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F})$ the full sub-∞-category of $\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}} - \mathbf{dg}^{fil}$ consisting of all filtered modules which are quasi-coherent as filtered $\mathcal{O}_X$-modules via restriction of scalars along the natural morphism of filtered dg-algebras $\mathcal{O}_X \longrightarrow \mathcal{D}^{\mathrm{fil}}_{\mathcal{F}}$. We will call $\mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F})$ the *∞-category of filtered quasi-coherent crystals along $\mathcal{F}$.*

Remark 4.1.2.4. Without the perfect condition the theorem 4.1.2.1 does not hold. In fact, in general the sheaf $\mathcal{D}_{\mathcal{F}}$ has no reasons of being quasi-coherent over $\mathcal{O}_X$, even if $\mathcal{F}$ is almost perfect.

4.1.3. Inverse image and induction ∞-functors. Let $u : \mathcal{F} \to \mathcal{G}$ be a morphism of derived foliations on a derived DM-stack X, which we assume to be perfect in order to be able to use our theorem 4.1.2.1. We have seen that it induces a morphism of sheaves of filtered dg-algebras over X

$$\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}} \longrightarrow \mathcal{D}^{\mathrm{fil}}_{\mathcal{G}}.$$

Associated to this, we have the usual forgetful and base-change adjunction on the corresponding ∞-categories of sheaves of filtered modules

$$u_! := \mathcal{D}^{\mathrm{fil}}_{\mathcal{G}} \otimes_{\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}}} - : \mathcal{D}^{\mathrm{fil}}_{\mathcal{F}} - \mathbf{dg}^{fil} \rightleftarrows \mathcal{D}^{\mathrm{fil}}_{\mathcal{G}} - \mathbf{dg}^{fil} : u^!. \tag{13}$$

The right adjoint is called the *inverse image ∞-functor*. The left adjoint $u_!$ is called the *induction along u* or *direct image ∞-functor*. By forgetting the filtrations, we have a corresponding non-filtered adjunction

$$u_! := \mathcal{D}_{\mathcal{G}} \otimes_{\mathcal{D}_{\mathcal{F}}} - : \mathcal{D}_{\mathcal{F}} - \mathbf{dg} \rightleftarrows \mathcal{D}_{\mathcal{G}} - \mathbf{dg} : u^!. \tag{14}$$

Both ∞-functors $u_!$ and $u^!$ preserve quasi-coherence, and thus induce an adjunction on quasi-coherent modules. By Theorem 4.1.2.1, this can also be interpreted as an adjunction on the ∞-category of quasi-coherent crystals

$$u_! : \mathbf{QCoh}(\mathcal{F}) \rightleftarrows \mathbf{QCoh}(\mathcal{G}) : u^!$$

where again, $u^!$ is called the inverse image ∞-functor, and $u_!$ the induction or direct image ∞-functor. Through Definition 4.1.2.3, the filtered adjunction (13) can be regarded as an adjunction on filtered crystals, as well

$$u_! : \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F}) \rightleftarrows \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{G}) : u^!.$$

The filtered and unfiltered versions of $u_!$ and $u^!$ are of course compatible with the underlying object ∞-functor, i.e. the following squares naturally commutes

$$\begin{tikzcd} \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F}) \arrow[r, "u_!"] \arrow[d, "(-)^u"'] & \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{G}) \arrow[d, "(-)^u"] \\ \mathbf{QCoh}(\mathcal{F}) \arrow[r, "u_!"'] & \mathbf{QCoh}(\mathcal{G}) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{G}) \arrow[r, "u^!"] \arrow[d, "(-)^u"'] & \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F}) \arrow[d, "(-)^u"] \\ \mathbf{QCoh}(\mathcal{G}) \arrow[r, "u^!"'] & \mathbf{QCoh}(\mathcal{F}) \end{tikzcd} \tag{15}$$

The same is true when the underlying object ∞-functor $(-)^u$ is replaced with the associated graded ∞-functor Gr.

$$\begin{tikzcd} \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F}) \arrow[r, "u_!"] \arrow[d, "(-)^u"'] & \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{G}) \arrow[d, "Gr"] \\ Gr(\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}}) - \mathbf{dg}^{gr} \arrow[r, "u_!"'] & Gr(\mathcal{D}^{\mathrm{fil}}_{\mathcal{G}}) - \mathbf{dg}^{gr} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{G}) \arrow[r, "u^!"] \arrow[d, "(-)^u"'] & \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F}) \arrow[d, "Gr"] \\ Gr(\mathcal{D}^{\mathrm{fil}}_{\mathcal{G}}) - \mathbf{dg}^{gr} \arrow[r, "u^!"'] & Gr(\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}}) - \mathbf{dg}^{gr} \end{tikzcd} \tag{16}$$

Remark 4.1.3.1. Tracking back the equivalence of Theorem 4.1.2.1 it is easy to see that the inverse image functor $u^! : \mathbf{QCoh}(\mathcal{F}) \longrightarrow \mathbf{QCoh}(\mathcal{G})$ may also be identified with the base change at the level of graded mixed dg-modules

$$\mathbf{DR}(\mathcal{F}) \otimes_{\mathbf{DR}(\mathcal{G})} - : \mathbf{DR}(\mathcal{G}) - \epsilon - \mathbf{dg}^{gr}_X \longrightarrow \mathbf{DR}(\mathcal{F}) - \epsilon - \mathbf{dg}^{gr}_X$$

for the morphism $\mathbf{DR}(\mathcal{G}) \longrightarrow \mathbf{DR}(\mathcal{F})$ corresponding to $u : \mathcal{F} \to \mathcal{G}$ in $\mathcal{F}ol(X)$. In other words, it does coincide with the !-pull-back of quasi-coherent crystals previously defined in Section 3.1 along the morphism of pairs $(id_X, u) : (X, \mathcal{F}) \to (X, \mathcal{G})$.

We will now examine two specific important cases of adjunctions (13) and (14): when either $\mathcal{F}$ is the initial foliation, or $\mathcal{G}$ is the final foliation (so that the morphism u is uniquely defined in both cases).

Let us first consider the morphism $u : 0_X \to \mathcal{F}$. We know that $\mathbf{QCoh}(0_X)$ is naturally equivalent to $\mathbf{QCoh}(X)$. The corresponding induction ∞-functor

$$u_! : \mathbf{QCoh}(X) \longrightarrow \mathbf{QCoh}(\mathcal{F})$$

is then simply induced by $\mathcal{D}_{\mathcal{F}} \otimes_{\mathcal{O}_X} -$. We warn the reader that this ∞-functor does not have an easy description on the level of graded mixed modules, and this shows a particular instance of the usefulness of Theorem 4.1.2.1. For example, $u_!$ sends $\mathcal{O}_X$ to $\mathcal{D}_{\mathcal{F}}$ which is a rather big and complicated object inside $\mathbf{QCoh}(\mathcal{F})$, not concentrated in degree 0 (except if $\mathcal{F}$ and X are both assumed to be smooth). The ∞-functor $u_!$ will play an important role for us later and will be referred to as *the induction ∞-functor for $\mathcal{F}$*. There is also a corresponding filtered version

$$u_! : \mathbf{QCoh}^{\mathrm{fil}}(X) \longrightarrow \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F}),$$

which takes a filtered object E in $\mathbf{QCoh}(X)$ and sends it to $\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}} \otimes_{\mathcal{O}_X} E$.

DEFINITION 4.1.3.2. *Let $\mathcal{F} \in \mathcal{F}ol(X)$ be a perfect derived foliation and $u : 0_X \to \mathcal{F}$ the canonical morphism. The* induction ∞-functor for $\mathcal{F}$ *is the ∞-functor*

$$\mathsf{Ind}_{\mathcal{F}} := u_! : \mathbf{QCoh}(X) \longrightarrow \mathbf{QCoh}(\mathcal{F}).$$

The filtered induction ∞-functor for $\mathcal{F}$ *is the ∞-functor*

$$\mathsf{Ind}^{\mathrm{fil}}_{\mathcal{F}} := u_! : \mathbf{QCoh}^{\mathrm{fil}}(X) \longrightarrow \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F}).$$

A direct consequence of the existence of the induction ∞-functor is the following important fact.

COROLLARY 4.1.3.3. *Let X be a derived DM-stack over lk. If the ∞-category $\mathbf{QCoh}(X)$ admits a compact generator, then the same is true for the ∞-categories $\mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F})$ and $\mathbf{QCoh}(\mathcal{F})$.*

Proof. Pick a family of compact generators $\{E_i\}_{i\in I}$ in $\mathbf{QCoh}(X)$. It is formal to check that the family of induced objects $\mathsf{Ind}_{\mathcal{F}}(E_i) \in \mathbf{QCoh}(\mathcal{F})$ is a family of compact generators. In the filtered case, we first consider the family of objects $(i_n)_!(E_i) \in \mathbf{QCoh}^{\mathrm{fil}}(X)$, for $\in \mathbb{Z}$ and $i \in I$. Here $(i_n)_!(E_i)$ is the filtered object in $\mathbf{QCoh}(X)$ whose filtration layers are all zero below n and are equal to E_i in layers above n. In other words, we have

$$\mathbf{Map}_{\mathbf{QCoh}^{\mathrm{fil}}(X)}((i_n)_!(E_i), E') \simeq \mathbf{Map}_{\mathbf{QCoh}(X)}(E_i, F^nE').$$

The family of objects $\{(i_n)_!E_i\}_{n,i}$ is a family of compact generators of $\mathbf{QCoh}^{\mathrm{fil}}(X)$, and it is formal to check that the induced family $\mathsf{Ind}^{\mathrm{fil}}_{\mathcal{F}}(\{(i_n)_!E_i)\}_{n,i}$ is a again a family of compact generators of $\mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F})$. $\square$

COROLLARY 4.1.3.4. *Let X be either a quasi-compact and quasi-separated derived k-scheme, or a separated and quasi-compact derived Deligne-Mumford k-stack, and let $\mathcal{F} \in \mathcal{F}ol^p(X/k)$ a perfect derived foliation on X. Then the ∞-categories $\mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F})$ and $\mathbf{QCoh}(\mathcal{F})$ are both compactly generated.*

Proof. If X is a quasi-compact and quasi-separated derived scheme then $\mathbf{QCoh}(X)$ is compactly generated by [**Toë12**, Thm. 3.7]. When X is a separated and quasi-compact derived DM-stack it possesses a coarse moduli space $\pi : X \to M$ where M is a separated and quasi-compact derived algebraic space (see for instance [**AHPS23**, Thm. 3.12]). We can write $\mathbf{QCoh}(X) \simeq \Gamma(M_{et}, \pi_*(\mathbf{QCoh}))$, the global sections of the quasi-coherent stack $\pi_*(\mathbf{QCoh})$ on M_{et}, the small étale site of M. Locally on M_{et} the derived stack X is equivalent to a quotient stack $[V/G]$ with V affine and G a finite group, and thus $\pi_*(\mathbf{QCoh})$ is locally compacty generated as a quasi-coherent stack on M_{et}. We can thus apply [**Lur22b**, Thm. 10.3.2.1], which implies that $\Gamma(M_{et}, \pi_*(\mathbf{QCoh})) \simeq \mathbf{QCoh}(X)$ is compactly generated.

We thus have seen that in both cases we can apply the corollary 4.1.3.3 and therefore have proven the result. $\square$

REMARK 4.1.3.5. Slightly more is true for corollary 4.1.3.4, as we also know that the compact objects in $\mathbf{QCoh}(X)$ are the perfect complexes, are both results [**Toë12**, Thm. 3.7] and [**Lur22b**, Thm. 10.3.2.1] identify the compact objects in the global section ∞-categories as the objects being compact locally.

The other interesting special case of (14) is that of the unique morphism $u : \mathcal{F} \to *_X$ to the final foliation $*_X$. By definition $\mathcal{D}_{*_X} = \mathcal{D}_X$ is the sheaf of differential operators on X, which coincides with the usual sheaf of differential operators when X is smooth (see proposition 4.1.1.5). The induction ∞-functor, in this situation, produces a ∞-functor

$$u_! : \mathbf{QCoh}(\mathcal{F}) \longrightarrow \mathcal{D}_X - \mathbf{dg}^{qcoh},$$

from quasi-coherent crystals along $\mathcal{F}$ to quasi-coherent $\mathcal{D}_X$-modules on X. The $\mathcal{D}_X$-modules of the form $u_!(E)$ will be called *induced from the foliation* $\mathcal{F}$. One important example is the induced $\mathcal{D}_X$-module $u_!(\mathcal{O}_X)$. This is a canonical $\mathcal{D}_X$-module on X associated to the derived foliation $\mathcal{F}$ which contains interesting informations about $\mathcal{F}$. For instance, when $\mathcal{F}$ is smooth, then $u_!(\mathcal{O}_X)$ is a coherent $\mathcal{D}_X$-module. However, this is not true anymore for non-smooth derived foliations $\mathcal{F}$. Measuring the defect of coherence of $u_!(\mathcal{D}_X)$ is a very interesting question related to invariants of singularities of derived foliations, generalizing classical invariants such as Milnor numbers. We will touch the general study of singularities of derived foliations in this book and details will appear in a future work.

4.2. Direct and inverse images of filtered crystals

In this Section we define (filtered and unfiltered) direct image functors between quasi-coherent crystals, along proper and quasi-smooth maps. We also prove the important result that filtered direct and filtered inverse images commutes both with the underlying and the associated graded objects functors.

4.2.1. Direct images. We have seen in §3.1 that $(X, \mathcal{F}) \mapsto \mathbf{QCoh}(\mathcal{F})$ is a contraviant ∞-functor by using !-pull-backs. This functoriality can be extended to the filtered case as follows. Let $f = (g, u) : (X, \mathcal{F}) \longrightarrow (Y, \mathcal{G})$ be a morphism of pairs consisting of derived DM-stacks endowed with perfect derived foliations. Thus f is given by a morphism $g : X \to Y$ of derived stacks, and a morphism $u : \mathcal{F} \to g^*(\mathcal{G})$ of derived foliations on X (i.e. a morphism $\mathbf{DR}_{g^*(\mathcal{G})} \to \mathbf{DR}_{\mathcal{F}}$ of graded mixed cdga's over X). This pair induces a morphism on the corresponding sheaves of differential operators

$$\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}} \to \mathcal{D}^{\mathrm{fil}}_{g^*(\mathcal{G})}.$$

We consider the sheaf of filtered k-modules on X defined by

$$\mathcal{D}^{\mathrm{fil}}_f := |\mathbb{R}\underline{\mathcal{H}om}_{g^{-1}(\mathbf{DR}_{\mathcal{G}})-\epsilon-\mathbf{dg}^{gr}}(g^{-1}(\mathcal{O}_Y), \mathcal{O}_X)|^t.$$

This obviously carries a right filtered $g^{-1}(\mathcal{D}^{\mathrm{fil}}_{\mathcal{G}})$-module structure by via the canonical morphism of sheaves filtered dg-algebras

$$g^{-1}(\mathcal{D}^{\mathrm{fil}}_{\mathcal{G}}) = g^{-1}|(\mathbb{R}\underline{\mathcal{H}om}_{\mathbf{DR}_{\mathcal{G}}-\epsilon-\mathbf{dg}^{gr}}(\mathcal{O}_Y, \mathcal{O}_Y)|^t \to |\mathbb{R}\underline{\mathcal{H}om}_{g^{-1}(\mathbf{DR}_{\mathcal{G}})-\epsilon-\mathbf{dg}^{gr}}(g^{-1}(\mathcal{O}_Y), g^{-1}(\mathcal{O}_Y))|^t.$$

The filtered complex $\mathcal{D}_f^{\mathrm{fil}}$ also carries a left filtered by $\mathcal{D}_{g^*(\mathcal{G})}^{\mathrm{fil}}$ by means of the natural morphism of sheaves of filtered dg-algebras

$$\mathcal{D}_{g^*(\mathcal{G})}^{\mathrm{fil}} = |\mathbb{R}\underline{\mathcal{H}om}_{(\mathbf{DR}_X \otimes_{g^{-1}(\mathbf{DR}_Y)} g^{-1}(\mathbf{DR}_{\mathcal{G}}))-\epsilon-\mathbf{dg}^{gr}}(\mathcal{O}_X, \mathcal{O}_X)|^t \longrightarrow |\mathbb{R}\underline{\mathcal{H}om}_{(g^{-1}(\mathbf{DR}_{\mathcal{G}}))-\epsilon-\mathbf{dg}^{gr}}(\mathcal{O}_X, \mathcal{O}_X)|^t.$$

The filtered bi $(\mathcal{D}_{g^*(\mathcal{G})}^{\mathrm{fil}}, g^{-1}(\mathcal{D}_{\mathcal{G}}^{\mathrm{fil}})$-module $\mathcal{D}_f^{\mathrm{fil}}$ defines a pull-back ∞-functor on filtered modules

$$\phi : \mathcal{D}_{\mathcal{G}}^{\mathrm{fil}} - \mathbf{dg}^{fil} \longrightarrow \mathcal{D}_{g^*(\mathcal{G})}^{\mathrm{fil}} - \mathbf{dg}^{fil}$$

by the standard formula

$$\phi(E) := \mathcal{D}_f^{\mathrm{fil}} \otimes_{g^{-1}(\mathcal{D}_{\mathcal{G}}^{\mathrm{fil}})} g^{-1}(E).$$

Finally, we can compose this with the forgetful ∞-functor for the morphism of filtered dg-algebras $u : \mathcal{D}_{g^*(\mathcal{G})}^{\mathrm{fil}} \to \mathcal{D}_{\mathcal{F}}^{\mathrm{fil}}$ in order to get the requires pull-back ∞-functor

$$f^! : \mathcal{D}_{\mathcal{G}}^{\mathrm{fil}} - \mathbf{dg}^{fil} \longrightarrow \mathcal{D}_{\mathcal{F}}^{\mathrm{fil}} - \mathbf{dg}^{fil}.$$

On the underlying $\mathcal{O}$-modules, the ∞-functor $f^!$ acts as the usual pull-back, as shown by the following lemma.

Lemma 4.2.1.1. *With the notations and assumptions above, we have a natural equivalence of filtered $\mathcal{O}_X$-modules*

$$\mathcal{D}_f^{\mathrm{fil}} \simeq \mathcal{O}_X \otimes_{g^{-1}(\mathcal{O}_Y)} g^{-1}(\mathcal{D}_{\mathcal{G}}^{\mathrm{fil}}).$$

In particular, the underlying $\mathcal{O}_X$-module of $f^!(E)$ is naturally equivalent to $f^*(E)$ and is therefore quasi-coherent. As a consequence we have a commutative square of ∞-functors

$$\begin{tikzcd} \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{G}) \arrow[r, "f^!"] \arrow[d] & \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F}) \arrow[d] \\ \mathbf{QCoh}^{\mathrm{fil}}(Y) \arrow[r, "g^*"'] & \mathbf{QCoh}^{\mathrm{fil}}(X), \end{tikzcd}$$

where the vertical ∞-functors are the forgetful ∞-functors to filtered quasi-coherent complexes.

Proof of the lemma. This computations have already been seen in the lemma 4.1.2.2, because $\mathcal{G}$ is perfect we have an equivalence of sheaves of graded mixed complexes

$$\mathbb{R}\underline{\mathcal{H}om}_{g^{-1}(\mathbf{DR}_{\mathcal{G}})-\epsilon-\mathbf{dg}^{gr}}(g^{-1}(\mathcal{O}_Y), \mathcal{O}_X) \simeq \mathbb{R}\underline{\mathcal{H}om}_{g^{-1}(\mathbf{DR}_{\mathcal{G}})-\epsilon-\mathbf{dg}^{gr}}(g^{-1}(\mathcal{O}_Y), g^{-1}(\mathcal{O}_Y)) \otimes_{g^{-1}(\mathcal{O}_Y)} \mathcal{O}_X.$$

Passing to the realization provides the wanted result

$$\mathcal{D}_f^{\mathrm{fil}} \simeq \mathcal{O}_X \otimes_{g^{-1}(\mathcal{O}_Y)} g^{-1}(\mathcal{D}_{\mathcal{G}}^{\mathrm{fil}}).$$

The rest of the lemma follows directly from the first assumption as the quasi-coherent pull-back is given by the standard formula

$$g^*(E) \simeq g^{-1}(E) \otimes_{g^{-1}(\mathcal{O}_Y)} \mathcal{O}_X.$$

□

Acoording to the previous lemma we do have constructed the pull-back of filtered crystals

$$f^! : \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{G}) \longrightarrow \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F}).$$

Definition 4.2.1.2. *The ∞-functor constructed above is called the* filtered pull-back of filtered crystals.

A first important property is the existence of a left adjoint to the !-pull back, given by the !-pushfoward. It will exists under sufficient properness and quasi-smoothness conditions to insure that the standard direct image of quasi-coherent sheaves preserves perfect complexes.

Propositon 4.2.1.3. *Let $f = (g, u) : (X, \mathcal{F}) \to (Y, \mathcal{G})$ be a morphism between derived DM-stacks endowed with perfect derived foliations. We assume that $g : X \to Y$ is proper and quasi-smooth, and that Y is either a quasi-compact and quasi-separated derived scheme, or a separated and quasi-compact derived DM-stack. Then, the ∞-functor*

$$f^! : \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{G}) \longrightarrow \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F})$$

admits a left adjoint

$$f_! : \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F}) \longrightarrow \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{G}).$$

Proof. We already know from Corollary 4.1.3.3 that both ∞-categories are compactly generated. For the existence of $f_!$ it is thus enough to check that $f^!$ commutes with limits. For this, we use that $f^!$ is compatible with the usual pull-backs of filtered $\mathcal{O}$-modules along g (see Lemma 4.2.1.1)

$$\begin{array}{ccc} \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{G}) & \longrightarrow & \mathbf{QCoh}^{\mathrm{fil}}(Y) \\ {\scriptstyle f^!}\big\downarrow & & \big\downarrow{\scriptstyle (g^*)^{\mathrm{fil}}} \\ \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F}) & \longrightarrow & \mathbf{QCoh}^{\mathrm{fil}}(X). \end{array}$$

The horizontal ∞-functors are clearly conservative and commute with limits and colimits. Therefore, to check that $f^!$ preserves limits, it is enough to show that $(g^*)^{\mathrm{fil}}$ does so. Again, as $(g^*)^{\mathrm{fil}}$ is obtained by extending the usual pull-back $g^* : \mathbf{QCoh}(Y) \longrightarrow \mathbf{QCoh}(X)$ to filtered objects, we are reduced to show that g^* commutes with limits.

This last step follows easily from the assumption that g is proper and quasi-smooth. Indeed, let $E \in \mathbf{QCoh}(X)$ be a compact generator (thus a perfect complex on X), and $\{F_i\}_{i\in I}$ a diagram in $\mathbf{QCoh}(Y)$. Using that E is dualizable and the projection formula, we have

$$Map(E, g^*(\lim_i F_i)) \simeq Map(\mathcal{O}_X, E^\vee \otimes_{\mathcal{O}_X} g^*(\lim_i F_i)) \simeq Map(\mathcal{O}_Y, g_*(E^\vee) \otimes_{\mathcal{O}_Y} (\lim_i F_i)).$$

Now we use that g is proper and quasi-smooth, so that $g_*(E^\vee)$ is again perfect and thus dualizable, and therefore the functor $g_*(E^\vee) \otimes_{\mathcal{O}_Y} -$ commutes with limits. So we have

$$Map(\mathcal{O}_Y, g_*(E^\vee) \otimes_{\mathcal{O}_Y} (\lim_i F_i)) \simeq \lim_i Map(\mathcal{O}_Y, g_*(E^\vee) \otimes_{\mathcal{O}_Y} F_i) \simeq \lim_i Map(E, g^*(F_i)).$$

This shows that the canonical map

$$Map(E, g^*(\lim_i F_i)) \longrightarrow \lim_i Map(E, g^*(F_i)) \simeq Map(E, \lim_i g^*(F_i))$$

is indeed in equivalence, so that g^* preserves limits (since E is a generator of $\mathbf{QCoh}(Y)$). $\square$

The proposition 4.2.1.3 allows us to give the following definition.

DEFINITION 4.2.1.4. *Let $f = (g, u) : (X, \mathcal{F}) \longrightarrow (Y, \mathcal{G})$ be a morphism with $g : X \to Y$ proper and quasi-smooth and satisfying the condition of Proposition 4.2.1.3. The* filtered direct image *is the ∞-functor*

$$f_! : \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F}) \longrightarrow \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{G}),$$

left adjoint to the pull-back ∞-functor $f^!$.

Note that for $f = (\mathrm{id}_X, u : 0_X \to \mathcal{F})$, we clear have $f_! = \mathsf{Ind}^{\mathrm{fil}}_{\mathcal{F}}$ (see Definition 4.1.3.2).

REMARK 4.2.1.5. Note that, although it is possible, we do not try to define direct images for non-proper or non-quasi-smooth morphisms (this would require working with Ind-coherent crystals rather than quasi-coherent ones). In the rest of the book, we will only need direct images for proper and quasi-smooth morphisms.

We also define the unfiltered direct image

$$f_! : \mathbf{QCoh}(\mathcal{F}) \longrightarrow \mathbf{QCoh}(\mathcal{G})$$

as being the left adjoint to the unfiltered version of $f^!$ whose existence is proved similarly.

The direct image functors satisfy the usual pseudo-functoriality properties, $(ff')_! \simeq f_! f'_!$ (for composeable, proper and quasi-smooth f and f'). An important consequence of this property is the following result.

PROPOSITON 4.2.1.6. *Let $f = (g, u) : (X, \mathcal{F}) \longrightarrow (Y, \mathcal{G})$ be a morphism with g proper and quasi-smooth, $\mathcal{F}$ and $\mathcal{G}$ perfect, and Y either a qcqs derived scheme or a separated and quasi-compact derived DM-stack. Then, the following diagram naturally commutes*

$$\begin{tikzcd}
\mathbf{QCoh}^{\mathrm{fil}}(X) \arrow[r, "\mathsf{Ind}^{\mathrm{fil}}_{\mathcal{F}}"] \arrow[d, "g_*(\omega_{X/Y}\otimes -)[d]"'] & \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F}) \arrow[d, "f_!"] \\
\mathbf{QCoh}^{\mathrm{fil}}(Y) \arrow[r, "\mathsf{Ind}^{\mathrm{fil}}_{\mathcal{G}}"'] & \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{G}),
\end{tikzcd}$$

where $\omega_{X/Y}$ is the relative canonical line bundle of X over Y, and d the relative dimension of X over Y.

Proof. We have a commutative diagram of pairs

$$\begin{tikzcd} (X, 0_X) \arrow[r, "u"] \arrow[d, "g"'] & (X, \mathcal{F}) \arrow[d, "f"] \\ (Y, 0_Y) \arrow[r, "v"'] & (Y, \mathcal{G}) \end{tikzcd}$$

where u and v are the unique morphisms from the initial foliation. We get from this a natural isomorphism of ∞-functors

$$f_! u_! \simeq v_! g_!.$$

The proposition then follows from the fact that, by definition $u_!$ and $v_!$ are the induction ∞-functors, and from the explicit formula for $g_! : \mathbf{QCoh}(X) \longrightarrow \mathbf{QCoh}(Y)$ in terms of relative Serre duality. □

An interesting application of proposition 4.2.1.6 is to the direct image of $\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}}$ itself. Keeping the same notations we obtain

$$f_!(\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}}) \simeq \mathcal{D}^{\mathrm{fil}}_{\mathcal{G}} \otimes_{\mathcal{O}_Y} g_*(\omega_{X/Y})[d].$$

When X is proper and quasi-smooth over k, and $f : (X, \mathcal{F}) \longrightarrow (\operatorname{Spec} k, 0)$ is the projection to the point (endowed with its trivial foliation), we see in particular that $f_!(\mathcal{D}_{\mathcal{F}})$ computes $H^{*+d}(X, \omega_X)$, that is *coherent homology of X with coefficients in $\mathcal{O}_X$.*

The following result is a direct consequence of the fact that $f^!$ commutes with colimits (and can also be deduced by using proposition 4.2.1.6 and the proof of corollary 4.1.3.3).

Corollary 4.2.1.7. *Let $f = (g, u) : (X, \mathcal{F}) \to (Y, \mathcal{G})$ be a morphism of derived schemes endowed with perfect derived foliations, with g proper and quasi-smooth and Y being either a qcqs derived scheme or a separated quasi-compact derived DM-stack. Then, the direct image ∞-functors*

$$f_! : \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F}) \longrightarrow \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{G}) \qquad f_! : \mathbf{QCoh}(\mathcal{F}) \longrightarrow \mathbf{QCoh}(\mathcal{G})$$

preserve compact objects.

4.2.2. Compatibility with underlying and associated graded objects. We conclude this section with the important result that filtered direct images commute with both taking underlying and associated graded objects. For this, let us consider a morphism $f = (g, u) : (X, \mathcal{F}) \longrightarrow (Y, \mathcal{G})$ with g proper and quasi-smooth, and assume the usual assumptions of proposition 4.2.1.3. We have the corresponding adjunction on filtered crystals

$$f^{\mathrm{fil}}_! : \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F}) \rightleftarrows \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{G}) : f^!_{\mathrm{fil}},$$

and its unfiltered version

$$f_! : \mathbf{QCoh}(\mathcal{F}) \rightleftarrows \mathbf{QCoh}(\mathcal{G}) : f^!.$$

We may also consider $\mathcal{F}^{\epsilon=0}$ which is the derived foliation whose underlying graded mixed cdga is $\mathbf{DR}(\mathcal{F})^{\epsilon=0}$ (i.e. with trivial mixed structure $\epsilon = 0$), and the same for $\mathcal{G}^{\epsilon=0}$. For such derived foliations, we have now a graded push-forward

$$f_!^{\mathrm{gr}} : \mathbf{QCoh}^{\mathrm{gr}}(\mathcal{F}^{\epsilon=0}) \longrightarrow \mathbf{QCoh}^{\mathrm{gr}}(\mathcal{G}^{\epsilon=0})$$

defined as the left adjoint to the graded pull-back $f^!_{\mathrm{gr}} : \mathbf{QCoh}^{\mathrm{gr}}(\mathcal{G}^{\epsilon=0}) \to \mathbf{QCoh}^{\mathrm{gr}}(\mathcal{F}^{\epsilon=0})$. Note that the ∞-categories $\mathbf{QCoh}^{\mathrm{gr}}(\mathcal{F}^{\epsilon=0})$ and $\mathbf{QCoh}^{\mathrm{gr}}(\mathcal{G}^{\epsilon=0})$ can also be identified with the categories of graded $Sym_{\mathcal{O}_X}(\mathbb{T}_{\mathcal{F}})$-modules over X and of graded $Sym_{\mathcal{O}_Y}(\mathbb{T}_{\mathcal{G}})$-modules over Y, respectively (where $\mathbb{T}_{\mathcal{F}}$ and $\mathbb{T}_{\mathcal{G}}$ both sit in weight 1). We leave to the reader the precise construction of these graded functorialities, which are obtained using similarly as the filtered case.

By putting all these functors together, we may write the following diagram of vertical adjunctions

(17)
$$\begin{tikzcd}
\mathbf{QCoh}^{\mathrm{gr}}(\mathcal{F}^{\epsilon=0}) \arrow[d, shift right, "f_!^{\mathrm{gr}}"'] & \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F}) \arrow[l, "\mathrm{Gr}"'] \arrow[r, "(-)^u"] \arrow[d, shift right, "f_!^{\mathrm{fil}}"'] & \mathbf{QCoh}(\mathcal{F}) \arrow[d, shift right, "f_!"'] \\
\mathbf{QCoh}^{\mathrm{gr}}(\mathcal{G}^{\epsilon=0}) \arrow[u, shift right, "f^!_{\mathrm{gr}}"'] & \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{G}) \arrow[l, "\mathrm{Gr}"] \arrow[r, "(-)^u"'] \arrow[u, shift right, "f^!_{\mathrm{fil}}"'] & \mathbf{QCoh}(\mathcal{G}) \arrow[u, shift right, "f^!"']
\end{tikzcd}$$

We already know that this diagram, when restricted to the inverse images only, naturally commutes. This implies that the diagram restricted to direct images is naturally lax commutative. In fact, the induced natural transformations

$$f_!^{\mathrm{gr}} \circ \mathrm{Gr} \Rightarrow \mathrm{Gr} \circ f_!^{\mathrm{fil}} \qquad f_! \circ (-)^u \Rightarrow (-)^u \circ f_!^{\mathrm{fil}}$$

turn out to be equivalences. Indeed, as all ∞-functors involved commute with colimits, it is enough to check this property on compact generators, and we can thus use proposition 4.2.1.6 and the proof of corollary 4.1.3.3 to conclude. As a consequence of the commutativity of (17), we are allowed to (and will from now on) simply write $f_!$ and $f^!$ *without* any decorations $(-)^{\mathrm{fil}}$ or $(-)^{\mathrm{gr}}$. Because of its importance, and for later reference, we state this result in the following corollary.

Corollary 4.2.2.1. *With the notations and assumptions above, the filtered direct and filtered inverse images of quasi-coherent crystals commute with taking the underlying or the associated graded objects, i.e. either considering its $f^!$ part or its $f_!$ part, the following diagram naturally commutes*

(18)
$$\begin{tikzcd}
\mathbf{QCoh}^{\mathrm{gr}}(\mathcal{F}^{\epsilon=0}) \arrow[d, shift right, "f_!"'] & \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F}) \arrow[l, "\mathrm{Gr}"'] \arrow[r, "(-)^u"] \arrow[d, shift right, "f_!"'] & \mathbf{QCoh}(\mathcal{F}) \arrow[d, shift right, "f_!"'] \\
\mathbf{QCoh}^{\mathrm{gr}}(\mathcal{G}^{\epsilon=0}) \arrow[u, shift right, "f^!"'] & \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{G}) \arrow[l, "\mathrm{Gr}"] \arrow[r, "(-)^u"'] \arrow[u, shift right, "f^!"'] & \mathbf{QCoh}(\mathcal{G}) \arrow[u, shift right, "f^!"']
\end{tikzcd}$$

Here is an equivalent way of looking at the leftmost adjunction $f_!^{\mathrm{gr}} : \mathbf{QCoh}^{\mathrm{gr}}(\mathcal{F}^{\epsilon=0}) \leftrightarrows \mathbf{QCoh}^{\mathrm{gr}}(\mathcal{G}^{\epsilon=0}) : f^{\mathrm{gr}\,!}$ of (17) that avoids introducing the auxiliary foliations $\mathcal{F}^{\epsilon=0}$ and $\mathcal{G}^{\epsilon=0}$. First of all, we have the associated graded object ∞-functor

$$\mathrm{Gr} : \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F}) = \mathcal{D}_{\mathcal{F}}^{\mathrm{fil}} - \mathbf{dg}_{qcoh}^{fil} \longrightarrow Gr(\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}}) - \mathbf{dg}_{qcoh}^{gr},$$

from $\mathcal{O}_X$-quasi-coherent sheaves of filtered $\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}}$-dg-modules on X to $\mathcal{O}_X$-quasi-coherent sheaves of graded $Gr(\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}})$-dg-modules on X (and similarly for $\mathcal{G}$ and Y). Note that $\mathrm{Gr}(\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}}) \simeq Sym_{\mathcal{O}_X}(\mathbb{T}_{\mathcal{F}})$ and $\mathrm{Gr}(\mathcal{D}_{\mathcal{G}}^{\mathrm{fil}}) \simeq Sym_{\mathcal{O}_Y}(\mathbb{T}_{\mathcal{G}})$ in the ∞-category of sheaves of graded associative dg-algebras $\mathrm{Alg}(\mathbf{dg}^{gr}(X))$ and $\mathrm{Alg}(\mathbf{dg}^{gr}(Y))$ (with $\mathbb{T}_{\mathcal{F}} = \mathbb{L}_{\mathcal{F}}^{\vee}$ and $\mathbb{T}_{\mathcal{G}} = \mathbb{L}_{\mathcal{G}}^{\vee}$ of pure weight 1). Now, we proceed as in §4.2.1 and consider

$$\mathrm{Gr}(\mathcal{D}_f^{\mathrm{fil}}) \simeq g^*(\mathrm{Gr}(\mathcal{D}_{\mathcal{G}}^{\mathrm{fil}})) = \mathcal{O}_X \otimes_{g^{-1}(\mathcal{O}_Y)} g^{-1}(\mathrm{Gr}(\mathcal{D}_{\mathcal{G}}^{\mathrm{fil}})).$$

which is a graded $(\mathrm{Gr}(\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}}), g^{-1}(\mathcal{D}_{\mathcal{G}}^{\mathrm{fil}}))$-bi-module over X, and as such, it defines a pull-back ∞-functor on graded modules

$$f_{\mathrm{gr}}^! : Gr(\mathcal{D}_{\mathcal{G}}^{\mathrm{fil}}) - \mathbf{dg}^{gr} \longrightarrow Gr(\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}}) - \mathbf{dg}^{gr}$$

by

$$f_{\mathrm{gr}}^!(E) := \mathrm{Gr}(\mathcal{D}_f^{\mathrm{fil}}) \otimes_{g^{-1}(\mathrm{Gr}(\mathcal{D}_{\mathcal{G}}^{\mathrm{fil}}))} g^{-1}(E).$$

This ∞-functor respects the property of being quasi-coherent over X and Y, so it induces a ∞-functor

$$f_{\mathrm{gr}}^! : Gr(\mathcal{D}_{\mathcal{G}}^{\mathrm{fil}}) - \mathbf{dg}_{qcoh}^{gr} \longrightarrow Gr(\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}}) - \mathbf{dg}_{qcoh}^{gr}$$

which has a left adjoint

$$f_!^{\mathrm{gr}} : Gr(\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}}) - \mathbf{dg}_{qcoh}^{gr} \leftrightarrows Gr(\mathcal{D}_{\mathcal{G}}^{\mathrm{fil}}) - \mathbf{dg}_{qcoh}^{gr} : f_{\mathrm{gr}}^!,$$

that coincides with the leftmost adjunction of (17), via the natural equivalenes $\mathbf{QCoh}^{\mathrm{gr}}(\mathcal{F}^{\epsilon=0}) \simeq Gr(\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}}) - \mathbf{dg}_{qcoh}^{gr}$, and $\mathbf{QCoh}^{\mathrm{gr}}(\mathcal{G}^{\epsilon=0}) \simeq Gr(\mathcal{D}_{\mathcal{G}}^{\mathrm{fil}}) - \mathbf{dg}_{qcoh}^{gr}$.

4.3. Characteristic cycles

In this section we introduce the notion of *characteristic cycle* of a quasi-coherent crystal (or of a $\mathcal{D}_{\mathcal{F}}$-module thanks to the equivalence of Theorem 4.1.2.1) along a derived foliation $\mathcal{F}$. For this we first introduce the global cotangent stack $T^*\mathcal{F}$ of a derived foliation $\mathcal{F}$, which is a derived Artin n-stack, where n is the tor-amplitude of the perfect complex $\mathbb{L}_{\mathcal{F}}$. We will then discuss the notion of *bounded coherent crystals* and of *good filtrations* on them. By definition, the associated graded to a good filtration will be a $\mathbb{G}_m$-equivariant perfect complex on $T^*\mathcal{F}$, that will be used to define *characteristic cycles*. We investigate the existence of good filtrations and prove some independence (of good filtrations) results for characteristic cycles.

4.3.1. Cotangent stacks of derived foliations. Let X be a derived scheme and E be a perfect complex on X of tor-amplitude contained in $[0, n]$ for some non-negative integer n (see [**TV07**] for the notion of tor-amplitude of perfect complexes). Recall from §2.5.1 that the linear stack $\mathbb{V}(E)$ associated to E comes equipped with its natural $\mathbb{G}_m$-action, covering the projection $\pi : \mathbb{V}(E) \to X$. We pass to the corresponding morphism on quotient stacks

$$\pi^{\mathrm{gr}} : [\mathbb{V}(E)/\mathbb{G}_m] \longrightarrow X \times B\mathbb{G}_m.$$

The direct image along this morphism is a symmetric lax monoidal ∞-functor

$$\pi^{\mathrm{gr}}_* : \mathbf{QCoh}^{\mathrm{gr}}(\mathbb{V}(E)) := \mathbf{QCoh}([\mathbb{V}(E)/\mathbb{G}_m]) \longrightarrow \mathbf{QCoh}(X \times B\mathbb{G}_m) =: \mathbf{QCoh}^{\mathrm{gr}}(X). \tag{19}$$

Since the structure sheaf $\mathcal{O} := \mathcal{O}_{[\mathbb{V}(E)/\mathbb{G}_m]}$ is the monoidal unit in $\mathbf{QCoh}^{\mathrm{gr}}(\mathbb{V}(E))$, the lax-monoidal the ∞-functor π^{gr}_* in (19) factors via a ∞-functor (denoted by the same symbol)

$$\pi^{\mathrm{gr}}_* : \mathbf{QCoh}^{\mathrm{gr}}(\mathbb{V}(E)) \longrightarrow \pi^{\mathrm{gr}}_*(\mathcal{O}) - \mathbf{dg}^{gr}_{qcoh},$$

from graded quasi-coherent complexes on $\mathbb{V}(E)$ to graded $\pi^{\mathrm{gr}}_*(\mathcal{O})$-modules over X which are quasi-coherent as $\mathcal{O}_X$-modules.

Propositon 4.3.1.1. *There exists a fully faithful ∞-functor*

$$Sym_{\mathcal{O}_X}(E) - \mathbf{dg}^{gr}_{\mathrm{perf}} \longrightarrow \mathbf{QCoh}^{\mathrm{gr}}(\mathbb{V}(E))$$

from perfect graded $Sym_{\mathcal{O}_X}(E)$-modules over X, sending the i-th weight twist $Sym_{\mathcal{O}_X}(E)(i)$ of the tautological graded module $Sym_{\mathcal{O}_X}(E)$, to the i-th weight twist $\mathcal{O}_{\mathbb{V}(E)}(i)$ of the structure sheaf of $\mathbb{V}(E)$.

Proof. We start by assuming that $X = Spec\, A$ is affine, with A a connective cdga, and by [**Mon21**] we known that there is a natural equivalence graded cdga's over X

$$\pi^{\mathrm{gr}}_*(\mathcal{O}) \simeq Sym_{\mathcal{O}_X}(E).$$

The direct image ∞-functor

$$\pi^{\mathrm{gr}}_* : \mathbf{QCoh}^{\mathrm{gr}}(\mathbb{V}(E)) \longrightarrow \pi^{\mathrm{gr}}_*(\mathcal{O}) - \mathbf{dg}^{gr}_{qcoh,X},$$

from graded quasi-coherent complexes on $\mathbb{V}(E)$ to graded $\pi_*(\mathcal{O})$-modules over X which are quasi-coherent as $\mathcal{O}_X$-modules possesses a left adjoint π^*_{gr}

$$\pi^*_{gr} : \pi^{\mathrm{gr}}_*(\mathcal{O}) - \mathbf{dg}^{gr}_{qcoh} \simeq Sym_{\mathcal{O}_X}(E) - \mathbf{dg}^{gr}_{perf} \longrightarrow \mathbf{QCoh}^{\mathrm{gr}}(\mathbb{V}(E)).$$

As the unit of this adjunction

$$Sym_{\mathcal{O}_X}(E) \to \pi^{gr}_*(\pi^*_{gr}(Sym_{\mathcal{O}_X}(E)) = \pi^{gr}_*(\mathcal{O})$$

is an equivalence, the ∞-functor π^*_{gr}, when restricted to the thick triangulated sub-∞-category generated by the objects $\mathcal{O}(i)$, induces an equivalences with perfect graded $Sym_{\mathcal{O}_X}(E)$-modules over X. This proves the proposition when X is affine.

The case where X is a general derived Artin stack easily follows by descent from the affine case thanks to the commutative diagram

$$\begin{tikzcd} Sym_{\mathcal{O}_X}(E)-\mathbf{dg}^{gr}_{perf} \arrow[r] \arrow[d, "\simeq"'] & \mathbf{QCoh}^{\mathrm{gr}}(\mathbb{V}(E)) \arrow[d, "\simeq"] \\ \lim_{u:Spec\,A\to X}\left(Sym_A(u^*(E))-\mathbf{dg}^{gr}_{perf}\right) \arrow[r] & \lim_{u:Spec\,A\to X}\mathbf{QCoh}^{\mathrm{rm}}(\mathbb{V}(u^*(E))). \end{tikzcd}$$

$\square$

The proposition 4.3.1.1 will be applied later to the particular case where $E=\mathbb{T}_{\mathcal{F}}$, the tangent complex of a derived foliation $\mathcal{F}$ on X. The derived stack $V(\mathbb{T}_{\mathcal{F}})$ will be denoted by $T^*\mathcal{F}$, and will be called the *global derived cotangent stack* of $\mathcal{F}$.

Definition 4.3.1.2. *For a derived scheme X and a perfect derived foliation $\mathcal{F}\in\mathcal{F}ol^p(X)$, the* (derived) cotangent stack of $\mathcal{F}$ *is defined by*

$$T^*\mathcal{F}:=V(\mathbb{T}_{\mathcal{F}}).$$

equipped with is natural $\mathbb{G}_m$-action.

In the next section, we will use the functor of the proposition 4.3.1.1 in order to define *characteristic cycles* of crystals on $\mathcal{F}$ admitting a good filtration.

4.3.2. Coherent crystals and good filtrations. Let X be a separated and quasi-compact derived Deligne-Mumford stack and $\mathcal{F}\in\mathcal{F}ol^p(X)$ be a perfect derived foliation on X. We let $E\in\mathbf{QCoh}(\mathcal{F})$ be a quasi-coherent crystal. Via the equivalence of our theorem 4.1.2.1, we will freely identify E with a $\mathcal{D}_{\mathcal{F}}$-module quasi-coherent over X.

Definition 4.3.2.1. *Let X, $\mathcal{F}$ and E as above.*

(1) The crystal E is called coherent *if it is a compact object in $\mathbf{QCoh}(\mathcal{F})$. The full sub-$\infty$-category of coherent crystals is denoted by $\mathbf{Coh}(\mathcal{F})\subset\mathbf{QCoh}(\mathcal{F})$.*
(2) For $E\in\mathbf{Coh}(\mathcal{F})$, a good filtration *on E is the data of a compact object $E^{\mathrm{fil}}\in\mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F})$ together with an equivalence in $\mathbf{QCoh}(\mathcal{F})$*

$$(E^{\mathrm{fil}})^u\simeq E.$$

Compact objects in $\mathbf{QCoh}(\mathcal{F})$ and in $\mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F})$ can be easily characterized, either through the induction ∞-functors or as sheaves of $\mathcal{D}_{\mathcal{F}}$-modules.

Propositon 4.3.2.2. *Let X and $\mathcal{F}$ be as above. An object $E\in\mathbf{QCoh}(\mathcal{F})$ (resp. $E^{\mathrm{fil}}\in\mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F})$) is compact if and only if it satisfies one of the following two equivalent conditions.*

(1) The object E belongs to the thick triangulated sub-category generated by objects of the form $\mathsf{Ind}_{\mathcal{F}}(E_0)$ (resp. $\mathsf{Ind}^{\mathrm{fil}}_{\mathcal{F}}(E_0^{\mathrm{fil}})$) for a perfect complex of $\mathcal{O}_X$-modules E_0 (resp. a filtered perfect complex of $\mathcal{O}_X$-modules E_0^{fil}).

(2) E (resp. E^{fil}) is a perfect $\mathcal{D}_{\mathcal{F}}$-module (resp. a perfect $\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}}$-module) i.e. locally on $X_{\acute{e}t}$, E (resp. E^{fil}) is a retract of a finite cell $\mathcal{D}_{\mathcal{F}}$-module (resp. of a finite cell filtered $\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}}$-module).

Proof. Condition (1) have been already considered in the proof of Corollary 4.1.3.3. Clearly, condition (1) implies condition (2). Finally condition (2) clearly implies compactness when X is affine. The general case follows from the fact that X is assumed to be separated and quasi-compact and therefore 4.1.3.4 holds and from the fact that any object satisfying condition (2) is a compact object (see our remark 4.1.3.5). $\square$

Suppose that E is a coherent crystal along $\mathcal{F}$, equipped with a good filtration E^{fil} in the sense of definition 4.3.2.1 above. By proposition 4.3.2.2, $\mathrm{Gr}(E^{\mathrm{fil}})$ is a perfect graded $\mathrm{Gr}(\mathcal{D}^{\mathrm{fil}}_{\mathcal{F}}) = Sym_{\mathcal{O}_X}(\mathbb{T}_{\mathcal{F}})$-module over X. Now, proposition 4.3.1.1 implies that $\mathrm{Gr}(E^{\mathrm{fil}})$ defines a graded perfect complex on the stack $T^*\mathcal{F}$, and by forgetting the $\mathbb{G}_m$-action we get a perfect complex on $T^*\mathcal{F}$. We consider the K-theory spectrum $K(T^*\mathcal{F})$, defined as the K-theory of the ∞-category of perfect (not graded) modules over $T^*\mathcal{F}$. The perfect complex $\mathrm{Gr}(E^{\mathrm{fil}})$ on $T^*\mathcal{F}$ thus defines a class

$$Ch(E^{\mathrm{fil}}) \in K_0(T^*\mathcal{F}).$$

DEFINITION 4.3.2.3. *The* characteristic cycle *of E^{fil} is the element*

$$Ch(E^{\mathrm{fil}}) \in K_0(T^*\mathcal{F})$$

defined above .

Note that the above definition depends a priori on the choice of E^{fil}. We will see later that, in fact, modulo a class of *phantoms objects*, it does not (see §4.3.2.2, and proposition 4.3.2.7). When X and the derived foliation $\mathcal{F}$ are both smooth these phantoms are all trivial, but in general we have not been able to show that $Ch(E^{fil})$ only depends on E and not of the choice of the good filtration.

4.3.2.1. *Existence of good filtrations.* The existence of good filtrations in general seems to be a complicated question, and the authors do not know if good filtrations always exist for arbitrary coherent crystals, as it is the case for usual $\mathcal{D}$-modules on smooth varieties. It can be shown that they do exist for *smooth* foliations on *smooth* varieties, but the fact that, for a general foliation $\mathcal{F}$, its ring of differential operators $\mathcal{D}_{\mathcal{F}}$ is not concentrated in degree 0, creates complications in constructing good filtrations. The following result is therefore very useful in practice.

PROPOSITON 4.3.2.4. *Let $f = (g, u) : (X, \mathcal{F}) \longrightarrow (Y, \mathcal{G})$ be a morphism of separated and quasi-compact derivbed DM-stacks endowed with perfect derived foliations and assume that g is proper and quasi-smooth. If E^{fil} is a good filtration on a coherent crystal $E \in \mathbf{QCoh}(\mathcal{F})$, then $f_!(E^{\mathrm{fil}})$ is a good filtration on $f_!(E)$.*

Proof. This follows from Corollary 4.2.1.7 together with the fact that direct images commutes with taking the underlying object (see diagram (16)). $\square$

We can also single out a large class of coherent crystals for which existence of good filtrations is guaranteed: the *finite cell* crystals. For a derived scheme X and $\mathcal{F} \in \mathcal{F}ol(X)$, we define a non-thick triangulated sub-∞-category $\mathbf{Coh}^{cell}(\mathcal{F}) \subset \mathbf{Coh}(\mathcal{F})$, as being generated (by finite limits and shifts) by the objects of the form $Ind_{\mathcal{F}}(E)$ for E a perfect complex on X. An object of $\mathbf{Coh}^{cell}(\mathcal{F})$ will be called a *finite cell crystal*. There is an obvious filtered version, too. Finite cell crystals are obviously coherent, but the converse has no reason to be true in general. Finite cell crystals are however useful because of the following result.

PROPOSITON 4.3.2.5. *With the notation above, any object $E \in \mathbf{Coh}^{cell}(\mathcal{F})$ admits a good filtration. Moreover, a good filtration E^{fil} can be chosen to be a finite cell object in $\mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F})$.*

Proof. Let $E \in \mathbf{Coh}^{cell}(\mathcal{F})$. By definition of a finite cell object, there is a finite sequence of morphisms in $\mathbf{Coh}(\mathcal{F})$

$$E_{-1} = 0 \longrightarrow E_0 \longrightarrow \dots \longrightarrow E_i \longrightarrow E_{i+1} \longrightarrow \dots \longrightarrow E_n = E,$$

with the following property: for all i there is a perfect complex K_i on X and a cartesian square in $\mathbf{Coh}(\mathcal{F})$

$$\begin{tikzcd} E_i \arrow[r] & E_{i+1} \\ Ind_{\mathcal{F}}(K_i) \arrow[u, "u_i"] \arrow[r] & 0. \arrow[u] \end{tikzcd}$$

We will show, by induction, that each E_i has a good filtration. For this, assume that E_i has a good filtration E_i^{fil}. The morphism u_i is given, by adjunction, by a morphism $v_i : K_i \longrightarrow E_i = (E_i^{\mathrm{fil}})^u$ in $\mathbf{QCoh}(\mathcal{F})$. The quasi-coherent sheaf $(E_i^{\mathrm{fil}})^u$ is the filtered colimit $\mathrm{colim}_k F^k(E_i^{\mathrm{fil}})$, and as K_i is a compact object in $\mathbf{QCoh}(X)$, v_i can be factored as $K_i \xrightarrow{w_i} F^k(E_i^{\mathrm{fil}}) \longrightarrow E_i$, for some index k. By using the left adjoint $Ind_{\mathcal{F}}^{\mathrm{fil}}$, the morphism w_i corresponds to a morphism of filtered $\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}}$-modules

$$\alpha_i : Ind_{\mathcal{F}}^{\mathrm{fil}}(K_i)(-k) \longrightarrow E_i,$$

where $(-k)$ denotes the endofunctor of $\mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F})$ that shifts by $-k$ the filtration. The cone of α_i clearly defines a good filtration on E_{i+1}. $\square$

4.3.2.2. *Independence of the good filtration.* We already noticed that the characteristic cycle of definition 4.3.2.3 *depends*, a priori, on the good filtration E^{fil}. In order to solve this problem, we introduce a reduced K-group, and prove that the image of $Ch(E^{\mathrm{fil}})$ in this reduced K-group, only depends on the object E.

Let $\mathcal{F}$ be a derived foliation on a derived scheme X. We consider $\mathbf{Coh}^{\mathrm{fil}}(\mathcal{F})$, the ∞-category of compact objects $\mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F})$, and the underlying object ∞-functor $(-)^u : \mathbf{Coh}^{\mathrm{fil}}(\mathcal{F}) \longrightarrow \mathbf{Coh}(\mathcal{F})$. An object $E \in \mathbf{Coh}^{\mathrm{fil}}(\mathcal{F})$ will be called a *phantom* if $E^u \simeq 0$. We then give the following definition.

Definition 4.3.2.6. *With the notations above, the reduced K-group $K_0^{red}(T^*\mathcal{F})$ is the quotient of $K_0(T^*\mathcal{F})$ by the subgroup generated by the classes of $Gr(E)$ for $E \in \mathbf{Coh}^{\mathrm{fil}}(\mathcal{F})$ a phantom.*

We have the following independence result.

Propositon 4.3.2.7. *Let $\mathcal{F}$ be a derived foliation on a derived scheme X, and $E \in \mathbf{QCoh}(\mathcal{F})$ be a coherent crystal. Let E_1^{fil} and E_2^{fil} be two good filtrations on E in the sense of Definition 4.3.2.1. Then we have $Ch(E_1^{\mathrm{fil}}) = Ch(E_2^{\mathrm{fil}})$ in $K_0^{red}(T^*\mathcal{F})$.*

Proof. We start by the following lifting lemma.

Lemma 4.3.2.8. *Let $M, N \in \mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F})$ with M compact. Then any morphism $u : M^u \longrightarrow N^u$ in $\mathbf{QCoh}(\mathcal{F})$ can be lifted, via the ∞-functor $(-)^u$, to a morphism $v : M \longrightarrow N(k)$ in $\mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F})$, where $N(k)$ the k-weight shift of N, for some integer k.*

Proof of the lemma. Recall that $N(k)$ denotes the filtered crystal with the filtration shifted by k, i.e.

$$F^i(N(k)) := F^{i+k}N.$$

As M is compact, we know by the proof of Corollary 4.1.3.3, that M is a retract of a finite cell object in $\mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F})$ in the sense of 4.3.2.5. Clearly, if the lemma is true for M (and any N) it is also true for any of its retracts. We may therefore assume that M is a finite cell object. By induction on the number of cells, we reduce ourselves to the following statement. Assume that the lemma is true for M (and any N), and let us consider a push-out

$$\begin{tikzcd} M \arrow[r] & M' \\ Ind_{\mathcal{F}}(K) \arrow[u, "\alpha"] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

with K compact in $\mathbf{QCoh}(X)$. We must prove that the lemma remains true for M' (and any N). Let $v : (M')^u \longrightarrow N^u$ be a morphism. It consists of the data of a morphism $u : M^u \longrightarrow N^u$ and a homotopy to zero of $(u \circ \alpha) : Ind_{\mathcal{F}}(K) \longrightarrow N^u$, i.e., by adjunction, a homotopy to zero h of the induced morphism $\beta : K \longrightarrow N^u$. By assumption on u we can lift u to $w : M \longrightarrow N(k)$ for some k. Moreover, h defines a homotopy to zero of $v = (w)^u : M^u \longrightarrow N(k)^u \simeq N^u$. As K is compact and $N^u = \mathrm{colim}_k F^i N$, the homotopy h factors as a homotopy to zero h' of $K \longrightarrow F^{k'}N \longrightarrow N^u$ for some $k' \geq k$. This pair (w, h') then defines the required lift $M' \longrightarrow N(k')$. □

Let us go back to the proof of Proposition 4.3.2.7. As E_i^{fil}, $i = 1, 2$ are a good filtrations on the same crystal E, we have a natural equivalence in $\mathbf{QCoh}(F)$

$$u : (E_1^{\mathrm{fil}})^u \simeq (E_2^{\mathrm{fil}})^u.$$

By Lemma 4.3.2.8, u can be lifted to a morphism of filtered crystals $v : E_1^{\mathrm{fil}} \longrightarrow E_2^{\mathrm{fil}}$. We set M to be the cone of v in $\mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F})$. This is a compact object which is obviously a phantom, i.e. $M^u \simeq 0$. We thus have an $Ch(E_1^{\mathrm{fil}}) = Ch(E_2^{\mathrm{fil}})$ in $K_0^{red}(T^*\mathcal{F})$. □

Proposition 4.3.2.7 shows that the following notion is well defined.

Definition 4.3.2.9. *Let X be a quasi-compact and separated derived DM-stack and $\mathcal{F} \in \mathcal{F}ol^p(X)$ a perfect derived foliation. If $E \in \mathbf{Coh}(\mathcal{F})$ admits a good filtration E^{fil}, then its characteristic cycle is defined as*

$$Ch(E) = Ch(E^{\mathrm{fil}}) \in K_0^{red}(T^*\mathcal{F}).$$

It is possible to show that when X and $\mathcal{F}$ are both smooth, then the natural projection $K_0(T^*\mathcal{F}) \longrightarrow K_0^{red}(T^*\mathcal{F})$ is bijective, or, in other words, that the class of $Gr(E)$ is trivial in $K_0(T^*\mathcal{F})$ for any phantom $E \in \mathbf{Coh}^{\mathrm{fil}}(\mathcal{F})$. This relies on using regularity and Quillen's devissage techniques, that do not work in our general setting. We will not need this isomorphism $K_0(T^*\mathcal{F}) \simeq K_0^{red}(T^*\mathcal{F})$ in the rest of the book, so we omit its proof. However, the following very simple particular case will be useful later, in order to get numerical formulas out of our general Grothendieck-Riemann-Roch statement (Section 4.4).

Propositon 4.3.2.10. *Let X be a quasi-compact and separated derived DM-stacks and $\mathcal{F} = 0_X$ be the initial foliation, so that $T^*\mathcal{F} \simeq X$. Then, the natural projection*

$$K_0(X) \longrightarrow K_0^{red}(X)$$

is bijective.

Proof. In this case $\mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F}) \simeq \mathbf{QCoh}^{\mathrm{fil}}(X)$ is the ∞-category of filtered quasi-coherent complexes. The compact objects in $\mathbf{QCoh}^{\mathrm{fil}}(X)$ are therefore finite filtered perfect complexes, and thus $Gr(E)$ has only a finite number of non-trivial summand for any compact object $E \in \mathbf{QCoh}^{\mathrm{fil}}(X)$. As a result, if E is a phantom, we have $0 = [Gr(E)]$ in $K_0(X)$. □

Remark 4.3.2.11. A particularly important case of proposition 4.3.2.10 is when $X = \operatorname{Spec} k$, for which we find $K_0^{red}(\operatorname{Spec} k) \simeq \mathbb{Z}$.

4.4. Grothendieck-Riemann-Roch for derived foliations

In this Section we will state and prove the *Grothendieck-Riemann-Roch* (GRR) *formula* for proper maps between derived schemes endowed with derived foliations.

4.4.1. The GRR formula. Let $f = (g, u) : (X, \mathcal{F}) \longrightarrow (Y, \mathcal{G})$ be a morphism of derived stack endowed with derived foliaions, with g proper and quasi-smooth. We will make the standard assumption that Y is either a qcqs derived scheme, or a separated and quasi-compact derived DM-stack. Associated to f is the so-called "Japanese correspondence"

$$T^*\mathcal{F} \xleftarrow{q} T^*\mathcal{G} \times_Y X \xrightarrow{p} T^*\mathcal{G}.$$

Here q is induced by the morphism of $g^*(\mathbb{L}_{\mathcal{G}}) \to \mathbb{L}_{\mathcal{F}}$ perfect complexes on X induced by u while the morphism p is the first projection.

Define the push-forward on K-groups

$$f_! := p_!(q^*(-)) : K_0(T^*\mathcal{F}) \longrightarrow K_0(T^*\mathcal{G}).$$

where $p_!$ is the left adjoint of the pull-back of quasi-coherent sheaves $p^* : \mathbf{QCoh}(T^*\mathcal{G} \times_Y X) \longrightarrow \mathbf{QCoh}(T^*\mathcal{G})$. This left adjoint exists because p is representable, proper and quasi-smooth, and it is given explicitly by

$$p_!(E) = p_*(E \otimes \omega_p[d])$$

for $E \in \mathbf{Perf}(T^*\mathcal{G} \times_Y X)$, where the integer d is the relative dimension of X over Y, and ω_p is the relative canonical sheaf of the morphism p. Note that ω_p coincides with the pull-back of $\omega_{X/Y}$ along the projection $T^*\mathcal{G} \times_Y X \to X$.

We start by noticing that $f_!$ is compatible with the quotient defining reduced K-groups. Indeed, if $E \in \mathbf{Coh}^{\mathrm{fil}}(\mathcal{F})$ is a phantom, then so is $f_!(E)$, and the image by $f_!$ of $Gr(E)$ is $Gr(f_!E)$ (see Corollary 4.2.2.1). Therefore, $f_!$ induces a well defined map

$$f_! : K_0^{red}(T^*\mathcal{F}) \longrightarrow K_0^{red}(T^*\mathcal{G}).$$

THEOREM 4.4.1.1. *Let $f = (g, u) : (X, \mathcal{F}) \longrightarrow (Y, \mathcal{G})$ be a morphism of derived DM-stack schemes endowed with perfect derived foliations with g proper and quasi-smooth. We assume that either Y is separated and quasi-compact, or a qcqs derived scheme. For any coherent crystal $E \in \mathbf{Coh}(\mathcal{F})$ along $\mathcal{F}$ admitting a good filtration (e.g. a finite cell object by proposition 4.3.2.5), we have*

$$Ch(f_!(E)) = f_!(Ch(E))$$

in the group $K_0^{red}(T^*\mathcal{G})$.

Proof. This is a direct consequence of the fact that direct images commutes with taking associated graded for good filtrations (corollary 4.2.2.1). □

The following consequence of Theorem 4.4.1.1 is obtained when $Y = \operatorname{Spec} k$ and $\mathcal{G} = 0_Y$. Let $s : X \to T^*\mathcal{F}$ denotes the zero section of the cotangent stack of $\mathcal{F}$. When X is a proper and quasi-smooth derived DM-stack, we denote by $p : X \to Spec\, k$ the canonical map, and by

$$p_* =: \int_X : K_0(X) \longrightarrow K_0(k) \simeq \mathbb{Z}$$

the induced push-forward on perfect complexes.

Corollary 4.4.1.2. *Let $(X,\mathcal{F})$ be a quasi-smooth and proper derived DM-stack endowed with a perfect derived foliation $\mathcal{F} \in \mathcal{F}ol^p(X)$. Let $f:(X,\mathcal{F}) \to (Spec\, k, 0_{\mathrm{Spec}\, k})$ be the canonical morphism. For any coherent crystal E that admits a good filtration we have*

$$\chi(f_!(E)) = \int_X s^*(Ch(E)) \otimes \omega_X.$$

This corollary is a *Hirzeburch-Riemann-Roch* (HRR) *formula* for crystals along the foliation $\mathcal{F}$. The complex $f_!(E)$ is what should be called the *foliated cohomology of* $(X,\mathcal{F})$ *with coefficients in* E. If we denote this cohomology by $H^*_{\mathcal{F}}(X,E)$, the HRR formula reads

$$\chi(H^*_{\mathcal{F}}(X,E)) = \int_X s^*(Ch(E)) \otimes \omega_X.$$

4.4.2. The non-proper case: Fredholm crystals. We explain here briefly how to extend theorem 4.4.1.1 to *non-proper* maps, by introducing the notion of *Fredholm crystal*.

We start by extending direct images to the case of *compactifiable morphisms*. Assume that $f=(g,u):(X,\mathcal{F}) \to (Y,\mathcal{G})$ is a morphism of derived DM-stack endowed with perfect derived foliations, such that $g: X \to Y$ is quasi-smooth. A *quasi-smooth compactification* of f is the datum of a factorization

$$f:\ (X,\mathcal{F}) \xrightarrow{(j,v)} (\bar{X},\bar{\mathcal{F}}) \xrightarrow{\bar{f}} (Y,\mathcal{G}) \tag{20}$$

where j is an open embedding, v is an equivalence $\mathcal{F} \simeq j^*(\bar{\mathcal{F}})$, and $\bar{f}$ is a proper and quasi-smooth morphism. For such a compactification, we set

$$j_* : \mathbf{QCoh}(\mathcal{F}) \to \mathbf{QCoh}(\bar{\mathcal{F}})$$

to be the right adjoint to the ∞-functor $j^!$. We note here that j_* exists because $j^!$ commutes with colimits. Moreover, j_* is compatible with the usual push-forward of quasi-coherent sheaves through the forgeful ∞-functors, i.e. the following diagram naturally commutes

$$\begin{tikzcd}
\mathbf{QCoh}(\mathcal{F}) \arrow[r, "j_*"] \arrow[d] & \mathbf{QCoh}(\bar{\mathcal{F}}) \arrow[d] \\
\mathbf{QCoh}(X) \arrow[r, "j_*"'] & \mathbf{QCoh}(\bar{X}),
\end{tikzcd}$$

where the vertical ∞-functors are the natural forgetful functors. Since j is an open immersion and v is an equivalence, we have a natural equivalence $j^{-1}\mathcal{D}_{\bar{\mathcal{F}}} \simeq \mathcal{D}_{\mathcal{F}}$ of sheaves of dg-algebras on X, and a canonical adjunction morphism on $\bar{X}$

$$a: \mathcal{D}_{\bar{\mathcal{F}}} \longrightarrow j_*(\mathcal{D}_{\mathcal{F}}).$$

The ∞-functor $j_*: \mathbf{QCoh}(\mathcal{F}) \to \mathbf{QCoh}(\bar{\mathcal{F}})$ is then simply induced by the usual push-forward of sheaves along $j: X \hookrightarrow \bar{X}$: it sends a $\mathcal{D}_{\mathcal{F}}$-module E to $j_*(E)$, viewed as a $\mathcal{D}_{\bar{\mathcal{F}}}$-module via the map a above. This description shows that we also have a commutative square involving the induction ∞-functors

$$\begin{tikzcd} \mathbf{QCoh}(\mathcal{F}) \arrow[r, "j_*"] & \mathbf{QCoh}(\bar{\mathcal{F}}) \\ \mathbf{QCoh}(X) \arrow[u, "Ind_{\mathcal{F}}"] \arrow[r, "j_*"'] & \mathbf{QCoh}(\bar{X}). \arrow[u, "Ind_{\bar{\mathcal{F}}}"'] \end{tikzcd}$$

We define the *$*$-direct image along f* as

$$f_* = \bar{f}_! \circ j_* : \mathbf{QCoh}(\mathcal{F}) \longrightarrow \mathbf{QCoh}(\mathcal{G}).$$

A standard argument, by using the product embedding, proves that f_*, defined as above, does not depend on the choice of the factorization (20) (see e.g. [**AGV73**, Exp XVII] or [**FK88**, §8]). Indeed, f_* clearly commutes with colimits, so the independence of the factorization in its definition can be checked on compact $\mathcal{D}_{\mathcal{F}}$-modules of the form $Ind_{\mathcal{F}}(E)$, where it is easily deduced from the compatibility of push-forwards for quasi-coherent sheaves, and from the explicit formula of proposition 4.2.1.6.

DEFINITION 4.4.2.1. *Let $f = (g, u) : (X, \mathcal{F}) \to (Y, \mathcal{G})$ be a morphism of derived DM-stack endowed with perfect derived foliations. Suppose that f admits a quasi-smooth compactification. An object $E \in \mathbf{Coh}(\mathcal{F})$ is called f-*Fredholm *if it admits a good filtration $E^{\mathrm{fil}} \in \mathbf{Coh}^{\mathrm{fil}}(\mathcal{F})$ such that $j_*(E^{\mathrm{fil}})$ is a compact object in $\mathbf{QCoh}^{\mathrm{fil}}(\mathcal{F})$ for some quasi-smooth compactification $j : (X, \mathcal{F}) \hookrightarrow (\bar{X}, \bar{\mathcal{F}})$ of the morphism f.*

Suppose $E \in \mathbf{Coh}(\mathcal{F})$ is f-Fredholm; pick a quasi-smooth compactification $(\bar{X}, \bar{\mathcal{F}})$ and a filtration E^{fil} as in the definition above. We have the associated graded $Gr(E^{\mathrm{fil}}) \in \mathbf{Perf}(T^*\mathcal{F})$, and its direct image by j is by again a perfect complex on $T^*\bar{\mathcal{F}}$, so that, in particular, the support of $Gr(E^{\mathrm{fil}})$, $SS(E) \subset T^*\mathcal{F}$ is closed in $T^*\bar{\mathcal{F}}$. If we pull-back this support by the canonical map $T^*\mathcal{G} \times_Y X \to \mathbb{T}^*\mathcal{F}$, we get a closed subset in $T^*\mathcal{G} \times_Y X$ which is proper over $T^*\mathcal{G}$. In other words, the morphism f is proper when restricted to the support of $Gr(E^{\mathrm{fil}})$. In our situation, the notion of Fredholm is a priori stronger, it is unclear that properness of f on the support of $Gr(E^{\mathrm{fil}})$ is enough to recover the fact that $j_*(E^{\mathrm{fil}})$ remains a compact object. Therefore, our definition of being f-Fredholm might be hard to check in practice, already because it involves the existence of a compactification of f which seems a delicate question even when X and Y are both smooth quasi-projective varieties.

We can now state the Grothendieck-Riemann-Roch formula for possibly *non-proper* maps, and *Fredholm* coefficients.

THEOREM 4.4.2.2. *Let $f = (g, u) : (X, \mathcal{F}) \longrightarrow (Y, \mathcal{G})$ be a morphism of derived DM-stacks endowed with perfect derived foliations. We assume as usual that either Y is a qcqs derived schemes, or a separated and quasi-compact DM-stack. If f admits a quasi-smooth* *compactification, and* $E \in \mathbf{Coh}(\mathcal{F})$ *is an* f*-Fredholm crystal, then we have an equality*

$$Ch(f_*(E)) = f_*(Ch(E))$$

of elements in $K_0^{red}(T^*\mathcal{G})$*.*

Proof. Simply apply Theorem 4.4.1.1 to the morphism $\bar{f}$ and to the object $j_*(E)$ over $(\bar{X}, \bar{\mathcal{F}})$, which has a good filtration given by $j_*(E^{\mathrm{fil}})$ for E^{fil} given by Definition 4.4.2.1. □

4.5. Examples and applications

We finish this Chapter by giving a sample of examples and applications of the general Grothendieck-Riemann-Roch formula of Theorem 4.4.1.1.

4.5.1. Grothendieck-Riemann-Roch for derived $\mathcal{D}$-modules. The very first special case of Theorem 4.4.1.1 is when $\mathcal{F} = *_X$ and $\mathcal{G} = *_Y$ are the final derived foliations on X and Y respectively. The ∞-categories $\mathbf{QCoh}(*_X)$ and $\mathbf{QCoh}(*_Y)$ are then called the *∞-categories of derived $\mathcal{D}$-modules on X and Y*, and denoted by $\mathcal{D}_X - \mathbf{dg}_{qcoh}$ and $\mathcal{D}_Y - \mathbf{dg}_{qcoh}$, respectively. We think they coincide, for all X, with the derived $\mathcal{D}$-modules introduced and studied in [**Ber21**], where they are denoted by $\mathcal{D}^{der}(X)$ and $\mathcal{D}^{der}(Y)$. We refer the reader to [**Buc**] for precise comparison statements. However, we remark that, contrary to the notion of crystals introduced in [**GR14**], these ∞-categories of derived $\mathcal{D}$-modules are sensitive to derived structures (see remark 4.1.1.7).

In this situation, $T^*\mathcal{F} = T^*X$ and $T^*\mathcal{G} = T^*Y$ are the global derived cotangent stacks of X and Y, and the GRR formula is the equality

$$Ch(f_!(E)) = f_! Ch(E) \text{ in } K_0^{red}(T^*Y) \tag{21}$$

for a compact object $E \in \mathcal{D}_X - \mathbf{dg}_{qcoh}$ admitting a good filtration, and $f : X \to Y$ a proper and quasi-smooth morphism. When X and Y are smooth varieties, this recovers the well known GRR formula for $\mathcal{D}$-modules of [**Lau83**] (see also [**Sab12**]). However, already when X and Y are just underived k-schemes, formula (21) is new and provides a Grothendieck-Riemann-Roch formula for $\mathcal{D}$-modules on *possibly singular* schemes.

An interesting feature of this situation is that any morphism $f = (g, u) : (X, \mathcal{F}) \to (Y, \mathcal{G})$ fits in a commutative diagram

$$\begin{tikzcd}
(X,\mathcal{F}) \arrow[r, "f"] \arrow[d, "u"'] & (Y,\mathcal{G}) \arrow[d, "v"] \\
(X,*_X) \arrow[r, "g"'] & (Y,*_Y).
\end{tikzcd}$$

As explained at the end of §4.1.3, the push-fowards $u_!$ and $v_!$ define induction ∞-functors

$$u_! : \mathbf{QCoh}(\mathcal{F}) \to \mathcal{D}_X - \mathbf{dg}_{qcoh} \qquad v_! : \mathbf{QCoh}(\mathcal{G}) \to \mathcal{D}_Y - \mathbf{dg}_{qcoh}.$$

The GRR formula for f, which is an equality in $K^{red}(T^*\mathcal{G})$, can be pushed forward along v to a formula in $K_0^{red}(T^*Y)$:

$$v_! Ch(f_!(E)) = g_! Ch(u_! E))$$

where E is a coherent crystal along $\mathcal{F}$ admitting a good filtration, $u_!(E)$ the induced coherent $\mathcal{D}_X$-module, and $v_! : K_0^{red}(T^*\mathcal{G}) \to K_0^{red}(T^*Y)$ the direct image along the canonical morphism of foliations $\mathcal{G} \to *_Y$. This gives a formula for the push-forward of the characteristic cycle of a $\mathcal{D}_X$-module *induced from* the derived foliation $\mathcal{F}$. When $f = id$ the formula reads

$$v_! Ch(E) = Ch(u_! E)$$

and should be understood as a formula for the characteristic cycle on a $\mathcal{D}_X$-module induced from the derived foliation $\mathcal{F}$.

4.5.2. Smooth Lie algebroids. Let X be a smooth variety and $\mathcal{F} \in \mathcal{F}ol(X)$ a smooth foliation on X (i.e. $\mathbb{L}_{\mathcal{F}}$ is a vector bundle on X). As explained in theorem 2.2.0.7, $\mathcal{F}$ is determined by a Lie algebroid $\mathbb{T}_{\mathcal{F}} \to \mathbb{T}_X$. The sheaf $\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}}$ is then isomorphic to the universal enveloping algebra $U(\mathbb{T}_{\mathcal{F}})$ of $\mathbb{T}_{\mathcal{F}}$, equipped with its natural PBW filtration. As already noticed $K_0^{red}(T^*\mathcal{F}) \simeq K_0(T^*\mathcal{F})$, in this case (of a smooth foliation on a smooth variety). It can also be shown that any coherent crystal along $\mathcal{F}$ admits a good filtration. Moreover, $\mathcal{D}_{\mathcal{F}}$ satisfies all the conditions of Quillen theorem (see [**Qui73**]) and we thus have natural isomorphisms of K-groups

$$\tau_X : K_0(\mathbf{Coh}(\mathcal{F})) \simeq K_0(T^*\mathcal{F}) \simeq K_0(X).$$

The first of this isomorphisms is precisely given by $E \mapsto Gr(E^{\mathrm{fil}})$ for E^{fil} a good filtration on E, and the second isomorphism is the pull-back along the zero section $s : X \to T^*\mathcal{F}$.

The GRR formula 4.4.1.1 tells us that the isomorphism $\tau_X : K_0(\mathbf{Coh}(F)) \simeq K_0(X)$ is covariantly functorial in $(X, \mathcal{F})$ in the following sense: given a morphism $f = (g, u) : (X, \mathcal{F}) \to (Y, \mathcal{G})$ of smooth varieties endowed with smooth foliations, with g proper, the following diagram

$$\begin{tikzcd}
K_0(\mathbf{Coh}(\mathcal{F})) \arrow[r, "\tau_X"] \arrow[d, "f_!"'] & K_0(X) \arrow[d, "g_*(-\otimes\omega_{X/Y})[d]"] \\
K_0(\mathbf{Coh}(\mathcal{G})) \arrow[r, "\tau_Y"'] & K_0(Y)
\end{tikzcd}$$

commutes. This recovers the well known GRR formula for $\mathcal{D}$-modules on smooth varieties (see [**Lau83**]), and its natural extension to Lie algebroids. This extension to Lie algebroids is probably a folklore result (as its proof is word by word the same as for $\mathcal{D}$-modules), but we have not been able to find an appropriate reference in the literature.

4.5.3. Shifted Poisson structures. An important class of derived foliations are provided by *shifted Poisson structures* in the sense of [**CPT+17**]. Indeed, let X be a derived scheme endowed with a shifted Poisson structure of degree n. The Poisson bracket defines a morphism of perfect complexes on X

$$a : \mathbb{L}_X[-n] \to \mathbb{T}_X$$

making $\mathbb{L}_X[-n]$ into a dg-Lie algebroid over X. This defines (§ 2.2) a derived foliation $\mathcal{F}$ whose underlying graded mixed cdga is $Sym_{\mathcal{O}_X}(\mathbb{T}_X[n])$, where the mixed structure is induced the bracket $[-, p]$, p being the bivector defining the Poisson structure. We refer to [**Tom25**] where this construction is explained in details, as several technical complications related to homotopy coherences have to be overcome.

This derived foliation associated to a shifted Poisson structure is the derived analogue of the foliation by symplectic leaves of a classical Poisson strcuture on a smooth variety. Its *leaves*, in the sense of Chapter 6, are by definition the symplectic leaves of the shifted Poisson structure. When the Poisson structure is non-degenerate, then the foliation $\mathcal{F}$ is the final foliation. In general $\mathcal{F}$ is a very interesting derived foliation containing information about the Poisson structure. For instance, our notion $\mathbf{QCoh}(\mathcal{F})$ of quasi-coherent crystals provides a useful setting to study various versions of Poisson cohomology. As an example, we may define *derived Poisson cohomology* to be

$$H^{\mathrm{Poiss}}_{der}(X) := \mathbb{R}\underline{Hom}_{\mathbf{QCoh}(\mathcal{F})}(\mathcal{O}_X, \mathcal{O}_X),$$

which coincides with derived de Rham cohomology when the Poisson structure is a symplectic structure. It is also possible to consider the induced $\mathcal{D}_X$-module $u_!(\mathcal{O}_X) \in \mathcal{D}_X - \mathbf{dg}_{X,qcoh}$, where $u : \mathcal{F} \to *_X$ is the unique morphism to the final foliation. Then, we may use the $\mathcal{D}_X$-module $u_!(\mathcal{O}_X)$ in order to define a version of Poisson *homology* by

$$H_{\mathrm{Poiss}}(X) := p_*(u_!(\mathcal{O}_X)) \in \mathbf{dg}$$

where $p : X \to Spec\, k$ is the structure map (assuming here that X is quasi-smooth and p admits a quasi-smooth compactification as defined in §4.4.2). This offers a criterion for finiteness of Poisson homology by requiring $u_!(\mathcal{O}_X)$ to be Fredholm as a $\mathcal{D}_X$-module on X. When X is smooth and the Poisson structure is a classical Poisson structure of degree 0, the $\mathcal{D}_X$-module $u_!(\mathcal{O}_X)$ has been considered in [**PS18**], where it was used to define the notion of *holonomic Poisson varieties* and to get finiteness results for Poisson cohomology. The GRR formula 4.4.2.2, and more generally the general formalism of crystals along derived foliations, provides a way to extend these notions and results to shifted Poisson structures.

4.5.4. A foliated index formula. As a last application of Theorem 4.4.2.2 we propose an *index formula* for a weakly Fredholm differential operator along the leaves of a derived foliation on a quasi-smooth derived scheme. We like to think of it as an algebraic version of the longitudinal index theorem of [**CS84**], possibly valid outside the smooth setting, i.e. for non-smooth derived schemes and non-smooth derived foliation over them.

Let $(X, \mathcal{F})$, where X is a derived DM-stack and $\mathcal{F} \in \mathcal{F}ol^p(X)$ a perfect derived foliation. We assume that X has a quasi-smooth compactification $j : X \hookrightarrow \bar{X}$ (i.e. the projection $X \to Spec\, k$ admits a quasi-smooth compactification as in §4.4.2), which implies in particular that X is separated and quasi-compact. Note that we only assume the existence of $\bar{X}$, and we do not assume that $\mathcal{F}$ can be extended to $\bar{X}$. Let E and E' be two perfect complexes on X. A *differential operator P along $\mathcal{F}$ from E to E'* is by definition a morphism of quasi-coherent sheaves on X

$$P : E \longrightarrow Ind_{\mathcal{F}}(E') = \mathcal{D}_{\mathcal{F}} \otimes_{\mathcal{O}_X} E',$$

or, equivalently, a morphism of coherent crystals along $\mathcal{F}$

$$a_P : Ind_{\mathcal{F}}(E) \longrightarrow Ind_{\mathcal{F}}(E').$$

As E is compact in $\mathbf{QCoh}(X)$, we note that P must factor through a morphism $E \to \mathcal{D}_{\mathcal{F}}^{\leq i} \otimes_{\mathcal{O}_X} E'$ for some integer i. In this case, we say that P is of *order* $\leq i$.

We let $\mathcal{D}(P)$ be the cone of the morphism a_P inside $\mathbf{Coh}(\mathcal{F})$, and we want to apply Theorem 4.4.2.2 to the coherent crystal $\mathcal{D}_X \otimes_{\mathcal{D}_{\mathcal{F}}} \mathcal{D}(P)$. Note that this is also the cone of the morphism $\mathcal{D}_X \otimes_{\mathcal{D}_{\mathcal{F}}} E \to \mathcal{D}_X \otimes_{\mathcal{D}_{\mathcal{F}}} E'$, induced by the composition

$$P : \ E \longrightarrow \mathcal{D}_{\mathcal{F}} \otimes_{\mathcal{O}_X} E' \longrightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} E'.$$

In order to apply Theorem 4.4.2.2, we need to impose a condition on P ensuring that $\mathcal{D}_X \otimes_{\mathcal{D}_{\mathcal{F}}} \mathcal{D}(P)$ is a Fredholm object (Definition 4.4.2.1). We consider the least integer i such that P factors as

$$E \longrightarrow \mathcal{D}_{\mathcal{F}}^{\leq i} \otimes_{\mathcal{O}_X} E'.$$

Associated to this, we have a morphism of filtered $\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}}$-modules

$$Ind_{\mathcal{F}}^{\mathrm{fil}}(E)(-i) \to Ind_{\mathcal{F}}^{\mathrm{fil}}(E),$$

whose cone defines a good filtration $\mathcal{D}^{\mathrm{fil}}(P)$ on $\mathcal{D}(P)$, and thus, by base change, a good filtration $\mathcal{D}_X^{\mathrm{fil}} \otimes_{\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}}} \mathcal{D}^{\mathrm{fil}}(P)$ on $\mathcal{D}_X \otimes_{\mathcal{D}_{\mathcal{F}}} D(P)$.

Definition 4.5.4.1. *The operator P along $\mathcal{F}$ is* weakly Fredholm *if $j_*(\mathcal{D}_X^{\mathrm{fil}} \otimes_{\mathcal{D}_{\mathcal{F}}^{\mathrm{fil}}} \mathcal{D}^{\mathrm{fil}}(P))$ is a compact object in $\mathbf{QCoh}^{\mathrm{fil}}(\bar{X})$, for a quasi-smooth compactification $j : X \hookrightarrow \bar{X}$.*

Assuming that P is weakly Fredholm, we are thus able to apply Theorem 4.4.2.2 to get our index formula, that has values in $K_0(k) \simeq \mathbb{Z}$, and thus is an equality of two numbers we are now going to describe.

Let $f : X \to Spec\, k$ the projection, that we factor as $X \xrightarrow{j} \bar{X} \xrightarrow{\bar{f}} Spec\, k$. We first describe $f_*(\mathcal{D}(P)) \simeq \bar{f}_!(j_*(\mathcal{D}(P))) \simeq \bar{f}_!(\mathcal{D}_X \otimes_{\mathcal{D}_{\mathcal{F}}} D(P))$. This is a perfect complex of k-modules, which by definition, is the cone of the morphism induced by P

$$\Gamma(P) : \Gamma(X, E \otimes \omega_X)[d] \to \Gamma(X, E' \otimes \omega_X)[d].$$

Note that none of the two complexes $\Gamma(X, E \otimes \omega_X)$ or $\Gamma(X, E' \otimes \omega_X)$ is perfect, and only the cone of $\Gamma(P)$ is so. This is the effect of the weakly Fredholm property, implying that $\Gamma(P)$ is indeed a Fredlhom operator. Therefore, we have a first well defined number, called the *algebraic index of P $Ind(P)$*, which is the Euler characteristic of the cone of $\Gamma(P)$

$$Ind(P) := (-1)^d.\chi(cone\,(\Gamma(P) : \Gamma(X, E \otimes \omega_X) \to \Gamma(X, E' \otimes \omega_X))).$$

On the other hand, the object $\mathcal{D}_X \otimes_{\mathcal{D}_\mathcal{F}} \mathcal{D}(P)$ is endowed with the good filtration $\mathcal{D}_X^{\mathrm{fil}} \otimes_{\mathcal{D}_\mathcal{F}^{\mathrm{fil}}} \mathcal{D}^{\mathrm{fil}}(P)$, and by assumption its associated graded defines a perfect complex of T^*X which remains perfect on $T^*\bar{X}$. This associated graded can be described as follows. Recall that i is the least integer such that P defines a morphism in $\mathbf{QCoh}(X)$ $P : E \to \mathcal{D}_X^{\leq i} \otimes_{\mathcal{O}_X} E'$. By projection $\mathcal{D}_X^{\leq i} \to Sym_X^i(\mathbb{T}_X)$, we find

$$\sigma(P) : E \to Sym_X^i(\mathbb{T}_X) \otimes_{\mathcal{O}_X} E'$$

which is called the *symbol of P*. This extends to

$$Sym_X(\mathbb{T}_X) \otimes_{\mathcal{O}_X} E \to Sym_X(\mathbb{T}_X) \otimes_{\mathcal{O}_X} E$$

as a morphism of graded $Sym_{\mathcal{O}_X}(\mathbb{T}_X)$-modules

The cone of this morphism defines a perfect complex, still denoted by $\sigma(P)$, on T^*X, which remains perfect on $T^*\bar{X}$ by the weakly Fredholm assumption. If we denote by $s : X \to T^*X$ the zero section map, we thus get a well defined perfect complex of k-modules $\Gamma(X, s^*(\sigma(P)) \otimes \omega_X)[d]$. The Euler characteristic of this complex is called the *K-theoretical index of the operator P*, and denoted by

$$Ind_\sigma(P) := (-1)^d \chi(\Gamma(X, s^*(\sigma(P)) \otimes \omega_X)).$$

Theorem 4.4.2.2 then implies the following

COROLLARY 4.5.4.2. *With the notations above, we have*

$$Ind(P) = Ind_\sigma(P).$$

In plain words, Corollary 4.5.4.2, says that the *algebraic index of P*, computing the Fredholm index of P acting on global sections of E and E', equals the *K-theoretic index of P*, which should be understood as an intersection number between X and the cycle defined by the symbol of P inside the total cotangent stack T^*X.

CHAPTER 5

Analytic aspects

This chapter is devoted to introduce the notion of derived foliation in the holomorphic context. We remind some basic definitions on holomorphic rings and holomorphic cdga's, and the corresponding notion of derived analytic stacks. We then introduce sheaves of holomorphic graded mixed cdga, and use them to define holomorphic derived foliations on general derived analytic stacks, analogously to the algebraic setting. We study analytic integrability of quasi-smooth and rigid derived foliations, locally in the analytic topology. Finally, we construct an analytification functor and prove a version of GAGA relating derived algebraic foliations and derived analytic foliations on a proper derived Deligne-Mumford algebraic stack.

Throughout this chapter the base ring k will be the field of complex numbers $\mathbb{C}$.

5.1. Quick reminders on derived analytic geometry

We recall the definition of an algebraic theory hol, of holomorphic rings as follows (see [**Pir15**, **Pri20**, **Por19**] and [**CR13**, **Nui12**] for the $\mathcal{C}^\infty$-version). The algebraic theory hol is by definition the full sub-category of $\mathbb{C}Man$, of complex manifolds and holomorphic maps, consisting of all the objects $\mathbb{C}^n$ for $n \geq 0$. We will denote by $hol(n)$ the set of holomorphic maps $\mathbb{C}^n \to \mathbb{C}$, so that in the category hol the maps from $\mathbb{C}^n$ to $\mathbb{C}^m$ are in natural bijection with $hol(n)^m$.

By definition, a *holomorphic ring* A is an algebra over the algebraic theory hol, that is a product preserving functor $A : hol \to Sets$. In more concrete terms, such a holomorphic ring consists of a commutative $\mathbb{C}$-algebra A, endowed with extra operations: for any holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ we have an application

$$\gamma_f : A^n \to A,$$

satisfying some natural properties with respect to compositions of holomorphic maps. Morally, $\gamma_f(a_1, \ldots, a_n)$ stands for "$f(a_1, \ldots, a_n)$", the "evaluation of f at the elements a_i". When f is restricted to polynomials maps, the operations γ_f determines a commutative $\mathbb{C}$-algebra structure on A (the sum and multiplication being induced by the two holomorphic maps $\mathbb{C}^2 \to \mathbb{C}$, $(x, y) \mapsto x+y$ and $(x, y) \mapsto xy$). The typical example of a holomorphic ring is the set $\mathcal{O}_X(X)$ of (globally defined) holomorphic functions on a given complex analytic space X, where the operations γ_f are actually defined by $\gamma_f(a_1, \ldots, a_n) = f(a_1, \ldots, a_n)$, for $a_i \in \mathcal{O}_X(X)$.

Morphisms between holomorphic rings are maps commuting with all the operations γ_f (for all f as above). Holomorphic rings thus form a category denoted by $\mathbf{CR}^h$, which comes equipped with with a forgetful functor $N : \mathbf{CR}^h \to \mathbb{C} - \mathbf{CR}$ to commutative $\mathbb{C}$-algebras. We will often use the same notations for a holomorphic ring A and its underlying $\mathbb{C}$-algebra, but in situations where the difference is important the latter will be denoted by $N(A)$. The forgetful functor $\mathbf{CR}^h \to \mathbb{C} - \mathbf{CR}$ commutes with limits and filtered colimits, and thus possesses a left adjoint, denoted by $A \mapsto A^h$, sending a $\mathbb{C}$-algebra to its "holomorphication" . This left adjoint $A \mapsto A^h$ can be understood as a left Kan extension as follows. We start by the functor sending the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$ to the ring $hol(n)$ of holomorphic functions on $\mathbb{C}^n$ (with its natural holomorphic structure), and any $\mathbb{C}$-algebra morphism $\mathbb{C}[X_1, \ldots, X_n] \to \mathbb{C}[Y_1, \ldots, Y_m]$, defined by polynomials $P_1(Y), \ldots, P_n(Y)$, to the map $hol(n) \to hol(m)$ obtained by composition with $(P_1, \ldots, P_n) : \mathbb{C}^m \to \mathbb{C}^n$. We then obtain $(-)^h$ as the left Kan extension of this functor along the canonical inclusion of the category of polynomial algebras into the whole category of commutative $\mathbb{C}$-algebras. For any $\mathbb{C}$-algebra A the holomorphic ring A^h can thus be described by choosing a presentation of A by polynomial algebras, whose image by $(-)^h$ provides a presentation by holomorphic rings $hol(n)$. More explicitely, if A is finitely generated, we can write A as a co-equalizer

$$(\mathbb{C}[X_1, \ldots, X_n] \rightrightarrows \mathbb{C}[Y_1, \ldots, Y_m]) \longrightarrow A$$

and obtain A^h as the corresponding coequalizer in $\mathbf{CR}^h$

$$(hol(n) \rightrightarrows hol(m)) \longrightarrow A^h.$$

When A is not finitely presented we write $A = colim_i A_i$ as a filtered colimit of finite presented $\mathbb{C}$-algebras and we have $A^h = colim_i (A_i)^h$.

The forgetful functor $\mathbf{CR}^h \to \mathbf{CR}$ commutes with filtered colimits (and even sifted colimits), but does *not* commute with push-outs in general. In order to avoid confusions we will then use $\otimes^h$ to denote the categorical sum in $\mathbf{CR}^h$. For instance the push-out $hol(n) \otimes^h hol(m)$ is naturally isomorphic to $hol(n+m)$, simply because $hol(n)$ turns out to be the free holomorphic ring over n generators

$$Hom_{\mathbf{CR}^h}(hol(n), A) \simeq N(A)^n.$$

But the $\mathbb{C}$-alegbras $hol(n+m)$ is certainly not isomorphic, by the canonical map of $\mathbb{C}$-algebras, to the algebraic tensor product $hol(n) \otimes hol(m)$. However, as $(-)^h$ is left adjoint, we always have a canonical isomorphism of holomorphic rings

$$A^h \otimes^h B^h \simeq (A \otimes_{\mathbb{C}} B)^h$$

for any two commutative $\mathbb{C}$-algebras A and B.

Any holomorphic ring A possesses a module of (holomorphic) differential forms Ω^1_A, which is an A-module with the following universal property: A-modules maps $\Omega^1_A \to M$ are in one-to-one correspondence with *holomorphic* derivations $\delta : A \to M$. Recall here that a derivation

$\delta : A \to M$ is *holomorphic* if for all $f \in hol(n)$ we have the chain rule

$$\delta(\gamma_f(a_1, \ldots, a_n)) = \sum_i \frac{\partial f}{\partial z_i}(a_1, \ldots, a_n).\delta(a_i).$$

It is important here to make a difference between a holomorphic ring A and its underlying $\mathbb{C}$-algebra $N(A)$. Indeed, Ω^1_A is different from the usual model of Kähler differentials $\Omega^1_{N(A)/\mathbb{C}}$ of $N(A)$. There is an obvious morphism $\Omega^1_{N(A)/\mathbb{C}} \to \Omega^1_A$ coming from the fact that the universal holomorphic derivation $A \to \Omega^1_A$ is a derivation in the usual algebraic sense, but this is not an isomorphism in general. On the other hand, if we start with a commutative $\mathbb{C}$-algebra R, then there exists a natural isomorphism of R^h-modules (which follows from the fact that the forgetful functor $\mathbf{CR}^h \to \mathbb{C} - \mathbf{CR}$ preserves trivial square zero extensions)

$$R^h \otimes_R \Omega^1_{R/\mathbb{C}} \simeq \Omega^1_{R^h}.$$

When A is a holomorphic ring Ω^1_A will always mean the holomorphic differentials, while we will use the notation $\Omega^1_{N(A)}$ for the (non-holomorphic) Kähler differentials.

5.1.1. The ∞-category of holomorphic cdga's. The next definition is essentially similar to the notions used in [**Pir15**, **Pri20**]. Note however that in the *non-connective* case, our notion of holomorphic cdga differs slightly, as the holomorphic structure will exist on the cohomological degree 0 subalgebra (and not merely on the algebra of 0-cocyles). Note also the extra condition on the differential which does not appear in the above mentioned references. However, for connective cdga's it coincides with the notions of [**Pir15**, **Pri20**] (note that for connective cdga's, a holomorphic structure, as in Definition 5.1.1.1 below, is just a holomorphic ring structure on its degree 0 part). We will non-connective holomorphic cdga's in an essential manner, for instance in order to consider the holomorphic *de Rham complex* as a holomorphic cdga.

DEFINITION 5.1.1.1. *(1) Let $A \in \mathbf{cdga}_{\mathbb{C}}$ be a possibly non-connective $\mathbb{C}$-linear cdga. A* holomorphic structure on A *consists of a holomorphic ring structure on the $\mathbb{C}$-algebra A^0 of elements of cohomological degree 0, such that the cohomological differential $d : A^0 \to A^1$ is a holomorphic derivation. A* holomorphic cdga *is a $\mathbb{C}$-linear cdga endowed with a holomorphic structure.*

(2) A morphism $A \to B$, between two holomorphic cdga's*, is a morphism of (underlying) cdga's for which the induced morphism $A^0 \to B^0$ is a morphism of holomorphic rings.*

Holomorphic cdga's form a category $cdga^h$, endowed with a forgetful functor $cdga^h \to cdga_{\mathbb{C}}$, to the category of $\mathbb{C}$-linear cdga's. This functor is the right adjoint of an adjunction, whose left adjoint is denoted by $A \to A^h$. Explicitly, for $A \in cdga_{\mathbb{C}}$, A^h is defined as follows. We start by the graded $\mathbb{C}$-algebra $(A^0)^h \otimes_{A^0} A$, by extending A^0 to its holomorphication. The cohomological differential $d : A^0 \to A^1$ extends uniquely into a holomorphic derivation

$d^h : (A^0)^h \to (A^0)^h \otimes_{A^0} A^1$. We then define the cohomological differential d^h on the whole graded algebra $(A^0)^h \otimes_{A^0} A$ by the formula

$$d^h(a \otimes x) = d^h(a).x + a.d(x),$$

for any homogenuous $x \in A$ and $a \in (A^0)^h$. This defines a structure of cdga on A^h. Moreover, $(A^h)^0 = (A^0)^h$ comes equipped with its natural holomorphic ring structure making A^h into a holomorphic cdga in the sense of the above definition.

More generally, it is possible to describe the coproducts in the category $cdga^h$ as follows. For this, we start by the case of $\mathbb{Z}$-graded holomorphic rings $(\mathbf{CR}^h)^{\mathbb{Z}}$. These are, by definition, graded commutative $\mathcal{C}$-algebras A, with the usual sign rules $xy = (-1)^{|x||y|} = yx$, together with a holomorphic structure on the ring A^0. For $A \in (\mathbf{CR}^h)^{\mathbb{Z}}$, we have the corresponding notions of graded holomorphic derivations and graded module of holomorphic differential forms Ω^1_A. In particular, any holomorphic cdga A can be understood via its underlying graded holomorphic algebra $A^{gr} \in (\mathbf{CR}^h)^{\mathbb{Z}}$, together with its holomorphic differential that is uniquely determined by the corresponding morphism of graded A^{gr}-modules

$$d : \Omega^1_{A^{gr}} \longrightarrow A^{gr}(-1),$$

where $A^{gr}(-1)$ is the weight twist (§1.2.2) of the unit graded module A^{gr}: $(A^{gr}(-1))^n = (A^{gr})^{n+1}$. With these notations, coproducts in $(\mathbf{CR}^h)^h$ are easily described by the formula

$$(A \otimes^h B)^n = \bigoplus_{p+q=n} (A^p \otimes B^q) \otimes_{A^0 \otimes B^0} (A^0 \otimes^h B^0),$$

with the usual commutative multiplicative structure. Note that by construction $(A \otimes^h B)^0 = A^0 \otimes^h B^0$ is the coproduct of holomorphic rings in weight 0.

Suppose now that A and B are two objects in $cdga^h$, which we represent as graded holomorphic rings A^{gr} and B^{gr} together with graded holomorphic derivations corresponding to morphisms

$$d_A : \Omega^1_{A^{gr}} \to A^{gr}(-1) \qquad B_A : \Omega^1_{B^{gr}} \to B^{gr}(-1)$$

of graded A^{gr}-modules and graded B^{gr}-modules, respectively. We can form the coproduct in graded rings $A^{gr} \otimes^h B^{gr}$, and the two differentials d_A and d_B provide for us a canonical morphism of graded $A^{gr} \otimes^h B^{gr}$-modules

$$d_A \otimes id + id \otimes d_B : \Omega^1_{A^{gr} \otimes^h B^{gr}} \simeq (\Omega^1_{A^{gr}} \otimes_{A^{gr}} (A^{gr} \otimes^h B^{gr})) \oplus ((A^{gr} \otimes^h B^{gr}) \otimes_{B^{gr}} \Omega^1_{B^{gr}}) \longrightarrow (A^{gr} \otimes^h B^{gr})(-1).$$

This defines a differential on the graded holomorphic ring $A^{gr} \otimes^h B^{gr}$ which is holomorphic in degree 0, and thus defines a holomorphic cdga. This holomorphic cdga is the coproduct of A and B computed in the category $cdga^h$. The important point here is that the forgetful functor $cdga^h \to (\mathbf{CR}^h)^{\mathbb{Z}}$, from holomorphic cdga's to graded holomorphic rings, preserves coproducts.

The adjunction $cdga^h \leftrightarrows cdga_{\mathbb{C}}$ can be used in order to lift the model structure on $cdga_{\mathbb{C}}$ to a model structure on $cdga^h$, for which fibrations and equivalences are defined via the forgetful functor, as shown in the proposition below.

Propositon 5.1.1.2. *The above notions of equivalences and fibrations make $cdga^h$ into a model category for which the forgetful functor $cdga^h \to cdga$ is a right Quillen functor.*

Proof. We apply the criterion of transferring model structures along a right adjoint using the path object argument (see [**BM03**, 2.6]). The only non-trivial statement consists in proving that any object $A \in cdga^h$ admits a path object. We construct this path object explicitly has follows.

We introduce the object $\mathbb{C}[\Delta^1]^h \in cdga^h$ which is the image by $(-)^h$ of the cdga freely generated by the acyclic complex $\left(\mathbb{C} \xrightarrow{id} \mathbb{C} \right)$ concentrated in cohomological degrees $[0,1]$. It can be explicitly described as the holomorphic cdga with only two non-trivial degrees

$$(\mathbb{C}[\Delta^1]^h)^0 = (\mathbb{C}[\Delta^1]^h)^1 = hol(1) \qquad (\mathbb{C}[\Delta^1]^h)^i = 0 \, for \, i \neq 0,1$$

and whose differential in degree 0 is given by

$$hol(1) \to hol(1)$$

sending a holomorphic function f to its derivative $f' = \frac{\partial f}{\partial z}$. The holomorphic cdga $\mathbb{C}[\Delta^1]^h$ admits two augmentation to $\mathbb{C}$, by simply evaluating a function f at 0 and at 1

$$ev_0, ev_1 : \mathbb{C}[\Delta^1]^h \rightrightarrows \mathbb{C}.$$

We claim that for any $A \in cdga^h$ the induced diagram

$$A \xrightarrow{j} A \otimes^h \mathbb{C}[\Delta^1]^h \underset{ev_1}{\overset{ev_0}{\rightrightarrows}} A$$

is a path object for A: j is a weak equivalence, $ev_0 \times ev_1 : A \otimes^h \mathbb{C}[\Delta^1]^h \to A \times A$ is a fibration and $ev_i \circ j = id$. The last equality $ev_i \circ j = id$ is obvious and we focus on the two other properties.

As a start, we want to describe $A \otimes^h \mathbb{C}[\Delta^1]^h$ explicitly. We know that coproducts in $cdga^h$ can be described on the level of the underlying graded holomorphic rings by forgetting the differential. Therefore, in degree i we have

$$(A \otimes^h \mathbb{C}[\Delta^1]^h)^i \simeq (A \otimes^h hol(1))^i \oplus (A \otimes^h hol(1))^{i-1},$$

and the differential is given by

$$\begin{pmatrix} d & \partial \\ 0 & d \end{pmatrix} : (A \otimes^h hol(1))^i \oplus (A \otimes^h hol(1))^{i-1} \longrightarrow (A \otimes^h hol(1))^{i+1} \oplus (A \otimes^h hol(1))^i,$$

where d the differential of $A \otimes^h hol(1)$, and $\partial : (A \otimes^h hol(1))^{i-1} \to (A \otimes^h hol(1))^i$ is the natural morphism induced by the holomorphic derivation $\partial/\partial z : hol(1) \to hol(1)$ by base change to $A \otimes^h hol(1)$. In other words, the underlying complex of $\mathbb{C}$-modules of $A \otimes^h \mathbb{C}[\Delta^1]^h$ is the cocone of the morphism

$$\partial : A \otimes^h hol(1) \to A \otimes^h hol(1).$$

In order to prove that j is a weak equivalence, it is therefore enough to show that the natural inclusion $A \to A \otimes^h hol(1)$ induces a short exact sequence of complexes

$$0 \longrightarrow A \longrightarrow A \otimes^h hol(1) \xrightarrow{\partial} A \otimes^h hol(1) \longrightarrow 0. \tag{22}$$

The statement we want to prove here is independent of the differential of A and we can thus work entirely with the underlying graded holomorphic rings. We start by the degree 0 part. The holomorphic ring A^0 can be written as a filtered colimit in $\mathbf{CR}^h$ of finitely presented holomorphic rings $A^0 = colim_i B_i$. Each B_i is thus the holomorphic ring of closed analytic space $X_i \subset \mathbb{C}^n$ (for some n depending on i), given by a finite number of global holomorphic equations. For each i, we can identify $B_i \otimes^h hol(1) \simeq \mathcal{O}(X_i \times C)$. The diagram above is then isomorphic to the following

$$0 \longrightarrow colim_i \mathcal{O}(X_i) \longrightarrow colim_i \mathcal{O}(X_i \times \mathbb{C}) \xrightarrow{\partial} colim_i \mathcal{O}(X_i \times \mathbb{C}) \longrightarrow 0.$$

Moreover, $\partial : \mathcal{O}(X_i \times \mathbb{C}) \to \mathcal{O}(X_i \times \mathbb{C})$ turns out to be the partial derivative along the canonical coordinate on $\mathbb{C}$. It is then clear that the diagram above is a filtered colimit of short exact sequences and thus is an exact sequence. This shows that 22 is indeed exact in degree 0. Exactness in non-zero degrees is proven similarly. For i fixed, let $M = A^i$ considered as an A^0-module, and we can replace A by the graded holomorphic ring $A^0 \oplus M[-i]$ in order to prove that 22 is exact in degree i. We first reduce to the case where A^0 is of the form $\mathcal{O}(X)$ for some $X \subset \mathbb{C}^n$ as above. We then write M as a filtered colimit of finite presented $\mathcal{O}(X)$-modules, and thus reduce to the case where M is finite presented. In this case it corresponds to a well defined coherent $\mathcal{M}$ sheaf on the Stein space X with a global presentation $\mathcal{O}_X^q \to \mathcal{O}_X^p \to \mathcal{M} \to 0$. The diagram 22 in degree i then can be identified with

$$0 \longrightarrow \Gamma(X, \mathcal{M}) \longrightarrow \Gamma(X \times \mathbb{C}, \pi^*(\mathcal{M})) \xrightarrow{\partial} \Gamma(X \times \mathbb{C}, \pi^*(\mathcal{M})) \longrightarrow 0,$$

where $\pi : X \times \mathbb{C} \to X$ is the first projection and ∂ is again the partial derivative along the coordinate on $\mathbb{C}$, which exists canonically on $\pi^*(\mathcal{M})$. The above diagram is again a short exact sequence.

We have thus shown that the sequence 22 is exact, and thus that the morphism $j : A \to A \otimes^h \mathbb{C}[\Delta^1]^h$ is indeed a weak equivalence of holomorphic cdga's. It remains to show that

$$ev_0 \times ev_1 : A \otimes^h \mathbb{C}[\Delta^1]^h \to A \times A$$

is a fibration, or equivalently that for any i the induced map $(A \otimes^h \mathbb{C}[\Delta^1]^h)^i \to A^i \times A^i$ is surjective. This last map is the composition

$$(A \otimes^h hol(1))^i \oplus (A \otimes^h hol(1))^{i-1} \to (A \otimes^h hol(1))^i \to A^i \times A^i,$$

where the first morphism is the canonical projection on the first factor, and the second map is induced by $ev_{0,1} : hol(1) \to \mathbb{C} \times \mathbb{C}$ given by evaluating a function f at the subset $\{0, 1\} \subset \mathbb{C}$. As the projection is obviously surjective it remains to show that

$$(A \otimes^h hol(1))^i \to A^i \times A^i,$$

is always a surjective morphism for any $A \in cdga^h$. Again, this statement does not depend on the differential of A, so it can be proven more generally for any graded holomorphic ring A. As above, when $i = 0$ we reduce the question to the case where $A^0 = \mathcal{O}(X)$ for $X \subset \mathbb{C}^n$ globally defined by a finite number of holomorphic equations. Then, the map under consideration is the evaluation at $X \times \{0, 1\}$

$$\mathcal{O}(X \times \mathbb{C}) \to \mathcal{O}(X) \times \mathcal{O}(X).$$

This is surjective as a couple of functions (f, g) will be the image of $z.f + (1-z).g$ where z is the global coordinate on $\mathbb{C}$. The case where $i \neq 0$ is treated similarly as before, by reducing to the case where $A^0 = \mathcal{O}(X)$ and furthermore A^i is a finitely presented A^0-module corresponding to a coherent sheaf $\mathcal{M}$ on X which admits a global free resolution. The map under consideration in degree i is then

$$\Gamma(X \times \mathbb{C}, \mathcal{M}) \to \Gamma(X, \mathcal{M}) \times \Gamma(X, \mathcal{M}).$$

To prove the surjectivity of this last map we can write $\mathcal{M}$ as a quotient $\mathcal{O}_X^p \to \mathcal{M}$. As X is Stein, the induced morphism $\Gamma(X, \mathcal{O}_X^p) \to \Gamma(X, \mathcal{M})$ is surjective, and thus we are reduced to the case where $\mathcal{M}$ is free which follows from the case already treated $\mathcal{M} = \mathcal{O}_X$.

This concludes the proof that $A \otimes^h \mathbb{C}[\Delta^1]^h$ is a path object in $cdga^h$, and thus, by [**BM03**, 2.6], it also completes the proof of the proposition. □

The ∞-category obtained from $cdga^h$ by inverting the weak equivalences (the morphisms inducing quasi-isomorphisms on the underlying cdga's) will be denoted by $\mathbf{cdga}^h$. The Quillen adjunction of the above proposition produces an adjunction of ∞-categories

$$\mathbf{cdga}_{\mathbb{C}} \leftrightarrows \mathbf{cdga}^h,$$

whose left adjoint will be denoted by $A \mapsto A^h$. The forgetful right adjoint, if necessary, will be denoted by $A \mapsto N(A)$, but most of the time it will not be written explicitly (so that we will simply write A for the underlying cdga).

5.1.2. Cotangent complexes. As a first comment, for a holomorphic cdga $A \in \mathbf{cdga}^h$ an A-dg-module will be, by definition, a dg-module over $N(A)$ the underlying cdga of A. Their ∞-category is denoted as usual

$$\mathbf{dg}_A := \mathbf{dg}_{N(A)}.$$

Any holomorphic cdga A possesses a module of *holomorphic differential forms* Ω_A^1. This is an A-dg-module such that morphisms of A-dg-modules $\Omega_A^1 \to M$ are in one-to-one correspondence with holomorphic derivations $A \to M$. In order to be more precise, we introduce the holomorphic trivial square zero extension $A \oplus M$. This is a holomorphic cdga whose underlying cdga is the trivial square zero extension of $N(A)$ by M. The holomorphic structure on $A^0 \oplus M^0$ is defined by formula

$$\gamma_f(a_1 + m_1, \ldots, a_n + m_n) = \gamma_f(a_1, \ldots, a_n) + \sum_i \frac{\partial f}{\partial z_i}(a_1, \ldots, a_n).m_i.$$

The holomorphic derivations from A to M are then defined as sections of the natural projection $A \oplus M \to A$. They can be naturally identified with derivations $d : N(A) \to M$ on the underlying cdga which satisfies the further condition that $d^0 : A^0 \to M^0$ is a holomorphic derivation from the holomophic ring A^0 to the A^0-module M^0.

As in the algebraic case, the construction $A \mapsto \Omega^1_A$ can be left derived in order to produce a cotangent complex $\mathbb{L}_A$ for any $A \in \mathbf{cdga}^h$. This cotangent complex satisfies the usual properties of stability by base change and functoriality inside $\mathbf{cdga}^h$. For $A \in \mathbf{cdga}_{\mathbb{C}}$ we have a natural equivalence

$$A^h \otimes_A \mathbb{L}_A \simeq \mathbb{L}_{A^h},$$

as this follows by adjunction from the straight forward observation that $N(A \oplus M) \simeq N(A) \oplus M$ (i.e. trivial square zero extensions are preserved by the forgteful ∞-functor $\mathbf{cdga}^h \to \mathbf{cdga}_{\mathbb{C}}$).

Any object $A \in \mathbf{cdga}^h$ possesses a connective cover $\tau_{\leq 0}A$. This is the right adjoint of the inclusion ∞-functor $\mathbf{cdga}^h_c \hookrightarrow \mathbf{cdga}^h$, where $\mathbf{cdga}^h_c$ stands for the full sub-∞-category spanned by holomorphic cdga's whose underlying cdga are connective. The connective cover construction $A \mapsto \tau_{\leq 0}A$ is moreover compatible with the forgetful ∞-functor to $\mathbf{cdga}$. Indeed, we simply have to notice that if A is a holomorphic cdga, the kernel $Ker(d : A^0 \to A^{-1}) \subset A^0$ is a sub-holomorphic ring of A^0 (because d is a holomorphic derivation).

A connective holomorphic cdga $A \in \mathbf{cdga}^h_c$ is called a *finite cell object* if there is a finite sequence of morphisms in $\mathbf{cdga}^h_c$

$$0 = A_0 \longrightarrow A_1 \longrightarrow \dots \longrightarrow A_{n-1} \longrightarrow A_n = A,$$

where each A_{i+1} is obtained from A_i by choosing some elements $a_i \in A_i^{d_i}$ of degree n_i, and freely adding a finite number of variables y_i of degrees n_{i-1} with $d(y_i) = a_i$. In a slightly weaker manner, $A \in \mathbf{cdga}^h_c$ is an *almost finite cell object* if there is a countable sequence of morphisms

$$0 = A_0 \longrightarrow A_1 \longrightarrow \dots \longrightarrow A_{n-1} \longrightarrow A_n \longrightarrow \dots,$$

with $A = \mathrm{colim}_i A_i$ and where each $A_i \to A_{i+1}$ is as above but with the condition that the sequence of integers n_i tends to ∞. In other words, an almost finite cell object is a cell object with a finite number of cell in each dimension.

DEFINITION 5.1.2.1. *(1) A connective holomorphic cdga A is* of finite presentation *if it is a retract, in the ∞-category $\mathbf{cdga}^h_c$ of a finite cell object.*
(2) A connective holomorphic cdga A is almost of finite presentation *if it is a retract of an almost finite cell object.*

Using [**TV07**, Prop. 2.2], it can be shown that finitely presented holomorphic cdga's are the compact objects in $\mathbf{cdga}^h_c$. Similarly, almost finitely presented holomorphic cdga's are the object A for which $Map(A, -)$ commutes with filtered colimits of uniformly truncated objects: for any filtered system $\{A_i\}$, with $H^j(A_i) = 0$ for all $j \leq n$ (for some fixed integer n), we have $colim_i Map(A, A_i) \simeq Map(A, colim_i A_i)$.

5.1.3. Derived complex analytic spaces and stacks.

DEFINITION 5.1.3.1. *The* ∞-category of affine derived analytic spaces *is the full sub-∞-category of the opposite ∞-category of $\mathbf{cdga}_c^h$ formed by holomorphic cdga's almost of finite presentation. It is denoted by $\mathbf{dAff}^h$. The object in $\mathbf{dAff}^h$ corresponding to an object $A \in \mathbf{cdga}_c^h$ will be denoted symbolically by $\mathbf{Spec}^h A$.*

We note that for an affine derived analytic space $X = \mathbf{Spec}^h A$, the holomorphic ring $H^0(A)$ can be canonically identified with the ring of holomorphic functions on a closed analytic subspace X in $\mathbb{C}^n$. More precisely, we can write A as a connective cell object with finitely many cells in each dimension. For such a cell object, A^0 is a free holomorphic ring on n generators (for some n), and thus $H^0(A)$ becomes a quotient of the holomorphic ring $hol(n) = \mathcal{O}(\mathbb{C}^n)$ by a finitely number of relations. These relations are given by elements $f_i \in hol(n)$ and thus they define a complex analytic subspace $\tau_0(X)$ inside $\mathbb{C}^n$. The holomorphic ring $H^0(A)$ is then naturally isomorphic to the holomorphic ring $\mathcal{O}_X(X)$ of holomorphic functions on $\tau_0(X)$. The complex space $\tau_0(X)$ is called the *truncation* of X. In the same manner, the cohomology groups $H^i(A)$ are finitely generated $H^0(A)$-modules, and therefore they correspond to globally generated coherent sheaves on $\tau_0(X)$ (see [**Pri20**, Prop. 1.24, Lem. 1.25]).

The natural transcendent topology on $\mathbb{C}$ defines a Grothendieck topology on the ∞-categories $\mathbf{dAff}^h$. If $i : \mathbf{Spec}^h A \to \mathbf{Spec}^h B$ is a morphism of affine derived analytic spaces, we say that i is an open immersion if the induced morphism $i_0 : \mathbf{Spec}^h H^0(A) \to \mathbf{Spec}^h H^0(B)$ is an open immersion of analytic spaces, and if furthermore the natural morphisms $H^i(A) \otimes_{H^0(A)} H^0(B) \to H^i(B)$ are isomorphisms (i.e. the coherent sheaves $H^i(A)$ on $\mathbf{Spec}^h H^0(A)$ restrict to $H^i(B)$ along the open immersion i_0). An open covering of affine derived analytic spaces is then a family of open immersion $\{U_i \to X\}_i$ such that $\coprod_i \tau_0(U_i) \to \tau_0(X)$ is a surjective map of complex spaces.

We can also define smooth and étale morphisms, simply by using the holomorphic cotangent complexes. We say that $\mathbf{Spec}^h A \to \mathbf{Spec}^h B$ is smooth (resp. étale) if the relative cotangent complex $\mathbb{L}_{B/A}$ is a projective B-module of finite rank (resp. $\mathbb{L}_{B/A} \simeq 0$). We refer to [**Pir15**, **Pri20**, **Por19**] for more details on these notions.

We are now in a situation in which we have a Grothendieck ∞-site $\mathbf{dAff}^h$ together with a notion of smooth morphisms. We can thus define Artin stacks by using the formal geometricity procedure (see [**TV08**, Ch. 1.3], [**Toe14**, 3.3] and [**PV21**, §4]). In particular, we have the ∞-topos of stacks on this ∞-site, which will be denoted by $\mathbf{dSt}^h$ and whose objects are called *derived analytic stacks*. We also have the full sub-∞-category of Artin (aka geometric) stacks inside $\mathbf{dSt}^h$.

REMARK 5.1.3.2. Note that, by definition, the site $\mathbf{dAff}^h$ consists only of almost finitely presented objects, and thus Artin stacks in these settings will be automatically locally almost finitely presented. In particular, for any Artin derived analytic stack X, its truncation $\tau_0 X$

is an (underived) Artin analytic stack locally of finite presentation, and the homotopy groups sheaves $H^i(\mathcal{O}_X)$ always define coherent sheaves on $\tau_0(X)$.

5.2. De Rham algebras in the holomorphic context

We introduce here a category of *holomorphic* $\mathbb{C}$-linear *graded mixed* cdga's. Its objects are graded mixed cdga's B (see §1.4) together with a holomorphic structure on the weight 0 cdga $B^{(0)}$, and such that the mixed structure $\epsilon : B^{(0)} \to B^{(1)}[-1]$ is a holomorphic derivation on the holomorphic cdga $B^{(0)}$. We can endow this category with a model category structure simply by defining equivalences and fibrations on the underlying complexes $B^{(i)}$ (by forgetting the holomorphic structures).

PROPOSITON 5.2.0.1. *The category $(\epsilon - CAlg(k)^{gr})^h$ of holomorphic graded mixed cdgas, with the above notions of fibrations and equivalences, defines a model category structure such that the forgetful functor to graded mixed cdga's is right Quillen.*

Proof. This is similar to the proof of Proposition 5.1.1.2 (with model structure on $\epsilon - CAlg(k)^{gr}$ as in Definition 1.4.0.1), and in fact easier, since the forgetful functor

$$(\epsilon - CAlg(k)^{gr})^h \longrightarrow \epsilon - CAlg(k)^{gr}$$

commutes with push-outs. We leave the details to the reader. □

The ∞-category obtained from the model category of holomorphic graded mixed cdga's will be denoted by $(\epsilon - \mathbf{cdga}^{gr})^h$, and will be called the ∞-category of *holomorphic graded mixed cdga's.* For any holomorphic cdga A its holomorphic de Rham algebra is defined using the holomorphic cotangent complex

$$\mathbf{DR}(A) = Sym_A(\mathbb{L}_A[1]).$$

It is first a graded mixed cdga where the mixed structure is the de Rham differential. But, the de Rham differential being obviously holomorphic, $\mathbf{DR}(A)$ is an object in $(\epsilon - \mathbf{cdga}^{gr})^h$. We can state things differently, by considering the ∞-functor $(\epsilon - \mathbf{cdga}^{gr})^h \to \mathbf{cdga}^h$, sending B to $B^{(0)}$, and observe that it has a left adjoint given by $B \to \mathbf{DR}(B)$. This adjunction of ∞-categories is moreover induced by a Quillen adjunction on the level of model categories. Of this adjunction we retain the following property

PROPOSITON 5.2.0.2. *Let A be a holomorphic cdga, and $\mathbf{DR}(A)$ its holomorphic de Rham complex as an object in $(\epsilon - \mathbf{cdga}^{gr})^h$. For any $B \in (\epsilon - \mathbf{cdga}^{gr})^h$, the morphism induced on mapping spaces*

$$\mathbf{Map}_{(\epsilon - \mathbf{cdga}^{gr})^h}(\mathbf{DR}(A), B) \longrightarrow \mathbf{Map}_{\mathbf{cdga}^h}(A, B^{(0)})$$

is an equivalence.

REMARK 5.2.0.3. There is of course a big difference between $\mathbf{DR}(A)$ and $\mathbf{DR}(N(A))$, as already the holomorphic cotangent complex $\mathbb{L}_A$ differs from the algebraic cotangent complex $\mathbb{L}_{N(A)}$.

We will also make use of a second adjunction between holomorphic graded mixed cdga's and holomorphic cdga's. Namely, we consider the ∞-functor $A \mapsto A(0)$, sending a holomorphic cdga A to the holomorphic graded mixed cdga $A(0)$ consisting of A sitting purely in weight 0 and endowed with the trivial mixed structure. This defines an ∞-functor $\mathbf{cdga}^h \to (\epsilon - \mathbf{cdga}^{gr})^h$ which commutes with colimits, and so it admits a right adjoint. This right adjoint is called the *holomorphic realization* and is denoted by

$$| - | : (\epsilon - \mathbf{cdga}^{gr})^h \to \mathbf{cdga}^h.$$

PROPOSITON 5.2.0.4. *For any $B \in (\epsilon - \mathbf{cdga}^{gr})^h$, the underlying cdga $N(|B|)$ associated to $|B|$ is canonically equivalent to $|N(B)|$, the realization (as in Remark 1.3.4.5) of the underlying graded mixed cdga $N(B)$ associated to B.*

Proof. This is clear by adjunction and from the fact that $A(0)^h$ obviously identifies with $A^h(0)$. For $A \in \mathbf{cdga}_{\mathbb{C}}$, we have a chain of natural equivalences

$$Map_{\mathbf{cdga}_{\mathbb{C}}}(A, N(|B|)) \simeq Map_{\mathbf{cdga}^h}(A^h, |B|) \simeq Map_{(\epsilon - \mathbf{cdga}^{gr})^h}(A(0)^h, B)$$
$$\simeq Map_{\epsilon - \mathbf{cdga}^{gr}}(A(0), N(B)) \simeq Map_{\mathbf{cdga}_{\mathbb{C}}}(A, |N(B)|).$$

□

DEFINITION 5.2.0.5. *Let A be a (connective) holomorphic cdga and $\mathbf{DR}(A)$ its de Rham holomorphic graded mixed cdga. The* holomorphic derived de Rham complex of A *is defined by* $\widehat{C}^*_{DR}(A) := |\mathbf{DR}(A)| \in \mathbf{cdga}^h$.

REMARK 5.2.0.6.
1. There is a discrepancy between the notation of Definition 5.2.0.5 and Definition 1.4.2.1, as the former involved also a filtration. To be perfectly coherent, we should have used instead the notation $\widehat{C}^*_{DR}(A)^u$. However, we will not use the Hodge filtration in the holomorphic context, so we will keep the notation $\widehat{C}^*_{DR}(A)$ as above. It is clear that Definition 5.2.0.5 also has a refined filtered version, using the natural notion of a *filtered holomorphic cdga*. We will not pursue this direction and leave these details to the interested readers.
2. Moreover, in most cases we will only consider the underlying cdga of $\widehat{C}^*_{DR}(A)$ and will not use often its natural holomorphic structure. However, this structure will be useful for some specific arguments (see for instance the proof of Theorem 5.4.0.1). Because of this, we will still denote by $\widehat{C}^*_{DR}(A)$ the underlying cdga and simply call it the derived de Rham complex of A.

5.3. Analytic derived foliations

For any affine derived analytic space $\mathbf{Spec}^h A$, we remind that we have a holomorphic graded mixed cdga $\mathbf{DR}(A)$ whose underlying graded cdga is $Sym_A(\mathbb{L}_A[1])$, and for which the mixed structure in given by the de Rham differential. The construction $A \mapsto \mathbf{DR}(A)$ defines an ∞-functor

$$\mathbf{DR}(-) : (\mathbf{dAff}^h)^{op} \longrightarrow (\epsilon - \mathbf{cdga}^{gr})^h.$$

We then consider the ∞-functor

$$\mathcal{F}ol^{pr} : (\mathbf{dAff}^h)^{op} \longrightarrow \mathbf{Cat}_\infty,$$

sending $X = \mathbf{Spec}^h A \in \mathbf{dAff}^h$ to the ∞-category of holomorphic graded mixed $\mathbf{DR}(A)$-cdga's B such that the following two conditions are satisfied

(1) the natural morphism $A \to B^{(0)}$ is a quasi-isomorphism
(2) $B^{(1)}$ is an almost perfect complex of A-dg-modules
(3) the natural morphism of graded cdga's

$$Sym_A(B^{(1)}) \to B$$

is a graded quasi-isomorphism.

As opposed to the algebraic situation, the ∞-functor $\mathcal{F}ol^{pr} : (\mathbf{dAff}^h)^{op} \longrightarrow \mathbf{Cat}_\infty$ is only a prestack i.e. it does not satisfy descent. Indeed, if $X \subset \mathbb{C}^n$ is a closed analytic subspace, globally defined by a finite number of equations, and with holomorphic ring of functions A, then almost perfect A-modules correspond to almost perfect complexes on X than can be written as bounded complex of globally generated vector bundles on X. Of course, these are not *all* the almost perfect complexes on X. The associated stack $\mathcal{F}ol$ to $\mathcal{F}ol^{pr}$ can however be computed easily. For this, we sheafify everything on X. We first consider the small site of affine opens $U \subset X$. The holomorphic ring A sheafifies to a sheaf of holomorphic cdga's $\mathcal{O}_X$. We next consider $\mathbf{DR}_X$ the sheaf $(U = \mathbf{Spec}^h B \subset X) \mapsto \mathbf{DR}(B)$ of holomorphic graded mixed cdga's on the underlying topological space of X. The objects in $\mathcal{F}ol(X)$ can then be described as sheaves of holomorphic graded mixed cdga's B, together with a morphism $\mathbf{DR}_X \to B$ in $(\epsilon - \mathbf{cdga}^{gr})^h(X)$ such that

(a) the natural morphism $\mathcal{O}_X \to B^{(0)}$ is a quasi-isomorphism of sheaves of cdga's on X
(b) $B^{(1)}$ is a almost perfect complex of $\mathcal{O}_X$-dg-modules
(c) the natural morphism of sheaves of graded cdga's on X

$$Sym_{\mathcal{O}_X}(B^{(1)}) \to B$$

is a quasi-isomorphisms of sheaves of graded complexes.

By definition $\mathcal{F}ol(X)$ is a full sub-∞-category of the opposite of $\mathbf{DR}_X/(\epsilon - \mathbf{cdga}^{gr})^h(X)$ of sheaves of holomorphic graded mixed cdga's under $\mathbf{DR}_X$. We thus get an ∞-functor $\mathcal{F}ol : (\mathbf{dAff}^h)^{op} \to \mathbf{Cat}_\infty$ satisfying the descent property can be left Kan extended to a colimit preserving ∞-functor on the ∞-category of all derived analytic stacks

$$\mathcal{F}ol : (\mathbf{dSt}^h)^{op} \longrightarrow \mathbf{Cat}_\infty.$$

DEFINITION 5.3.0.1. *For a derived analytic stack $X \in \mathbf{dSt}^h$, the* ∞-category of derived foliations on X *is $\mathcal{F}ol(X)$. For a morphism of derived analytic stacks $f : X \to Y$ the induced ∞-functor $\mathcal{F}ol(Y) \to \mathcal{F}ol(X)$ is called the* pull-back of derived foliations *and is denoted by f^*.*

REMARK 5.3.0.2. By definition derived foliations in the analytic settings are always almost perfect by assumptions. This is only because we want to avoid technical complications related to the notion of quasi-coherent sheaves in the analytic setting, and moreover all the examples treated in this book will be almost perfect.

We finish this section by mentioning that most of the general results and notions we have seen for derived foliations on derived stacks remains valid in the analytic setting, possibly with mild modifications. As a start, if $\mathcal{F} \in \mathcal{F}ol(X)$ is a derived foliation on a derived analytic stack $X \in \mathbf{dSt}^h$, we can define its fake (or big) cotangent complex $\mathbb{L}_{\mathcal{F}}$, which is an $\mathcal{O}_X$-module, simply by sending $u : S := \mathbf{Spec}^h A \to X$ to the $\mathcal{O}_S$-module $\mathbb{L}_{u^*(\mathcal{F})}$. Here, $u^*(\mathcal{F}) \in \mathcal{F}ol(\mathbf{Spec}^h A)$ and can thus be represented by a sheaf of graded mixed $\mathbf{DR}_S$-algebras $\mathbf{DR}_{u^*(\mathcal{F})}$ satisfying conditions (a), (b), (c) above. The $\mathcal{O}_S$-module $\mathbb{L}_{u^*(\mathcal{F})}$ is then by definition $\mathbf{DR}^{(1)}_{u^*(\mathcal{F})}[-1]$, the (shifted) weight one piece of $\mathbf{DR}_{u^*(\mathcal{F})}$, which by assumption is an almost perfect $\mathcal{O}_S$-module. We get this way a family of $\mathcal{O}_S$-modules, functorial in $S \to X$, or in other words an $\mathcal{O}_X$-module denoted by $\mathbb{L}^{big}_{\mathcal{F}}$. It comes equipped with a canonical morphism $\mathbb{L}^{big}_X \to \mathbb{L}^{big}_{\mathcal{F}}$.

A first observation is that the cone of this morphism is an almost perfect $\mathcal{O}_X$-modules. Indeed, this is due to the fact that conormal complexes of derived foliations are stable by pull-backs. We denote this cone by $\mathcal{N}^*_{\mathcal{F}}$. Suppose now that X is a derived analytic Artin stack, and let us consider its cotangent complex $\mathbb{L}_X$, which comes equipped with its canonical morphism $\mathbb{L}^{big}_X \to \mathbb{L}_X$.

DEFINITION 5.3.0.3. *Let X be an analytic derived Artin stack and $\mathcal{F} \in \mathcal{F}ol(X)$ be a derived foliation on X. The* cotangent complex of $\mathcal{F}$, $\mathbb{L}_{\mathcal{F}}$, *is defined by the cartesian square of $\mathcal{O}_X$-modules*

$$\begin{tikzcd}
\mathbb{L}^{big}_X \arrow[r] \arrow[d] & \mathbb{L}^{big}_{\mathcal{F}} \arrow[d] \\
\mathbb{L}_X \arrow[r] & \mathbb{L}_{\mathcal{F}}.
\end{tikzcd}$$

We note that the cone of $\mathbb{L}_X \to \mathbb{L}_{\mathcal{F}}$ is again $\mathcal{N}^*_{\mathcal{F}}$, that we have seen to be almost perfect. Therefore, $\mathbb{L}_{\mathcal{F}}$ is itself an almost perfect $\mathcal{O}_X$-module. Also, from the definition it is easy to deduce the functorial nature of cotangent complexes. For a morphism $f : X \to Y$ of analytic derived Artin stacks, and $\mathcal{F} \in \mathcal{F}ol(Y)$, we have a Cartesian square of almost perfect $\mathcal{O}_Y$-modules

$$\begin{tikzcd}
f^*(\mathbb{L}_Y) \arrow[r] \arrow[d] & f^*(\mathbb{L}_{\mathcal{F}}) \arrow[d] \\
\mathbb{L}_X \arrow[r] & \mathbb{L}_{f^*(\mathcal{F})}.
\end{tikzcd}$$

Finally, we write $\mathbf{DR}_{\mathcal{F}}$ for the sheaf of holomorphic graded mixed cdga's associated to $\mathcal{F} \in \mathcal{F}ol(X)$, which is defined on the big site $\mathbf{dAff}^h/X$ of affine derived analytic spaces over X. We denote by $\widehat{C}^*_{DR}(\mathcal{F}) = |\mathbf{DR}_{\mathcal{F}}|$ its realization in the sense of Definition 5.2.0.5, which is a sheaf of holomorphic cdga's on X.

5.4. Analytic integrability and existence of derived enhancements

The following local integrability result is specific to the analytic setting, and is a version of the classical Frobenius theorem for transversely smooth and rigid derived foliations. It is a direct consequence of a result of Malgrange combined with the analytic version of Corollary 2.3.4.6. However, we still display the result as a theorem to insist on its importance.

As in the algebraic setting, we will say that a derived foliation $\mathcal{F} \in \mathcal{F}ol(X)$ on a derived analytic space X is *analytically integrable*, if there exists a morphism $f : X \to Y$ to another derived analytic space Y such that $\mathcal{F} \simeq f^*(0_Y)$, where $0_Y \in \mathcal{F}ol(Y)$ is the initial foliation on Y. When such an f only exists locally on X we say that $\mathcal{F}$ *locally analytically integrable.*

THEOREM 5.4.0.1. *Let X be smooth Deligne-Mumford analytic stack, and $\mathcal{F} \in \mathcal{F}ol(X)$ be derived foliation. We suppose that the following two conditions are satisfied.*

(1) The derived foliation $\mathcal{F}$ is transversely smooth and rigid.
(2) The derived foliation is smooth in codimension 2.

The, locally on X, $\mathcal{F}$ is integrable.

Proof. The result being local on X we can assume that X is affine and we will allow to shrink it furthermore when necessary. For any point $x \in X$, we consider the formal completion $\hat{X}_x$ of X at x, and the restriction $\hat{\mathcal{F}}_x$ of $\mathcal{F}$ on $\hat{X}_x$. We know by our results on formal integrability that $\hat{\mathcal{F}}_x$ is formally integrable (see Corollary 2.3.4.6).

We now consider the truncation K of $\mathcal{F}$ in the sense of Definition 2.4.1.2. This truncation is a sheaf of holomorphic graded mixed cdga of the form $Sym^{naive}_{\mathcal{O}_X}((\Omega^1_X/K)[1])$, where $K \subset \Omega^1_X$ is the differential ideal defined as the kernel of the morphism $\Omega^1_X \to H^0(\mathbb{L}_{\mathcal{F}})$. The sub-sheaf K is also the image of $\mathcal{N}^*_{\mathcal{F}} \to \Omega^1_X$. By generic smoothness the moprhism between vector bundles $\mathcal{N}^*_{\mathcal{F}} \to \Omega^1_X$ must be a monomorphism as it is a monomorphism on the dense open U. Therefore K is, as an abstract $\mathcal{O}_X$-module isomorphic to $\mathcal{N}^*_{\mathcal{F}}$. Shrinking X is necessary we can assume that $\mathcal{N}^*_{\mathcal{F}}$ is a free vector bundle, and thus that $K \simeq \mathcal{O}^d_X$. The image of the basis elements of $\mathcal{O}^d_X$ in Ω^1_X defines differential forms $(\omega_1, \dots, \omega_d)$ on X which are linearly independent over $\mathcal{O}_X$. We can thus apply [**Mal77**, Thm. 3.1, B] to the forms $(\omega_1, \dots, \omega_d)$ and get that (possibly after localizing on X), there exists a holomorphic map $f : X \to S$, where S is another complex manifold (that can be chosen to be an open in $\mathbb{C}^d$), such that f integrates K: $f^*(\Omega^1_S) = K \subset \Omega^1_X$. We now claim that f also integrates $\mathcal{F}$ in the sense that $\mathcal{F}$ is isomorphic to $f^*(0_S)$ as derived foliations on X.

For this, we consider $\mathcal{O}_{\mathcal{F}} \subset \mathcal{O}_X$ the subring of functions which are killed by the relative de Rham differential $d : \mathcal{O}_X \to \Omega^1_X/K$. This is also the 0-th naive de Rham cohomology group of the differential ideal K, denoted by $H^0_{dR,naive}(K)$. By our assumptions on the codimension of the singularities of $\mathcal{F}$, we know that the natural morphism from derived de Rham cohomology along $\mathcal{F}$ to naive de Rham cohomology of K induces an isomorphism of sheaves of holomorphic rings (see 2.4.2.5)

$$H^0_{dR}(\mathcal{F}) \longrightarrow H^0_{dR,naive}(K) = \mathcal{O}_{\mathcal{F}}.$$

In particular, as $\widehat{C}^*_{DR}(\mathcal{F}) = |\mathbf{DR}_{\mathcal{F}}|$ is the realization of $\mathbf{DR}_{\mathcal{F}}$, and because quasi-smoothness implies that $H^i(|\mathbf{DR}_{\mathcal{F}}|) \simeq 0$ for all $i < 0$, we have a canonical morphism of sheaves of holomorphic cdga's

$$\mathcal{O}_{\mathcal{F}} \simeq H^0_{dR}(\mathcal{F}) \longrightarrow |\mathbf{DR}_{\mathcal{F}}|,$$

which we can compose with the adjunction morphism $|\mathbf{DR}_{\mathcal{F}}| \to \mathbf{DR}_{\mathcal{F}}$ where the sheaf of holomorphic cdga $|\mathbf{DR}_{\mathcal{F}}|$ is considered with its trivial graded mixed structure (remind that $|-|$ is right adjoint to the ∞-functor remembering only the weight 0 piece, see after Proposition 5.2.0.2).

Lemma 5.4.0.2. *There exists a commutative square of sheaves of holomorphic graded mixed cdga's on X*

$$\begin{tikzcd} \mathbf{DR}_X \arrow[r] & \mathbf{DR}_{\mathcal{F}} \\ \mathbf{DR}_{\mathcal{O}_{\mathcal{F}}} \arrow[r] \arrow[u] & \mathcal{O}_{\mathcal{F}}, \arrow[u] \end{tikzcd}$$

where $\mathcal{O}_{\mathcal{F}} \to \mathbf{DR}_{\mathcal{F}}$ is the natural morphism described above, and $\mathbf{DR}_{\mathcal{O}_{\mathcal{F}}} \to \mathbf{DR}_X$ the morphism induced from the inclusion of sheaves of holomorphic rings $\mathcal{O}_{\mathcal{F}} \to \mathcal{O}_X$.

Proof of the lemma. The square of the lemma provides two possible composed morphisms $\mathbf{DR}_{\mathcal{O}_{\mathcal{F}}} \to \mathbf{DR}_{\mathcal{F}}$. These are morphisms of holomorphic graded mixed cdga's. By the universal property of $\mathbf{DR}_{\mathcal{O}_{\mathcal{F}}}$ (see Proposition 5.2.0.2), in order construct a homotopy between these two morphisms it is enough to see that they induce the homotopic morphisms of holomorphic rings in weight 0. But, by construction, these two morphisms in weight 0 both coincide with the canonical inclusion $\mathcal{O}_{\mathcal{F}} \to \mathcal{O}_X$ and thus are homotopic in a uniquely possible manner (the space of self homotopies of this inclusion is contractible). $\square$

By the previous lemma, we have constructed a natural morphism of sheaves of holomoprhic graded mixed cdga's on X

$$\mathbf{DR}_{\mathcal{O}_X/\mathcal{O}_{\mathcal{F}}} = \mathbf{DR}_X \otimes_{\mathbf{DR}(\mathcal{O}_{\mathcal{F}})} \mathcal{O}_{\mathcal{F}} \longrightarrow \mathbf{DR}_{\mathcal{F}}.$$

We claim that this morphism is an equivalence. Indeed, it is enough to check that it induces an equivalence on the associated graded, or equivalently (by multiplicativity) on weight 1. Now, in weight 1 the above morphism is the canonical morphism $\mathbb{L}_{\mathcal{O}_X/\mathcal{O}_{\mathcal{F}}} \to \mathbb{L}_{\mathcal{F}}$, where $\mathbb{L}_{\mathcal{O}_X/\mathcal{O}_{\mathcal{F}}}$ is the holomorphic relative cotangent complex of $\mathcal{O}_{\mathcal{F}} \subset \mathcal{O}_X$, and the morphism above is induced by the holomorphic derivation $\mathcal{O}_X \to \mathbf{DR}^{(1)}_{\mathcal{F}}$ which vanishes on $\mathcal{O}_{\mathcal{F}}$. To see that this morphism $\mathbb{L}_{\mathcal{O}_X/\mathcal{O}_{\mathcal{F}}} \to \mathbb{L}_{\mathcal{F}}$ is an equivalence we can work locally on X, and thus assume that the differential ideal K is integrable by a morphism $f : X \to S$ with S a smooth manifold. In this case, [Mal77, Thm. 2.1.1] shows that $\mathcal{O}_{\mathcal{F}} \simeq f^{-1}(\mathcal{O}_S)$, and thus $\mathbb{L}_{\mathcal{O}_X/\mathcal{O}_{\mathcal{F}}}$ is identified with the coherent sheaf Ω^1_f of relative 1-forms on X over S. The fact that $\Omega^1_f \simeq \mathbb{L}_{\mathcal{F}}$ then follows from the very definition of the fact that f integrates the differential ideal K.

To summarize, we have seen that under the hypothesis of the theorem $\mathbf{DR}_{\mathcal{F}}$ is of the form $\mathbf{DR}_{\mathcal{O}_X/\mathcal{O}_{\mathcal{F}}}$. Moreover, locally on X, $\mathbf{DR}_{\mathcal{O}_X/\mathcal{O}_{\mathcal{F}}}$ is the relative de Rham algebra $\mathbf{DR}_{X/S}$ for a morphism $f : X \to S$ integrating the differential ideal K. This shows, in particular, that f also integrates $\mathcal{F}$ as claimed. □

Also the following two existence and uniqueness results are specific to the analytic setting and does not have an algebraic counterpart (see however Corollary 5.5.0.8 in the proper case). They are very close to Theorem 5.4.0.1, and they are again consequences of the results of [**Mal77**].

Propositon 5.4.0.3. *Let X be a smooth complex manifold and $K \subset \Omega^1_X$ be a differential ideal such that K is a vector bundle.*

(1) If the coherent sheaf Ω^1_X/K is a vector bundle in codimension 2, then there exists at most one, up to equivalence, transversely smooth and rigid derived enhancement for K.
(2) If the coherent sheaf Ω^1_X/K is a vector bundle in codimension 3, then there exists a unique, up to equivalence, transversely smooth and rigid derived enhancement for K.

Proof. (1) This is a consequence of the proof of Theorem 5.4.0.1. Indeed, in that proof we have shown that if $\mathcal{F} \in \mathcal{F}ol(X)$ is a transversely smooth and rigid derived enhancement of K, then we must have $\mathbf{DR}_{\mathcal{F}} \simeq \mathbf{DR}_{\mathcal{O}_X/\mathcal{O}_{\mathcal{F}}}$, for $\mathcal{O}_{\mathcal{F}} \subset \mathcal{O}_X$ the holomorphic subring of functions killed by the de Rham differential $d : \mathcal{O}_X \to \Omega^1_X/K$. This shows indeed that K determines $\mathcal{F}$ and thus implies the uniqueness of $\mathcal{F}$.

(2) The point (1) already insures unicity, so it remains to prove existence. By unicity it is enven enough to prove existence locally on X. The argument is then the same as in the proof of Theorem 5.4.0.1. We let $\mathcal{O}_{\mathcal{F}} \subset \mathcal{O}_X$ be the holomorphic subring of functions killed by the de Rham differential $d : \mathcal{O}_X \to \Omega^1_X/K$. We set $\mathbf{DR}_{\mathcal{F}} := \mathbf{DR}_{\mathcal{O}_X/\mathcal{O}_{\mathcal{F}}}$ be the relative holomorphic de Rham algebra of the inclusion $\mathcal{O}_{\mathcal{F}} \subset \mathcal{O}_X$. It is endowed with the canonical morphism $\mathbf{DR}_X \to \mathbf{DR}_{\mathcal{F}}$, and we claim that as such it is a transversely smooth and rigid derived enhancement of K. For this we use case (A) of [**Mal77**, Thm. 3.1] which insures that K is locally integrable. During the proof of Theorem 5.4.0.1 we have seen that $\mathbf{DR}_{\mathcal{F}}$ is locally of the form $\mathbf{DR}_{X/S}$ for some morphism $f : X \to S$ with S smooth. This clearly implies that the sheaf of graded mixed $\mathbf{DR}_X$-cdga $\mathbf{DR}_{\mathcal{F}}$ satisfies the conditions of being a transversely smooth and rigid derived foliation and that the corresponding differential ideal is equal to K. □

5.5. Analytification of derived foliations: the GAGA theorem

We consider the full sub-∞-category $\mathbf{dSt}_{\mathbb{C}}^{aft} \subset \mathbf{dSt}_{\mathbb{C}}$, consisting of derived stacks which are locally almost of finite presentation over $\mathbb{C}$. These are the derived stacks that can be obtained as colimits of objects of the form $\mathbf{Spec}\, A$ with A a connective cdga which is almost of finite presentation over $\mathbb{C}$. The canonical embedding $\mathbf{dSt}_{\mathbb{C}}^{aft} \subset \mathbf{dSt}_{\mathbb{C}}$ can also be identified with the left Kan extension of the Yoneda embedding restricted to derived affine schemes locally almost of finite presentation $\mathbf{dAff}_{\mathbb{C}}^{aft} \subset \mathbf{dAff}_{\mathbb{C}} \subset \mathbf{dSt}_{\mathbb{C}}$. In particular, an alternative description of $\mathbf{dSt}_{\mathbb{C}}^{aft}$ is as the ∞-category of (hypercomplete) stacks on the étale ∞-site $\mathbf{dAff}_{\mathbb{C}}^{aft}$.

The analytification ∞-functor $\mathbf{cdga}_{c,\mathbb{C}}^{aft} \to \mathbf{cdga}_c^h$ of §5.1 induces an analytification ∞-functor

$$(-)^h : \mathbf{dSt}_{\mathbb{C}}^{aft} \longrightarrow \mathbf{dSt}^h$$

from the ∞-category of derived stacks locally almost of finite presentation to the ∞-category of derived analytic stacks. It is obtained as the left Kan extension of the analytification ∞-functor followed by the Yoneda embedding

$$\mathbf{dAff}_{\mathbb{C}}^{aft} \longrightarrow \mathbf{dAff}^h \longrightarrow \mathbf{dSt}^h$$

where the leftmost ∞-functor is the (opposite of the) analytification functor of §5.1, and it sends $\mathbf{Spec}\, A$ to $\mathbf{Spec}^h\, A^h$, where A^h is the holomorphic cdga generated by A. The analytification ∞-functor $(-)^h : \mathbf{dSt}_{\mathbb{C}}^{aft} \to \mathbf{dSt}^h$ is the left adjoint of the restriction ∞-functor, sending $X \in \mathbf{dSt}^h$ to the derived stack defined by $A \mapsto X(A^h)$. This restriction will be denoted by r so that we have an adjunction

$$(-)^h : \mathbf{dSt}_{\mathbb{C}}^{aft} \leftrightarrows \mathbf{dSt}^h : r.$$

Moreover, the right adjoint r is a geometric morphism of ∞-topoi, or equivalently $(-)^h$ commutes with finite limits. To avoid confusions with the various $(-)^h$ notations involved (for instance for graded mixed cdga's), we will denote this adjunction by

$$u^{-1} : \mathbf{dSt}_{\mathbb{C}}^{aft} \leftrightarrows \mathbf{dSt}^h : u_*.$$

Remind that to both ∞-topoi $\mathbf{dSt}_{\mathbb{C}}^{aft}$ and $\mathbf{dSt}^h$ are associated canonical stacks (in ∞-categories) of *almost perfect* derived foliations, denoted by $\mathcal{F}ol^{ap}$ in the algebraic setting (see Definition 2.1.4.5), and simply by $\mathcal{F}ol$ in the holomorphic setting. These can be compared by means of the analytification ∞-functor as shown in the following proposition.

Propositon 5.5.0.1. *There exists a canonical morphism of derived analytic stacks*

$$u^{-1}(\mathcal{F}ol^{ap}) \longrightarrow \mathcal{F}ol.$$

Proof. By adjunction it is enough to construct a morphism $\mathcal{F}ol^{ap} \to u_*(\mathcal{F}ol)$. For an object $\mathbf{Spec}\, A \in \mathbf{dAff}_{\mathbb{C}}^{aft}$, we consider the analytification construction $(-)^h : \mathcal{F}ol^{ap}(X) \to \mathcal{F}ol(X^h)$, sending a graded mixed cdga B with $B^{(0)} \simeq A$ to B^h, which is a holomorphic graded mixed cdga with $(B^h)^{(0)} \simeq A^h$ satisfying the conditions to define a holomorphic derived foliations on

X^h. This is clearly functorial in A, and thus defines a morphism $\mathcal{F}ol^{ap} \to u_*(\mathcal{F}ol)$ as required. □

The above proposition is needed in order to construct the analytification ∞-functor for derived foliations. Indeed, for $X \in \mathbf{dSt}_{\mathbb{C}}^{aft}$, we define an ∞-functor

$$(-)^h : \mathcal{F}ol^{ap}(X) \to \mathcal{F}ol(X^h)$$

by means of the analytification ∞-functor followed by the morphism of proposition

$$\mathcal{F}ol^{ap}(X) \simeq Map(X, \mathcal{F}ol^{ap}) \to Map(X^h, u^{-1}(\mathcal{F}ol)) \to Map(X^h, \mathcal{F}ol) \simeq \mathcal{F}ol(X^h).$$

Definition 5.5.0.2. *For $X \in \mathbf{dSt}_{\mathbb{C}}^{aft}$, the ∞-functor defined above*

$$(-)^h : \mathcal{F}ol^{ap}(X) \to \mathcal{F}ol(X^h)$$

is called the analytification ∞-functor for derived foliations*.*

It is easy to check that $(-)^h$ extends to a morphism of stacks $\mathcal{F}ol^{ap}(-) \to \mathcal{F}ol(-) \circ (-)^h$. i.e. it is functorial for pullbacks of derived foliations: for any $f : X \to Y$ in $\mathbf{dSt}_{\mathbb{C}}^{aft}$, there is an induced ∞-functor $f^* : \mathcal{F}ol(Y^h) \to \mathcal{F}ol(X^h)$, compatible with composition.

For the next proposition, remind from [**Por19**] the existence of an analytification ∞-functor for almost perfect complexes

$$(-)^h : \mathbf{APerf}(X) \to \mathbf{APerf}(X^h),$$

functorially in $X \in \mathbf{dSt}_k^{aft}$.

Propositon 5.5.0.3. *For $X \in \mathbf{dSt}_{\mathbb{C}}^{aft}$ a derived Artin stack locally of finite presentation, the analytification functor $(-)^h : \mathbf{APerf}(X) \to \mathbf{APerf}(X^h)$ preserves cotangent complexes of derived foliations, i.e. for any $\mathcal{F} \in \mathcal{F}ol^{ap}(X)$, we have a canonical isomorphism*

$$(\mathbb{L}_{\mathcal{F}})^h \simeq \mathbb{L}_{\mathcal{F}^h}$$

in $\mathbf{APerf}(X^h)$.

Proof. This is an easy consequence of [**PY20**, Thm. 5.21]. □

The following result is our version of GAGA theorem for (almost perfect) derived foliations.

Theorem 5.5.0.4. *Let X be a proper algebraic Deligne-Mumford stack. Then the ∞-functor*

$$(-)^h : \mathcal{F}ol^{ap}(X) \to \mathcal{F}ol(X^h)$$

from almost perfect algebraic derived foliations on X to almost perfect holomorphic derived foliations on X^h, is an equivalence of ∞-categories.

Proof. We start by enlarging quite a bit the ∞-categories $\mathcal{F}ol^{ap}(X)$ and $\mathcal{F}ol(X^h)$. For this, let C_X be the ∞-category of sheaves of graded mixed cdga's B on $X_{ét}$ together with an augmentation $B \to \mathcal{O}_X$ (of graded mixed cdga's for the trivial graded mixed structure on $\mathcal{O}_X$) and satisfying the following conditions.

(1) The augmentation $B \to \mathcal{O}_X$ induces an equivalence on weight zero parts $B^{(0)} \simeq \mathcal{O}_X$.
(2) The negative weight pieces of B vanish: $B^{(i)} \simeq 0$ for all $i < 0$.
(3) For all i, the $\mathcal{O}_X$-module $B^{(i)}$ is almost perfect.

Similarly, we define C_{X^h} the ∞-category of sheaves of holomorphic graded mixed cdga's B on X^h_{et} together with an augmentation $B \to \mathcal{O}_{X^h}$ (of holomorphic graded mixed cdga's for the trivial graded mixed structure on $\mathcal{O}_{X^h}$) and satisfying the following conditions.

(1) The augmentation $B \to \mathcal{O}_{X^h}$ induces an equivalence on weight zero parts $B^{(0)} \simeq \mathcal{O}_{X^h}$.
(2) The negative weight pieces of B vanish: $B^{(i)} \simeq 0$ for all $i < 0$.
(3) For all i, the $\mathcal{O}_{X^h}$-module $B^{(i)}$ is almost perfect.

As a first step, we observe the following finiteness result for cotangent complexes.

Lemma 5.5.0.5. *Let $B \in C_X$ (resp. $B \in C_{X^h}$), then the graded cotangent complex $\mathbb{L}_B$ is such that each individual weight piece $\mathbb{L}_B^{(i)}$ is almost perfect over $\mathcal{O}_X$ and is zero for $i < 0$.*

Proof of Lemma 5.5.0.5. We give the proof for objects in C_X, the holomorphic case can be treated similarly.

First of all, this is a statement about the underlying graded cdga so we can forget the mixed structure. We construct a "graded cell decomposition" of B in the usual way, for which cells in a given dimension will be parametrized by an almost perfect $\mathcal{O}_X$-modules. We construct, by induction, a sequence of sheaves of graded $\mathcal{O}_X$-cdga's

$$C(0) = \mathcal{O}_X \longrightarrow C(1) \longrightarrow \dots \longrightarrow C(n) \longrightarrow C(n+1) \longrightarrow \dots \longrightarrow B$$

such that $C(n) \to B$ induces an equivalence in weights less or equal to n, and for each n there exists a push-out square of sheaves of graded cdga's

$$\begin{tikzcd} C(n) \arrow[r] & C(n+1) \\ Sym_{\mathcal{O}_X}(E(n)) \arrow[u] \arrow[r] & \mathcal{O}_X \arrow[u] \end{tikzcd} \tag{23}$$

with $E(n)$ an almost perfect $\mathcal{O}_X$-module pure of weight $n+1$. Assuming that $C(n) \to B$ has been constructed, we consider the induced morphism $C(n)^{(n+1)} \to B^{(n+1)}$ on weight $n+1$ pieces and we denote by $E(n)$ is homotopy fiber, which is a graded $\mathcal{O}_X$-module pure of weight $(n+1)$. The graded $\mathcal{O}_X$-cdga $C(n+1)$ is then defined by the push-out above. Because the morphism $K(n) \to B$ is canonically homotopic to zero, the morphism $C(n) \to B$ factors canonically through $C(n+1) \to B$. Finally, $C(n)$ and $C(n+1)$ do coincide in weight less than n, so $C(n+1) \to B$ induces an equivalence in weights less than n. Moreover, the weight n part of $C(n+1)$ is by definition equivalent to the cone of $E(n) \to C(n)^{(n+1)}$, and thus is sent by an equivalence to $B^{(n+1)}$ as required.

The existence of the sequence C does imply the lemma, as $\mathbb{L}_B$ will coincide with $\mathbb{L}_{C(n)}$ in weight less than n. Moreover, by induction on n and from the push-out square (23), $\mathbb{L}_{C(n)}$ is seen to be a graded perfect $C(n)$-module for all n, and therefore is graded almost perfect over $\mathcal{O}_X$. $\Box$

A second observation is that C_X (resp. C_{X^h}) contains $\mathcal{F}ol^{ap}(X)$ (resp. $\mathcal{F}ol(X^h)$) as a full sub-∞-category.

LEMMA 5.5.0.6. *The natural forgetful ∞-functors*

$$\mathcal{F}ol^{ap}(X) \longrightarrow C_X \qquad \mathcal{F}ol(X^h) \longrightarrow C_{X^h}$$

are fully faithful. Their essential images consists of $B \in C_X$ (resp. $B \in C_{X^h}$) such that $Sym_{\mathcal{O}_X}(B^{(1)}) \simeq B$ (resp. $Sym_{\mathcal{O}_{X^h}}(B^{(1)}) \simeq B$) as graded cdga's.

Proof of Lemma 5.5.0.6. This follows by simply unraveling the various definitions, and the universal property of the de Rham algebras. We give the argument for C_X, the case of C_{X^h} being completely analogous.

For two derived foliations $\mathcal{F}$ and $\mathcal{F}'$ in $\mathcal{F}ol^{ap}(X)$, corresponding to sheaves of graded mixed $\mathbf{DR}_X$-cdga's $\mathbf{DR}_{\mathcal{F}}$ and $\mathbf{DR}_{\mathcal{F}'}$, the mapping space $Map(\mathcal{F}, \mathcal{F}')$ is by definition the homotopy fiber, taken at the identity of $\mathcal{O}_X$, of the natural morphism

$$Map_{\epsilon-\mathbf{cdga}^{gr}(X)}(\mathbf{DR}_{\mathcal{F}'}, \mathbf{DR}_{\mathcal{F}}) \longrightarrow Map_{\epsilon-\mathbf{cdga}^{gr}(X)}(\mathbf{DR}_X, \mathbf{DR}_{\mathcal{F}}) \simeq$$

$$\simeq \mathbf{Map}_{\mathbf{cdga}(X)}(\mathcal{O}_X, \mathbf{DR}^{(0)}_{\mathcal{F}}) \simeq Map_{\mathbf{cdga}(X)}(\mathcal{O}_X, \mathcal{O}_X)$$

(where we have denoted $\epsilon - \mathbf{cdga}^{gr}_X$ and $\mathbf{cdga}_X$ the ∞-categories of sheaves of graded mixed cdga's and of cdga's on $X_{ét}$). Analogously, when $\mathbf{DR}_{\mathcal{F}}$ are considered as augmented towards $\mathcal{O}_X$ via their natural augmentation to $\mathbf{DR}^{(0)}_{\mathcal{F}} \simeq \mathcal{O}_X \simeq \mathbf{DR}^{(0)}_{\mathcal{F}'}$, their mapping space in C_X is the homotopy fiber, taken at the identity, of the natural morphism

$$Map_{\epsilon-\mathbf{cdga}^{gr}(X)}(\mathbf{DR}_{\mathcal{F}'}, \mathbf{DR}_{\mathcal{F}}) \longrightarrow Map_{\epsilon-\mathbf{cdga}^{gr}(X)}(\mathbf{DR}_{\mathcal{F}'}, \mathbf{DR}^{(0)}_{\mathcal{F}}) \simeq$$

$$\simeq Map_{\mathbf{cdga}(X)}(\mathbf{DR}^{(0)}_{\mathcal{F}'}, \mathbf{DR}^{(0)}_{\mathcal{F}}) \simeq Map_{\mathbf{cdga}(X)}(\mathcal{O}_X, \mathcal{O}_X).$$

These two morphisms are easily seen to be equivalent, and thus so are their homotopy fibers. This shows that the ∞-functors of the lemma are fully faithful. Similarly, if $B \to \mathcal{O}_X$ is an object in C_X, choosing an inverse of the equivalence $B^{(0)} \simeq \mathcal{O}_X$ provides a morphism $\mathcal{O}_X \to B^{(0)}$, and thus a morphism $\mathbf{DR}_X \to B$ by the universal property of de Rham alegbras. The graded mixed cdga B, together with this morphism from $\mathbf{DR}_X$ clearly defines an object in $\mathcal{F}ol^{ap}(X)$, whose image by the forgetful ∞-functor is equivalent to B, showing the essential surjectivity. $\Box$

The analytification ∞-functor $(-)^h : \mathcal{F}ol^{ap}(X) \to \mathcal{F}ol(X^h)$ extends to $(-)^h : C_X \to C_{X^h}$, simply by sending a graded mixed cdga to the corresponding holomorphic graded mixed cdga by the analytification construction. Concretely, this sends a sheaf of graded mixed cdga B on $X_{ét}$ to the sheaf (on $X^h_{ét}$) of graded holomorphic cdga $\mathcal{O}_{X^h} \otimes_{u^{-1}(\mathcal{O}_X)} u^{-1}(B)$ endowed with its canonical mixed structure together with its natural augmentation to $\mathcal{O}_{X^h}$. To prove the theorem. It is therefore enough to prove that this extended ∞-functor $(-)^h : C_X \to C_{X^h}$ is an equivalence of ∞-categories.

In order to show this, we consider the forgetful ∞-functor to graded cdga's by forgetting the mixed structures. We let C_X^o be the ∞-category of sheaves of graded cdga's B on $X_{ét}$ together with an augmentation $B \to \mathcal{O}_X$ and satisfying the graded analogues of the previous three conditions: $B^{(0)} \simeq \mathcal{O}_X$, $B^{(i)}$ are zero for $i < 0$ and are almost perfect for $i > 0$. Similarly we have $C_{X^h}^o$ consisting of sheaves of graded cdga's B on $X_{ét}^h$ together with an augmentation $B \to \mathcal{O}_{X^h}$ and satisfying the graded analogues of the previous three conditions: $B^{(0)} \simeq \mathcal{O}_{X^h}$, $B^{(i)}$ are zero for $i < 0$ and are almost perfect for $i > 0$. We have forgetful ∞-functors $C_X \to C_X^o$ and $C_{X^h} \to C_{X^h}^o$ which commute with analytification functors

$$\begin{tikzcd} C_X \arrow[r, "(-)^h"] \arrow[d] & C_{X^h} \arrow[d] \\ C_X^o \arrow[r, "(-)^h"'] & C_{X^h}^o. \end{tikzcd} \tag{24}$$

Lemma 5.5.0.7. *The analytification ∞-functor*

$$(-)^h : C_X^o \to C_{X^h}^o$$

is an equivalence of ∞-categories.

Proof of Lemma 5.5.0.7. This is a simple application of the GAGA theorem for almost perfect complexes of proper derived Artin stack of [**Por19**]. Indeed, GAGA implies that the analytification $(-)^h : \mathbf{APerf}(X) \to \mathbf{APerf}(X^h)$ is an equivalence (note that $\mathbf{APerf}$ is denoted by Coh^- in [**Por19**]). We deduce that it also induces an equivalence on the ∞-categories of $\mathbb{N}$-graded objects $\mathbf{APerf}^{\mathbb{N}}(X) \to \mathbf{APerf}^{\mathbb{N}}(X^h)$. As this equivalence is moreover an equivalence of symmetric monoidal ∞-categories, we get an induced equivalence on commutative algebra objects. Finally, considering commutative algebra augmented towards $\mathcal{O}_X$ and $\mathcal{O}_{X^h}$ we get the statement of the lemma. □

In order to deduce the theorem from the lemma we use the Barr-Beck theorem [**Lur22a**, Theorem 4.7.3.5]. Indeed, the forgetful ∞-functors $C_X \to C_X^o$ and $C_X \to C_{X^h}$ are both monadics. The left adjoint to $C_X \to C_X^o$ sends an augmented sheaf of graded algebras $B \to \mathcal{O}_X$ to $Sym_B(\mathbb{L}_B[1])$, the graded de Rham algebra of B endowed with its total graduation and its natural mixed structure induced by the de Rham differential. The weight 0 part of $Sym_B(\mathbb{L}_B[1])$ is clearly $\mathcal{O}_X$, and thus we do get an object in C_X. Similarly, the forgetful ∞-functor $C_{X^h} \to C_{X^h}^o$ has a left adjoint given by the graded holomorphic de Rham algebra construction. Because cotangent complexes commutes with analytification (see Proposition 5.5.0.3), we have that the commutative square 24 is adjointable and the adjoint square

$$\begin{tikzcd} C_X \arrow[r, "(-)^h"] & C_{X^h} \\ C_X^o \arrow[u] \arrow[r, "(-)^h"'] & C_{X^h}^o \arrow[u] \end{tikzcd}$$

naturally commutes (the canonical natural transformation between the two composition is an equivalence). Therefore, the equivalence $(-)^h : C_X^o \simeq C_{X^h}^o$ preserves the monads defining C_X and C_{X^h}, and we thus have that the induced ∞-functor $(-)^h : C_X \to C_{X^h}$ is an equivalence as required. $\square$

Combining Proposition 5.4.0.3 and Theorem 5.5.0.4 we get the following important corollary.

Corollary 5.5.0.8. *Let X be a smooth and proper algebraic complex variety and $K \subset \Omega_X^1$ be an algebraic differential ideal such that K is a vector bundle. If the coherent sheaf Ω_X^1/K is a vector bundle in codimension* 3*, then there exists a unique, up to equivalence, transversely smooth and rigid derived enhancement for K in the sense of Definition 2.4.1.2.*

The theorem 5.5.0.4 also has a linear version, for crystals which are almost perfect as $\mathcal{O}_X$-modules. The proof of the theorem below can be formally deduced from Theorem 5.5.0.4 by interpreting graded mixed $\mathbf{DR}_{\mathcal{F}}$-modules (and in particular thus crystals) as trivial square zero extensions of $\mathbf{DR}_{\mathcal{F}}$. Its proof will thus be only sketched.

Theorem 5.5.0.9. *Let X be a proper algebraic Deligne-Mumford stack, $\mathcal{F} \in \mathcal{F}ol^{ap}(X)$ an almost perfect derived foliation on X, and $\mathcal{F}^h \in \mathcal{F}ol(X^h)$ its analytification. The analytification ∞-functor*

$$(-)^h : \mathbf{DR}_{\mathcal{F}} - \epsilon - \mathbf{dg}^{gr}(X) \to \mathbf{DR}_{\mathcal{F}^h} - \epsilon - \mathbf{dg}^{gr}(X^h)$$

induces an equivalence on the full sub-∞-category of crystals which are almost perfect as $\mathcal{O}_X$-modules and $\mathcal{O}_{X^h}$-modules respectively.

Proof. We leave to the reader the precise construction of the analytification ∞-functor

$$(-)^h : \mathbf{QCoh}(\mathcal{F}) \to \mathbf{QCoh}(\mathcal{F}^h)$$

along the same lines as the construction $\mathcal{F} \mapsto \mathcal{F}^h$ explained at the beginning of this section. It is such that the following diagram of ∞-categories commutes

$$\begin{tikzcd} \mathbf{DR}_{\mathcal{F}} - \epsilon - \mathbf{dg}^{gr}(X) \arrow[r] \arrow[d] & \mathbf{DR}_{\mathcal{F}^h} - \epsilon - \mathbf{dg}^{gr}(X^) \arrow[d] \\ \mathcal{O}_X - \mathbf{dg}^{gr}(X) \arrow[r] & \mathcal{O}_{X^h} - \mathbf{dg}^{gr}(X^h), \end{tikzcd}$$

where the vertical ∞-functor forget the module structures and only remember the graded $\mathcal{O}$-module structures. The commutative square above is adjointable, the monads being here given by $\mathbf{DR}_{\mathcal{F}} \otimes_{\mathcal{O}_X} -$ and $\mathbf{DR}_{\mathcal{F}} \otimes_{\mathcal{O}_{X^h}} -$ respectively. Using the GAGA theorem for almost perfect $\mathcal{O}$-modules (see [**Por19**]) we easily deduce the theorem. $\square$

CHAPTER 6

Leaf spaces and holonomy

In this chapter we pursue our study of holomorphic derived foliations focusing now of the notion of analytic leaf and analytic leaf spaces. We start by discussing the notion of leaves in a very general setting. In the specific case of quasi-smooth and rigid derived foliation we state natural conditions, principally related to the singularities of the derived foliation, insuring that a leaf space exists in the analytic category. The existence of this leaf space is used to define the holonomy, as well as the relative homotopy type describing the change of topology of the leaves. We provide two main applications of the existence of the leaf space, a Riemann-Hilbert correspondence relating algebraic crystals and relative local systems, and an algebraic integrability result based on notion of Reed stability.

We set $k = \mathbb{C}$ throughout this chapter.

6.1. Formal, algebraic and analytic leaves

In Definition 2.3.3.4 we have seen the notion of formal leaves at a given global point. We remind that, for any derived Artin stack X locally of finite presentation, any perfect derived foliation $\mathcal{F} \in \mathcal{F}ol^p(X/\mathbb{C})$, and any $x \in X(\mathbb{C})$, the formal completion $\hat{\mathcal{F}}_x$ of $\mathcal{F}$ at x is determined by a morphism of perfect dg-Lie algebras $p : \ell_{X,x} \to \ell_{\mathcal{F}_x}$. The *formal leaf* of $\mathcal{F}$ at x is defined to be the formal moduli problem determined by the $\mathbb{C}$-dg-Lie algebra which is the fiber of p. We denote by $\hat{\mathcal{L}}_x$ this formal moduli problem (or the corresponding derived stack by the full embedding of Proposition 2.3.1.2). In this section we will extend this formal notion to more global algebraic and analytic versions.

As a start, we introduce the ∞-category $\mathbf{dSt}_{\mathbb{C}}^{fol}$, of *foliated derived stacks*. Its objects are pairs $(X, \mathcal{F})$, where X is a derived stack and $\mathcal{F}$ is a derived foliation on X. A morphism $(X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ is defined to be a pair (f, u), where $f : X \to Y$ is a morphism of derived stacks, and $u : \mathcal{F}_X \to f^*(\mathcal{F}_Y)$ is a morphism in $\mathcal{F}ol(X/\mathbb{C})$. Formally, $\mathbf{dSt}_{\mathbb{C}}^{fol}$ is obtained as the Grothendieck construction for the ∞-functor of (7)

$$\mathcal{F}ol(-/\mathbb{C}) : \mathbf{dSt}_{\mathbb{C}}^{op} \longrightarrow \mathbf{Cat}_\infty.$$

We let X be a derived Artin stack locally of finite presentation over $\mathbb{C}$, and $\mathcal{F} \in \mathcal{F}ol^p(X/\mathbb{C})$ a perfect derived foliation over X, and consider $(X, \mathcal{F})$ as an object in $\mathbf{dSt}_{\mathbb{C}}^{fol}$. For any derived

Artin stack L, we consider a morphism of foliated derived stacks

$$(f,u):(L,*_L)\longrightarrow(X,\mathcal{F}).$$

Such a morphism can be referred to as a *preleaf of $\mathcal{F}$*. Considering the induced morphisms on tangent complexes, we find the following cartesian square of perfect complexes on L (see Proposition 2.1.4.3)

$$\begin{tikzcd} f^*(\mathbb{T}_{\mathcal{F}}) \arrow[r] & f^*(\mathbb{T}_X) \\ \mathbb{T}_{f^*(\mathcal{F})} \arrow[u] \arrow[r] & \mathbb{T}_L. \arrow[u] \end{tikzcd}$$

The morphism u determines a section of $\mathbb{T}_{f^*(\mathcal{F})}\to\mathbb{T}_L$, and thus a morphism of perfect complexes $\theta_u:\mathbb{T}_L\to f^*(\mathbb{T}_{\mathcal{F}})$.

Definition 6.1.0.1. *Let X be a derived Artin stack locally of finite presentation over $\mathbb{C}$ and $\mathcal{F}\in\mathcal{F}ol^p(X/\mathbb{C})$ a perfect derived foliation over X. With the notations and assumptions as above, a preleaf $(f,u):(L,*_L)\longrightarrow(X,\mathcal{F})$ is called an* (algebraic) leaf of $\mathcal{F}$*, if L is non-empty and the induced morphism $\theta_u:\mathbb{T}_L\to f^*(\mathbb{T}_{\mathcal{F}})$ is an equivalence.*

We can organise the leaves of $\mathcal{F}$ into an ∞-category, by considering the full sub-∞-category in $\mathbf{dSt}_{\mathbb{C}}^{fol}/(X,\mathcal{F})$ of objects of the forms $(L,*_L)$, with L a derived Artin stack, and such that the structure morphism $(L,*_L)\to(X,\mathcal{F})$ satisfies the condition of the previous definition. A first important observation is that a morphism $(L,*_L)\to(L',*_{L'})$ between two leaves of $\mathcal{F}$ induces automatically an étale morphism $L\to L'$. As opposed to the formal notion of leaves of Definition 2.3.3.4, *the* leaf passing through a given point $x\in X(\mathbb{C})$ is not a well defined notion, as we could always modify L by an étale cover to obtain another, non-equivalent, leaf. Under some extra conditions (e.g. for transversely smooth and rigid derived foliations), there is a canonical choice, namely the "maximal leaf" passing through a point, which will be studied later with the introduction of the *leaf space*. We have however the following comparison between the formal and global versions, showing that formally at each point L is equivalent to the formal leaf defined previously.

Propositon 6.1.0.2. *We let X be a derived Artin stack locally of finite presentation over $\mathbb{C}$ and $\mathcal{F}\in\mathcal{F}ol(X/\mathbb{C})$ a perfect derived foliation over X. Let $(L,*_L)\to(X,\mathcal{F})$ be a leaf of $\mathcal{F}$, and $x\in L(\mathbb{C})$ be a global point. Then, the formal completion $(L,x)^\wedge$ of L at x is, as a formal moduli problem, naturally equivalent to the formal leaf $\hat{\mathcal{L}}_x$ as in Definition 2.3.3.4.*

Proof. First of all, any morphism $f:L\to X$, induces a morphism of derived foliations over $\mathbf{Spec}\,k$ (or of dg-Lie algebras according to Corollary 2.3.3.3) $x^*(0_L)\to x^*(0_X)$. The derived foliation $\mathcal{F}$ itself determines a second morphism $x^*(0_X)\to x^*(\mathcal{F})$. Now, a lift of f to a morphism of foliated derived stacks $(L, *_L) \to (X, \mathcal{F})$ defines a commutative diagram in $\mathcal{F}ol(*)$

$$\begin{tikzcd} x^*(0_L) \arrow[r] \arrow[d] & x^*(0_X) \arrow[d] \\ 0 \arrow[r] & x^*(\mathcal{F}). \end{tikzcd}$$

Finally, if we require the lift to be a leaf in the sense of Definition 7.1.0.1, then the square above is moreover cartesian. This implies that $x^*(0_L)$ is equivalent to the fiber of $x^*(0_X) \to x^*(\mathcal{F})$. This proves the Proposition, as via the equivalence of Corollary 2.3.3.3, $x^*(0_L)$ corresponds to the formal completion of L at x, and the fiber of $x^*(0_X) \to x^*(\mathcal{F})$ is by definition the formal leaf at x. $\square$

Analytic leaves are defined in exactly the same way.

Definition 6.1.0.3. *We let X be a derived analytic Artin stack locally of finite presentation over $\mathbb{C}$ and $\mathcal{F} \in \mathcal{F}ol(X/\mathbb{C})$ a perfect derived foliation over X. A morphism $(f, u) : (L, *_L) \longrightarrow (X, \mathcal{F})$ of foliated derived analytic Artin stacks is called an* analytic leaf of $\mathcal{F}$, *if L is non-empty and the induced morphism $\theta_u : \mathbb{T}_L \to f^*(\mathbb{T}_{\mathcal{F}})$ is an equivalence of perfect complexes on L*

Of course, by analytification, an algebraic leaf $(L, *_L) \to (X, \mathcal{F})$ produces an analytic leaf $(L^h, *_{L^h}) \to (X^h, \mathcal{F}^h)$. Algebraic leaves exist very rarely, as opposed to analytic leaves.

Example 6.1.0.4. Let $f : X \to Y$ be a morphism of derived Artin stacks locally of finite presentation. Then, for any global point $y \in Y(\mathbb{C})$, the canonical morphism $(f^{-1}(y), *_{f^{-1}(y)}) \to (X, f^*(0_Y) \simeq *_{X/Y})$ identifies the fibers of f with leaves of the foliation $\mathcal{F} = f^*(0_Y) \in \mathcal{F}ol(X)$. The same argument applies in the analytic setting. In other words, when the derived foliation is integrable by the morphisms f, then the fibers of f are naturally leaves of $\mathcal{F}$, as expected.

6.2. Existence of leaf space and holonomy

In this section we will restrict ourselves to the following specific setting. We let X be a separated and connected (smooth) complex manifold. Let $\mathcal{F} \in \mathcal{F}ol(X/\mathbb{C})$ be a derived foliation on X which is transversely smooth and rigid (i.e. satisfies Definition 2.3.4.1 in the analytic setting). We will study the existence of an analytic leaf space for $\mathcal{F}$, namely we would like to contract all the leaves to points and produce the quotient space. This quotient does not exist in general, and we start by studying extra conditions on $\mathcal{F}$ that will ensure its existence.

6.2.1. Some conditions. We start by discussing some conditions on $\mathcal{F}$ that will later on ensure the existence of a leaf space.

Codimension 2. Because $\mathcal{F}$ is assumed transversely smooth and rigid, the cotangent complex $\mathbb{L}_{\mathcal{F}}$ is a length two complex of vector bundles on X of the form $\mathcal{N}^*_{\mathcal{F}} \to \Omega^1_{X/\mathbb{C}}$, where

$\mathcal{N}^*_{\mathcal{F}}$ is the conormal bundle of $\mathcal{F}$. The *codimension* 2 *condition* for $\mathcal{F}$ states that there exists a closed analytic subset $Z \subset X$, of codimension 2 or higher, such that $\mathbb{L}_{\mathcal{F}}$ restricts on $U = X - Z$ to a vector bundle. Equivalently, $\mathcal{N}^*_{\mathcal{F}} \to \Omega^1_{X/\mathbb{C}}$ is injective, and is a sub-bundle when restricted to U. In particular, the perfect complex $\mathbb{L}_{\mathcal{F}}$ is equivalent to a single coherent sheaf in degree zero, namely $\Omega^1_{X/\mathbb{C}}/\mathcal{N}^*_{\mathcal{F}}$, which is a vector bundle outside of Z. The closed subset Z is called the *singular set of* $\mathcal{F}$.

When the codimension 2 condition is satisfied for $\mathcal{F}$, we will also say that $\mathcal{F}$ *is smooth in codimension* 2.

Flatness. For any point $x \in X$, we have seen that there exists a formal leaf $\hat{\mathcal{L}}_x(\mathcal{F})$ passing through x (Definition 2.3.3.4). This formal leaf is a formal moduli problem given by a dg-Lie algebra whose underlying complex is $\mathbb{T}_{\mathcal{F},x}[-1]$, the fiber of the tangent complex of $\mathcal{F}$ at x, shifted by -1. As $\mathcal{F}$ is transversely smooth and rigid, $\mathbb{T}_{\mathcal{F},x}[-1]$ is cohomologicaly concentrated in degrees $[1,2]$, and therefore corresponds to a pro-representable formal moduli problem (see [**Lur22b**, Prop. 13.3.3.1]): there exists a pro-Artinian connective local cdga $\underline{A} = \text{"}\lim\text{"} A_i$ such that $\hat{\mathcal{L}}_x(\mathcal{F}) \simeq \mathsf{Spf}\underline{A}$, where $\mathsf{Spf}\underline{A}$ is the formal spectrum of A.

The *flatness condition* states that, if $A = \lim_i A_i$ is the realization of $\underline{A}$, then $H^i(A) = 0$ for all $i < 0$. Equivalently, this means that $\hat{\mathcal{L}}_x(\mathcal{F})$ is flat over $\mathbb{C}$. As A is naturally quasi-isomorphic to the Chevalley complex of the dg-Lie algebra $\mathbb{T}_{\mathcal{F},x}[-1]$, the flatness condition can also be stated as the following vanishing for dg-Lie algebra cohomology

$$H^i(\mathbb{T}_{\mathcal{F},x}[-1], \mathbb{C}) \simeq 0 \qquad \forall\, i < 0.$$

Local connectedness. The local connectedness condition only makes sense under the codimension 2 condition. Indeed, we have already seen that when $\mathcal{F}$ is smooth in codimension 2, it is locally (analytically) integrable on X (Theorem 5.4.0.1). Therefore, there exists a basis $\mathcal{B}$ for the topology of X, and for all open $U \in \mathcal{B}$, a holomorphic map $f : U \to \mathbb{C}^d$, such that $\mathcal{F}_{|U} \simeq f^*(0)$. The *local connectedness coinditon* states that B and the holomorphic maps f above can be moreover chosen so that the non-empty fibers of f are all connected. This condition is of topological nature and is not always satisfied. When $d = 1$, it is always satisfied as f is smooth in codimension 2 and thus it is well known that the Milnor fibers of f are always connected (by [**KM75**] the Milnor fiber is $(n-2-k)$ connected, where n is the dimension of X and k the dimension of the singular locus of $\mathcal{F}$). When $d > 1$, there are examples of holomorphic maps f such that the fibers close to a singular fibers are connected or not depending on the direction we approach the singular values. A simple example is given in [**Sab83**], explictly given by $f : \mathbb{C}^3 \to \mathbb{C}^2$ given by $f(x, y, z) = (x^2 - y^2 z, y)$.

Our existence theorem below requires all three conditions, codimension 2, flatness and local connectedness. Before going further we would like to make some comments on these conditions.

REMARK 6.2.1.1. *(1)* As already reminded before the codimension 2 condition implies that $\mathcal{F}$ is locally integrable (Theorem 5.4.0.1). So, for any given point $x \in X$, there is an open $x \in U$ and a holomorphic map $f : U \to \mathbb{C}^d$ such that $f^*(0) \simeq \mathcal{F}_{|U}$. This implies that the formal leaf $\hat{\mathcal{L}}_x(\mathcal{F})$ is the formal completion of $f^{-1}(f(x))$ at the point x, where $f^{-1}(f(x))$ denotes here the derived fiber of f passing through x. As formal completion are faithfully flat (see Corollary A.3.0.6), the flatness of the formal moduli space $\hat{\mathcal{L}}_x(\mathcal{F})$ implies that $f^-1(f(x))$ is flat over $\mathbf{Spec}\,\mathbb{C}$ locally at x. Therefore, flatness implies that the derived fibers are all flat over $\mathbb{C}$, and thus that f must be a flat morphism. The converse is obviously true. Therefore, we see that under the codimension 2 condition, flatness is equivalent to the fact that the all the local holomorphic maps integrating $\mathcal{F}$ are actually flat holomorphic maps.

(2) If $\mathcal{F}$ is moreover smooth, and thus is given by an integrable subbundle of $\mathbb{T}_X$, then all three conditions above are all automatic. Indeed, the local holomorphic maps f must be smooth and thus their fibers will be themselves smooth. In particular, the local connectedness is true because the maps f are submersions. Finally, the formal leaves are the formal completions of smooth sub-varieties (the fibers of f), and are thus all equivalent to formal affine spaces $\widehat{\mathbb{A}}^{n-d}$ (where $n = dimX$), and so they are flat.

6.2.2. Existence theorem. The following result establishes, under appropriate hypotheses, the existence of a *leaf space*.

THEOREM 6.2.2.1. *Let X be a smooth, connected and separated complex manifold and $\mathcal{F} \in \mathcal{F}ol(X/\mathbb{C})$ be a derived foliation which is flat, smooth in codimension 2 and locally connected. There exists a smooth and 1-truncated Deligne-Mumford analytic stack $X /\!\!/ \mathcal{F}$, together with a holomorphic morphism $\pi : X \to X /\!\!/ \mathcal{F}$ with the following properties.*

(1) The morphism π is flat, surjective, and we have $\pi^*(0) \simeq \mathcal{F}$.
(2) The Deligne-Mumford stack $X /\!\!/ \mathcal{F}$ is quasi-separated, effective and is of dimension d the codimension of $\mathcal{F}$.
(3) For any global point $x : * \to X /\!\!/ \mathcal{F}$ the fiber of π at x

$$\pi^{-1}(x) := * \times_{X /\!\!/ \mathcal{F}} X$$

is a connected and separated analytic space.
(4) For any global point $x : * \to X /\!\!/ \mathcal{F}$, the isotropy group H_x of $X /\!\!/ \mathcal{F}$ at x acts freely and properly on $\pi^{-1}(x)$.
(5) The morphism π is relatively connected in the sense of topos theory: the pull-back ∞-functor $\pi^{-1} : St(X /\!\!/ \mathcal{F}) \to St(X)$ is fully faithful when restricted to 0-truncated objects (i.e. sheaves of sets).

Before giving a proof of Theorem 6.2.2.1 some of the terms in (2) require some explanations. First of all, *quasi-separated* here means that the diagonal of $X /\!\!/ \mathcal{F}$ is a separated morphism. In more concrete terms, if the stack $X /\!\!/ \mathcal{F}$ is presented as an étale groupoid object $G \rightrightarrows U$, with U a disjoint union of Stein manifolds, then the smooth analytic space G must be separated. *Effectiveness* of $X /\!\!/ \mathcal{F}$ means that the action of G on U, by germs of biholomorphic maps, is faithful (see [**Car19**, Def. 2.4]). Equivalently, the groupoid G is a sub-groupoid of the

Haefliger's groupoid of germs of biholomorphic isomorphisms. As shown in [**Car19**, Def. 2.9], any Deligne-Mumford stack Y possesses a universal effective quotient Y^{eff}. We will see later than $X /\!/ \mathcal{F}$ is itself the effective quotient of a bigger stack which is pro-étale over it, given as the relative pro-homotopy type of X over $X /\!/ \mathcal{F}$ (see §6.3).

Finally, Theorem 6.2.2.1 allows to set the following useful terminology.

Definition 6.2.2.2. *Let X be a smooth complex manifold and $\mathcal{F}$ be a derived foliation on X, and assume the conditions of Theorem 6.2.2.1 hold.*

(1) The stack $X /\!/ \mathcal{F}$ is called the leaf space of the derived foliation $\mathcal{F}$*.*
(2) For any $x \in X$, the quotient analytic space $\pi^{-1}(\pi(x))/H_{\pi(x)}$ is called the maximal leaf passing through x*.*
(3) The $H_{\pi(x)}$-covering $\pi^{-1}(\pi(x)) \to \pi^{-1}(\pi(x))/H_{\pi(x)}$ is called the holonomy covering*, and the group $H_{\pi(x)}$ the* holonomy group *of $\mathcal{F}$ at x.*

The maximal leaf will be denoted by $\mathcal{L}_x^{max}(\mathcal{F})$, and the holonomy group by $Hol_{\mathcal{F}}(x)$. The holonomy covering of $\mathcal{L}_x^{max}(\mathcal{F})$ will be denoted by $\widetilde{\mathcal{L}}_x^{max}(\mathcal{F})$.

We now prove Theorem 6.2.2.1, except for (3): the connectedness of $\pi^{-1}(x)$ will be proved at the very end of the section, as a consequence of an explicit description of the underlying groupoid of $X /\!/ \mathcal{F}$.

Construction of $X /\!/ \mathcal{F}$. We let $\mathcal{B}$ be the set of all open subsets $U \subset X$ such that there exists a flat holomorphic function $f_U : U \to \mathbb{C}^d$ with connected fibers, and with $f_U^*(0) \simeq \mathcal{F}_{|U}$. As f_U is flat, it has an open image $V \subset \mathbb{C}^d$, and thus factors $U \to V \subset \mathbb{C}^d$. For each $U \in \mathcal{B}$ we fix once for all $f_U : U \to V \subset \mathbb{C}^d$, where V is open, f_U is flat, surjective with connected fibers, and such that $f_U^*(0) \simeq \mathcal{F}_{|U}$. Our conditions on $\mathcal{F}$ ensure that the set $\mathcal{B}$ forms a basis for the topology of X.

Before going further, we note that $\mathcal{B}$ consists of all opens $U \subset X$ with the required properties, so the only choices that have been made here are the choices of the functions f_U. In order to understand the impact of these choices we will need the following lemma. By the codimension 2 condition we know that the perfect complex $\mathbb{L}_{\mathcal{F}}$ is a coherent sheaf, denoted by $\Omega^1_{\mathcal{F}} \simeq H^0(\mathbb{L}_{\mathcal{F}})$. The de Rham differential for $\mathcal{F}$ therefore induces a morphism $dR_{\mathcal{F}} : \mathcal{O}_U \to \Omega^1_{\mathcal{F}}$ of sheaves on U. Because $f_U^*(0) \simeq \mathcal{F}$, this de Rham differential can be identified with the relative de Rham differential $dR_{U/V} : \mathcal{O}_U \to \Omega^1_{U/V}$. In particular, the composite morphism $f^{-1}(\mathcal{O}_V) \to \mathcal{O}_U \to \Omega^1_{\mathcal{F}}$ is zero.

Lemma 6.2.2.3. *The natural morphism of sheaves of rings on U*

$$f^{-1}(\mathcal{O}_V) \to Ker(dR_{\mathcal{F}})$$

is an isomorphism.

Proof of Lemma 6.2.2.3. Because f is flat and surjective the induced morphism $f^{-1}(\mathcal{O}_V) \to \mathcal{O}_U$ is a monomorphism of sheaves. Therefore, $f^{-1}(\mathcal{O}_V) \to Ker(dR_{\mathcal{F}})$ is also mono. It remains to show that a local section of $Ker(dR_{\mathcal{F}})$, locally descend to a local holomorphic function on V.

SUB-LEMMA 6.2.2.4. *Let $f : X \to Y$ be a flat surjective morphism of complex manifolds which is smooth in codimension 2. If an application $u : Y \to \mathbb{C}$ is such that $f \circ u : X \to \mathbb{C}$ is holomorphic then u is holomorphic.*

Proof of sublemma 6.2.2.4. Let $U_0 \subset X$ be the open on which f is smooth, so that $Z = X - U_0$ is a closed analytic subset of codimension 2 or higher. The image of U_0 by f is an open $V_0 \subset Y$, because f is flat, which is dense by Sard's theorem. As $f \circ u : U_0 \to V_0$ is a holomorphic submersion, it has local sections, and thus we deduce that u is holomorphic when restricted to V_0. Moreover, f being flat and surjective implies that the topology of Y is the quotient of the topology on X induced by the map f. This shows that u is also continuous on Y. The application u is continuous on Y and holomorphic on a dense open, and so is holomorphic on the whole Y. $\square$

Now, let s a be local section of $Ker(dR_{\mathcal{F}})$ defined on an open $U' \subset U$. By possibly shrinking U' and using the local connectedness condition, we can assume that the restriction of f_U on U' has connected fibers. The codimension 2 condition implies that the canonical map from derived de Rham cohomology of U over V coincides with the naive de Rham cohomology in degree 0 (see Corollary 2.4.2.5). In particular, $dR_{\mathcal{F}}(s) = 0$ implies that s determines a natural element in $H^0_{dR}(U'/V)$, the derived de Rham cohomology of U' relative to V. We can write this as $s \in |\mathbf{DR}(U/V)|(U')$, where $\mathbf{DR}(U/V)$ is the sheaf on U of graded mixed cdga's of relative de Rham theory of U over V. The sheaf $|\mathbf{DR}(U/V)|$ is naturally a module over $f^{-1}(\mathcal{O}_V)$. Moreover, if $y \in V$ is a point, corresponding to a morphism of sheaves of rings $f^{-1}(\mathcal{O}_V) \to \mathbb{C}$, then we have an equivalence of sheaves of cdga's

$$|\mathbf{DR}(U/V)| \otimes_{f^{-1}(\mathcal{O}_V)} \mathbb{C} \simeq |\mathbf{DR}(U/V) \otimes_{f^{-1}(\mathcal{O}_V)} \mathbb{C}| \simeq j_*(|\mathbf{DR}(U_y/\mathbb{C})|),$$

where $j : U_y \hookrightarrow U$ is the inclusion of the fiber $U_y = f^{-1}(y)$, and where the first equivalence follows from the fact that $\mathbb{C}$ is a perfect $f^{-1}(\mathcal{O}_V)$-module (so that the tensor operation $\otimes_{f^{-1}(\mathcal{O}_V)}\mathbb{C}$ commutes with limits and with the realization). To summarize, the restriction of s on U'_y, lies in derived de Rham cohomology $H^0_{dR}(U'_y/\mathbb{C})$. Now U'_y is connected, so that by the comparison between derived de Rham cohomology and Betti cohomology (see Proposition 1.4.2.2 which also holds in the complex analytic setting), we have that $H^0_{dR}(U'_y/\mathbb{C}) \simeq \mathbb{C}$, and thus the restriction of s on U'_y is a constant function. As $f : U' \to V$ is flat with connected fibers, the function s descends to an application defined on the open $V' = f(U') \subset V$, which is moreover holomorphic by the sublemma 6.2.2.4. This shows that s comes from a local section of $f^{-1}(\mathcal{O}_V)$ as required. $\square$

Lemma 6.2.2.3 has the following important corollary, showing that the choices of the local morphisms $f : U \to V$ are essentially unique.

COROLLARY 6.2.2.5. *Let $f : U \to V$ and $g : U \to V'$ be two flat and surjective holomorphic morphisms of smooth manifolds with $f^*(0) \simeq g^*(0)$. Assume that V' is isomorphic to an open in $\mathbb{C}^d$. If the fibers of f are connected Then, there exists a unique étale holomorphic morphism $\alpha : V \to V'$ such that $\alpha f = g$.*

Proof of Corollary 6.2.2.5. We apply Lemma 6.2.2.3 to local coordinate functions of $\mathbb{C}^d$ restricted on V', and get a unique factorization $\alpha : V \to V'$ such that $\alpha f = g$. Because f is surjective, α must be unique. Finally, as $f^*(0) \simeq g^*(0)$, comparing cotangent complexes show that α must be étale. ☐

We now come back to the base $\mathcal{B}$ of the topology. The set $\mathcal{B}$ is ordered by the inclusions and as such will be considered as a category. We define a functor

$$\mathcal{B} \to \mathbb{C}Man^d_{et},$$

from $\mathcal{B}$ to the category of complex manifolds of dimension d and étale morphisms between them. On objects, the functor ϕ sends U to the space V, the base of the morphism $f_U : U \to V$. If $U' \subset U$ is an inclusion of opens in $\mathcal{B}$, with morphisms $f_U : U \to V$ and $f_{U'} \to V'$. Corollary 6.2.2.5 implies that there exists a unique étale factorization $\alpha : V' \to V$ such that $f_{U'}\alpha = (f_U)_{|U'}$. This provides for each open inclusion $U' \subset U$ in $\mathcal{B}$, a natural morphism $\phi(U') = V' \to \phi(U) = V$ in $\mathbb{C}Man^d_{et}$. This defines the functor ϕ.

Smooth Deligne-Mumford stacks are stable arbitrary colimits of étale morphisms. We therefore consider the colimit of the functor ϕ, and take its universal effective quotient defined in [**Car19**, Def. 2.9]

$$X /\!\!/ \mathcal{F} := (\mathrm{colim}_{U\in\mathcal{B}} V)^{eff}.$$

By construction $X /\!\!/ \mathcal{F}$ is an effective smooth Deligne-Mumford stack of dimension d. Moreover, as $\mathcal{B}$ is a basis for the topology of X, we have that $X \simeq \mathrm{colim}_{U\in B} U$. The local morphisms $f_U : U \to V$ therefore define the projection

$$\pi : X \simeq \mathrm{colim}_{U\in B} U \to \mathrm{colim}_{U\in\mathcal{B}} V \to (\mathrm{colim}_{U\in\mathcal{B}} V)^{eff} = X /\!\!/ \mathcal{F}.$$

The morphism π is flat surjective and $\pi^*(0_{X/\!\!/\mathcal{F}}) \simeq \mathcal{F}$. By construction, the morphism $\pi : X \to X /\!\!/ \mathcal{F}$, restricted to an open $U \in \mathcal{B}$ factorizes as $\pi_{|U} :\ U \xrightarrow{f_U} V \xrightarrow{p_V} X /\!\!/ \mathcal{F}$, where p_V is the composition of the natural morphism $V \to \mathrm{colim}_{U\in B} V$, with the canonical projection to the effective quotient $\mathrm{colim}_{U\in B} V \to X /\!\!/ \mathcal{F}$. The morphism p_V is etale, and f_U is flat, so we see that $\pi_{|U}$ is a flat morphism for all $U \in \mathcal{B}$. As $\mathcal{B}$ is a basis for the topology of X, this shows that π is flat.

To see that π is surjective, it is enough to notice that the universal effective quotient map is always surjective (because it does not the change the space of objects), and moreover that $\coprod_{U\in\mathcal{B}} V \to \mathrm{colim}_{U\in\mathcal{B}} V$ is surjective. As a result, we see that the composition $\coprod_{U\in\mathcal{B}} U \to X \to X /\!\!/ /\mathcal{F}$ is a surjective morphism, and thus π is surjective.

Finally, let us consider the pullback foliation $\pi^*(0_{X/\!\!/\mathcal{F}})$. We thus have two derived foliations on X, $\pi^*(0_{X/\!\!/\mathcal{F}})$ and $\mathcal{F}$. For all $U \in \mathcal{B}$, we know that $f_U^*(0) \simeq \mathcal{F}_{|U}$, but as $p_V : V \to X /\!\!/ \mathcal{F}$ is étale, we have that $f_U^*(0) \simeq \pi^*(0)_{|U}$. Therefore, the two derived foliations $\pi^*(0)$ and $\mathcal{F}$ coincide on each open U. This implies in particular that the corresponding differential ideals $K_{\mathcal{F}}$ and $K_{\pi^*(0)}$, obtained as their truncations (Definition 2.4.1.2), agree on each U, and thus globally agree on X. This means that $\mathcal{F}$ and $\pi^*(0)$ are two transversely smooth and rigid derived enhancements of the same differential ideal $K_{\mathcal{F}}$. It follows from the uniqueness statement of derived enhancements in codimension 2 (Proposition 5.4.0.3) that $\mathcal{F} \simeq \pi^*(0)$ as required.

Quasi-separatedness. The quasi-separatedness property for $X /\!\!/ \mathcal{F}$ follows from the general lemma below.

Lemma 6.2.2.6. *Any smooth effective Deligne-Mumford stack is quasi-separated.*

Proof. Let Y be an effective Deligne-Mumford stack and $U \to Y$ an étale atlas with U a disjoint union of Stein manifolds (hence separated). The nerve of $U \to Y$ defines an étale groupoid $G = U \times_Y U$ acting on U. The quasi-separatedness of Y is equivalent to the fact that the analytic space G is separated.

Let $Haf(U)$ be the *Haefliger groupoid* on U. As a set, $Haf(U)$ consists of triplets (x, y, u), with x and y points in U and u is a germ of biholomorphic isomorphism from x to y. A basis for the topology on $Haf(U)$ is defined as follows. Fix $V \subset U$ and $W \subset U$ two opens, and $u : V \simeq W$ and isomorphism. Associated to V, W and u we have a subset $Hal(U)_{V,W,u} \subset Haf(U)$ consisting of all points $(x, u(x), u_x)$, where $x \in V$, $u(x) \in W$ its image and u_x the germ of isomorphism at x from x to $u(x)$. By definition, a basis for the topology consists of all $Hal(U)_{V,W,u}$ where (V, W, u) varies as above. The groupoid structure on $Haf(U)$ is the obvious one: the source (resp. target) map sends (x, y, u) to x (resp. y), and the composition is given by composing germs of isomorphisms. Since we are working in the holomorphic context, the space $Haf(U)$ is automatically separated, as two holomorphic maps agreeing on an open agree globally (on the connected components meeting this open).

Finally, there is a canonical morphism of groupoids over U, $\phi : G \to Haf(U)$. To a point $u \in G$, with source x and target y, we associate $\phi(u)$ the germs of isomorphisms defined by the diagram

$$U \longleftarrow G \longrightarrow U \ ,$$

which, G being an étale groupoid, induces an isomorphism on germs

$$U_x \xleftarrow{\simeq} G_u \xrightarrow{\simeq} U_y \ .$$

The isomorphism $U_x \simeq U_y$ obtained this way is $\phi(u)$. Now, Y being effective is equivalent to the fact that the morphism ϕ is a monomorphism (see [**Car19**, Def. 2.4]). So $G \hookrightarrow Haf(U)$ is here an injective holomorphic map. As $Haf(U)$ is separated, this implies that G is also separated, as claimed. $\square$

Actions of isotropy groups. Let $x : * \to X /\!\!/ \mathcal{F}$ be a point in $X /\!\!/ \mathcal{F}$, and $H_x = aut(x)$ be the isotropy group of the DM-stack $X /\!\!/ \mathcal{F}$ at x. We have a commutative diagram with cartesian squares

$$\begin{tikzcd}
\pi^{-1}(x) \arrow[r] \arrow[d] & \pi^{-1}(BH_x) \simeq [\pi^{-1}(x)/H_x] \arrow[r] \arrow[d] & X \arrow[d] \\
\bullet \arrow[r] & BH_x \arrow[r] & X /\!\!/ \mathcal{F}.
\end{tikzcd}$$

The canonical morphism $BH_x \to X /\!\!/ \mathcal{F}$ is a monomorphism, and thus $[\pi^{-1}(x)/H_x] \to X$ is also a monomorphism. This implies that the stack $[\pi^{-1}(x)/H_x]$ is homotopically 0-truncated and thus is an analytic space. Moreover, as X is separated, so is $[\pi^{-1}(x)/H_x]$. Therefore, H_x acts freely and properly on $\pi^{-1}(x)$.

Relative connectedness. Let us denote by $Sh(Z) \subset St(Z) := \lim_{S \to X} St(S)$ the full sub-category of 0-truncated objects (for some stack Z), i.e. the category of sheaves of sets. By construction, the functor $\pi^{-1} : Sh(X /\!\!/ \mathcal{F}) \to Sh(X)$ is obtained as a functor on limit categories

$$\lim_{U \in \mathcal{B}} f_U^{-1} : Sh(X /\!\!/ \mathcal{F}) \simeq \lim_{U \in B} Sh(V) \to \lim_{U \in B} Sh(U) \simeq Sh(X).$$

Therefore, in order to prove that π^{-1} is fully faithful, it is enough to prove that $f_U^{-1} : Sh(V) \to Sh(U)$ is fully faithful. But this follows easily from the fact that f_U is a flat surjective morphism with connected fibers, as shown by the next lemma.

LEMMA 6.2.2.7. *Let $f : U \to V$ be an open and surjective morphism such that the topology of V is the quotient of the topology of U (i.e. a subset $T \subset V$ is open if and only if $f^{-1}(T) \subset U$ is open). Then, if f has connected fibers, for any sheaf of sets E on V the canonical morphism*

$$E \to f_* f^{-1}(E)$$

is an isomorphism.

Proof. Let us represent E by its espace étalé $p : E \to V$. As the statement is local on V, it is enough to show that the natural map $E(V) \to f^{-1}(E)(U)$ is bijective. Now, $E(V)$ is the set of continuous sections of $E \to V$, and $f^{-1}(E)(U)$ is the set of continuous maps $s : U \to E$ such that $ps = f$. The map $E(V) \to f^{-1}(E)(U)$ simply sends a section $s : V \to E$ to $sf : U \to E$. As f is surjective this map is clearly injective. Moreover, if $s : U \to E$ is an element $f^{-1}(E)(U)$, we can restrict s to the fiber of f at a point $v \in V$. This provides a continuous map $f^{-1}(v) \to E_v$, where E_v is the fiber of E at v endowed with the discrete topology. As $f^{-1}(v)$ is connected we see that the restriction of s along all the fibers of f is a constant morphism. Because f is a topological quotient, this implies that $s : U \to E$ descend to a continuous map $V \to E$, and thus s is the image of an element in $E(V)$. □

An étale groupoid description. It is possible to describe the stack $X /\!\!/ \mathcal{F}$ by a (more or less explicit) étale groupoid acting on $W := \coprod_{U \in \mathcal{B}} V$. Recall that we have chosen, for any $U \in \mathcal{B}$, a smooth surjective holomoprhic map with connected fibers $f_U : U \to V$ which integrates $\mathcal{F}_{|U}$. We consider all pairs of embeddings

$$U_1 \supset U_0 \subset U_2$$

of opens $U_i \in \mathcal{B}$. Because of Corollary 6.2.2.5, there is a corresponding pair of étale morphisms

$$V_1 \xleftarrow{b} V_0 \xrightarrow{s} V_2.$$

Therefore, for any $v \in V_0$, we deduce a germ of isomorphism $\alpha(v)$ between $(V_1, s(v))$ and $(V_2, b(v))$. When v runs in V_0, we get this way a family of elements $\{(s(v), b(v), \alpha(v))\}_{v \in V_0}$ in $Haf(W)$, the Haefliger groupoid on $W = \coprod_{U \in \mathcal{B}} V$ (see the proof of 6.2.2.6 for the definition of Haf).

Lemma 6.2.2.8. *Let $G \to W$ be the étale groupoid obtained as the nerve of the canonical morphism $W \to X /\!\!/ \mathcal{F}$. Then, G is isomorphic, as a groupoid over W, to the subgroupoid of $Haf(W)$ generated by the elements $(s(v), b(v), \alpha(v))$ for all possible choices of $U_1 \supset U_0 \subset U_2$ in $\mathcal{B}$ and of $v \in V_0$.*

Proof. This follows from a more general formula for the effective part of a colimit of étale morphisms. Let $V_* : I \to \mathbb{C}Man^d_{et}$ be a diagram of smooth complex manifolds of dimension d with étale transition map. Let $X = (\mathrm{colim} U_i)^{eff}$ and $W = \coprod V_i \to X$ be the canonical map. Then, the nerve $W \to X$ is isomorphic, as a groupoid over W, to the subgroupoid of $Haf(W)$ generated by all the germs of isomorphisms generated by diagrams of the form $V_j \longleftarrow V_i \longrightarrow V_k$, image of $j \longleftarrow i \longrightarrow k$ in I by V_*. This last general fact is obvious according to the universal properties of colimits and of the effective part. □

Connectedness of the leaves and their holonomy coverings. We prove here Theorem 6.2.2.1 (3). We first prove that the leaves are connected. For this, let x and y be two points in X such that $\pi(x)$ and $\pi(y)$ are isomorphic in $X /\!\!/ \mathcal{F}$. By Lemma 6.2.2.8 we know that there exists a finite sequence of open inclusions in $\mathcal{B}$

$$U_1 \supset U_{1,2} \subset U_2 \supset U_{2,3} \subset U_3 \dots U_{n-1} \supset U_{n-1,n} \subset U_n,$$

with points $x_{i,j} \in U_{i,j}$, and such that

(1) $x \in U_1$ and $y \in U_n$
(2) $f_1(x_{1,2}) = f_1(x)$ and $f_n(x_{n-1,n}) = f_n(y)$
(3) $f_i(x_{i-1,i}) = f_i(x_{i,i+1})$

where $f_i : U_i \to V_i$ is our chosen map integrating $\mathcal{F}_{|U_i}$. As the fibers of f_i are all connected, it is possible to find a path γ_i joining $x_{i-1,i}$ with $x_{i,i+1}$ and such that $f_i(\gamma_i)$ is constant (for all $i \neq 1, n$). In the same manner, we can choose paths α from x to $x_{1,2}$ with $f_1(\alpha)$ constant, and β from $x_{n-1,n}$ to y with $f_n(\beta)$ constant. The concatenation of the γ_i with α and β provides a path $\gamma : [0,1] \to X$, from x to y and such that for all $t \in [0,1]$ the object $\pi(\gamma(t))$ are all isomorphic. By definition, this means that γ factors through the image of the monomorphism

$\mathcal{L}_x^{max}(\mathcal{F}) \hookrightarrow X$, and thus it is a continuous path in $\mathcal{L}_x^{max}(\mathcal{F})$ joining x and y. This implies that $\mathcal{L}_x^{max}(\mathcal{F})$ is path connected.

It remains to show that the H_x-covering $\widetilde{\mathcal{L}}_x^{max}(\mathcal{F}) \to \mathcal{L}_x^{max}(\mathcal{F})$ is connected or, equivalently, that the corresponding classifying morphism $\pi_1(\mathcal{L}_x^{max}(\mathcal{F}), x) \to H_x$ is a surjective morphism of groups. This is proven using the same argument as before with $x = y$. Indeed, an element $h \in H_x$ is determined by a finite sequence of open inclusions in $\mathcal{B}$

$$U_1 \supset U_{2,3} \subset U_2 \supset U_{2,3} \subset U_3 \dots U_{n-1} \supset U_{n-1,n} \subset U_n,$$

with points $x_{i,j} \in U_{i,j}$, and such that

(1) $x \in U_1 \cap U_n$
(2) $f_1(x_{1,2}) = f_1(x)$ and $f_n(x_{n-1,n}) = f_n(x)$
(3) $f_i(x_{i-1,i}) = f_i(x_{i,i+1})$.

We have seen that there exists a loop γ in X, pointed at $x \in X$, and such that $\pi(\gamma(t))$ is independent of $t \in [0,1]$, up to isomorphism, in $X /\!\!/ \mathcal{F}$. The image of γ by the morphism $\pi_1(\mathcal{L}_x^{max}(\mathcal{F}), x) \to H_x$ is h by construction.

We conclude this § with the following statement, concerning the functoriality of the leaf space construction, and thus of its characterization by a universal property.

Propositon 6.2.2.9. *Let $f : X \to Y$ be a morphism between smooth, connected and separated complex manifolds and $\mathcal{F}_Y \in \mathcal{F}ol(Y)$ and $\mathcal{F}_X := f^*(\mathcal{F}_Y)$. We assume that both derived foliations $\mathcal{F}_Y$ and $\mathcal{F}_X$ are transversely smooth, flat, smooth in codimension 2, and locally connected. Then, there exists a morphism, unique up to a unique isomorphism, $u : X /\!\!/ \mathcal{F}_X \to Y /\!\!/ \mathcal{F}_Y$ such that the following diagram commutes up to isomorphism*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi_X"'] & Y \arrow[d, "\pi_Y"] \\ X /\!\!/ \mathcal{F}_X \arrow[r, "u"'] & Y /\!\!/ \mathcal{F}_Y. \end{tikzcd}$$

Proof. Left to the reader. $\square$

6.3. Relative homotopy theory

Let $f : \mathcal{T} \to \mathcal{T}'$ be a geometric morphism of ∞-topoi. The inverse image $f^{-1} : \mathcal{T}' \to \mathcal{T}$ is exact by definition, and thus admits a pro-left adjoint[1]

$$f_\sharp : \mathcal{T} \to Pro(\mathcal{T}').$$

[1] We recall that this means that $Pro(f^{-1}) : Pro(\mathcal{T}') \to Pro(\mathcal{T})$ has a genuine left adjoint, and $f_\sharp$ is then the composition of $\mathcal{T} \to Pro(\mathcal{T})$ with such a genuine left adjoint.

DEFINITION 6.3.0.1. *The pro-object* $f_\sharp(*) \in Pro(\mathcal{T}')$ *is called the* relative homotopy type of $\mathcal{T}$ over $\mathcal{T}'$. *It is denoted either by* $\mathcal{H}_f$ *or* $\mathcal{H}_{\mathcal{T}/\mathcal{T}'}$.

By definition, the relative homotopy type satisfies the following universal property: for any object $X \in \mathcal{T}'$, we have $Map_{Pro(\mathcal{T}')}(\mathcal{H}_f, X) \simeq Map_{\mathcal{T}}(*, f^{-1}(X))$. This must be understood as the fact that $\mathcal{H}_f$ pro-represents non-abelian cohomology of $\mathcal{T}$ with coefficients coming from $\mathcal{T}'$. When $\mathcal{T}' = \widehat{*} = \mathbf{Top}$ is the punctual ∞-topos, the pro-object $\mathcal{H}_f$ has been introduced in [**TV02**] as the *shape of* $\mathcal{T}$ (see also [**Lur09**, §7.1.6], [**Hoy18**, §2] and [**Vol21**, §3]).

We will restrict ourselves to the case where $\mathcal{T}$ and $\mathcal{T}'$ are ∞-topoi of stacks on nice topological spaces. In particular, all these spaces will be locally CW complexes of finite dimension, and thus locally contractible and locally of finite homotopical dimension. Therefore, all our ∞-topoi appearing below will be hypercomplete, making it easier to check that morphisms are equivalences by just looking at fibers at each points.

Let $f : X \to Y$ be a holomorphic morphism between analytic Stein varieties. In this case, X and Y are CW complexes of finite dimension by Hironaka (see [**Hir75**]), and thus are locally contractible, and locally of finite homotopical dimension. We still denote by $f : \mathbf{St}(X) \to \mathbf{St}(Y)$ the induced geometric morphism between the corresponding ∞-topoi of stacks on X and Y. The pro-object $\mathcal{H}_f \in Pro(\mathbf{St}(Y))$ is in general non-constant, but has the following remarkable properties.

PROPOSITON 6.3.0.2. *With the notation above, the following assertions hold.*

(1) Let $y \in Y$ *be a point, then the fiber of* $\mathcal{H}_f$ *at* y *is pro-constant as a pro-object in* $\mathbf{Top}$*, equivalent to* $Sing(f^{-1}(y))$*, the homotopy type of the fiber of* f *at* y.

(2) If f *is a smooth morphism, then the pro-object* $\mathcal{H}_f$ *is pro-constant.*

(3) Suppose that f *is topologically a locally trivial fibration with fiber* F_0*, then* $\mathcal{H}_f$ *is pro-constant, and a locally constant stack stack with fiber the homotopy type* $Sing(F_0)$.

Proof. (1) Let $i : \{y\} \hookrightarrow Y$ the inclusion map. Let $X_y := f^{-1}(y)$ be the fiber of f at y, $j : X_y \hookrightarrow X$ the inclusion and $f_y : X_y \to *$ the structure map. For any space $F \in \mathbf{St}(*) = \mathbf{Top}$, we have

$$Map_{Pro(\mathbf{Top})}(i^*(\mathcal{H}_f), F) \simeq Map_{Pro(\mathbf{St}(Y))}(\mathcal{H}_f, i_*(F)) \simeq Map_{\mathbf{St}(X)}(*, f^{-1}i_*(F)).$$

But, by proper base change, we have a canonical equivalence $f^{-1}i_*(F) \simeq (f_y)_*j^{-1}(F)$. Therefore, we have

$$Map_{\mathbf{St}(X)}(*, f^{-1}i_*(F)) \simeq Map_{\mathbf{St}(X_y)}(*, j^{-1}(F)) \simeq Map_{Pro(\mathbf{Top})}(\mathcal{H}_{f_y}, F).$$

However, X_y is Stein analytic space, and thus is a CW complex of finite dimension by [**Hir75**]. Using [**Vol21**, Cor. 3.2] combined with [**Lur09**, Cor. 7.2.3.7], we have that $\mathcal{H}_{f_y}$ is pro-constant and equivalent to the usual homotopy type $Sing(X_y)$. As a final result, we have functorial equivalences

$$Map_{Pro(\mathbf{Top})}(i^*(\mathcal{H}_f), F) \simeq Map_{\mathbf{Top}}(Sing(X_y), F),$$

which shows (1) by Yoneda.

(2) This is proven in [**Vol21**, Lem. 3.2], keeping in mind, as in the previous point, that Stein analytic spaces are essential in the sense of [**Vol21**] (as they are of finite covering dimension and locally contractible).

(3) The statement being local on Y, we may assume that $X = Y \times F_0$ with F_0 another Stein space, and $f : Y \times F_0 \to Y$ the natural projection. Let $K = Sing(F_0)$ and $\underline{K}_Y \in \mathbf{St}(Y)$ be the constant stack with fiber K on Y. There exists a canonical morphism $\mathcal{H}_f \to \underline{K}_Y$ defined as follows. Such a morphism, by previous (2), is in fact a morphism of stacks on Y and not of pro-stacks. By definition of $\mathcal{H}_f$, it is thus determined by a global section $* \to \underline{K}_X$ of the constant stack $\underline{K}_X$ with fiber K on X. As X is a CW complex, we know that the stack $\underline{K}_X$ simply sends an open $U \subset X$ to $Map_{\mathbf{Top}}(Sing(U), K)$. Therefore, a global section of $\underline{K}_X$ is determined by a morphism of simplicial sets $Sing(X) \to Sing(F_0)$, which we take as being the natural projection. This defines our morphism $\mathcal{H}_f \to \underline{K}_Y$. To check this is an equivalence we use that $\mathbf{St}(Y)$ is hypercomplete, and thus it is enough to check it is a fiberwise equivalence. The point (1) therefore implies the result. □

The relative homotopy theory mentionned above as several interesting applications, and one of them is the definition of a *monodromy groupoid*, bigger than the leave space $X \mathbin{/\!\!/} \mathcal{F}$ of Definition 6.2.2.2. It is gather in the following definition, and will be an essential object for our Riemann-Hilbert correspondence studied in the next section.

DEFINITION 6.3.0.3. *Let X be a smooth complex manifold and $\mathcal{F} \in \mathcal{F}ol(X)$ a derived foliation satisfying the condition of Theorem 6.2.2.1. The* monodromic leaf space *of X with respect to $\mathcal{F}$ is the relative homotopy type*

$$\mathcal{H}_\pi \in Pro(X \mathbin{/\!\!/} \mathcal{F}),$$

where $\pi : X \to X \mathbin{/\!\!/} \mathcal{F}$ is the natural projection. It is denoted by $\widetilde{X \mathbin{/\!\!/} \mathcal{F}}$.

The monodromic leaf space $\widetilde{X \mathbin{/\!\!/} \mathcal{F}}$ can be understood as a projective system of étale morphism of analytic stacks over $X \mathbin{/\!\!/} \mathcal{F}$, and thus as a smooth pro-Deligne-Mumford stack. The major difference between $\widetilde{X \mathbin{/\!\!/} \mathcal{F}}$ and $X \mathbin{/\!\!/} \mathcal{F}$ is that the former retains the variation of the homotopy types of the leaves of $\mathcal{F}$, whereas in $X \mathbin{/\!\!/} \mathcal{F}$ these leaves are contracted to stack points BH_x. By the base change formula Proposition 6.3.0.2 (1) the fiber of $\widetilde{X \mathbin{/\!\!/} \mathcal{F}} \to X \mathbin{/\!\!/} \mathcal{F}$ at a point x is pro-constant and equivalent to the usual homotopy type of the complex space $\widetilde{\mathcal{L}_x^{max}}(\mathcal{F})$.

REMARK 6.3.0.4. The expression *monodromic leaf space* is not standard but refers to the classical notion of the *monodromy groupoid* of a genuine foliation, which is an extension of the holonomy groupoids. If we were to recover these groupoid notions we would simply have to consider the nerve groupoids of the projections $\widetilde{\pi} : X \to \widetilde{X /\!\!/ \mathcal{F}}$ and $\pi : X \to X /\!\!/ \mathcal{F}$, which should be called the *monodromy groupoid and the holonomy groupoid of* $\mathcal{F}$. The new feature here is that the presence of possible singularities in $\mathcal{F}$ first implies that these groupoids are flat groupoids but are not smooth in general (except when $\mathcal{F}$ is a genuine smooth foliation). Moreover, the monodromy groupoid does not even exist as a genuine groupoid but only as a pro-object in groupoids.

6.4. The Riemann-Hilbert correspondence

In this section we propose a first application of the existence of the leaf spaces and monodromy leaf spaces discussed previously, by proving a Riemann-Hilbert correspondence relating on the one side analytic crystals along a derived foliation $\mathcal{F}$, and a certain ∞-category of relative local systems along $\mathcal{F}$ that we will define. For integrable smooth foliation this recovers Deligne's relative Riemann-Hilbert correspondence of [**Del70**], but our result also includes a relative correspondence for flat, possibly non-smooth, morphisms between complex manifolds, as well as its generalization to the case of derived foliations.

We fix X a separated smooth complex manifold and $\mathcal{F}$ be rigid and quasi-smooth derived foliation on X. We assume that the assumptions of Theorem 6.2.2.1 are satisfied so we have a leaf space $X /\!\!/ \mathcal{F}$ and a monodromic leaf space $\widetilde{X /\!\!/ \mathcal{F}}$, related by natural projections

$$\begin{tikzcd} X \arrow[r, "\widetilde{\pi}"] \arrow[dr, "\pi"'] & \widetilde{X /\!\!/ \mathcal{F}} \arrow[d, "p"] \\ & X /\!\!/ \mathcal{F}. \end{tikzcd}$$

Recall that π is flat surjective and integrates $\mathcal{F}$, $\pi^*(0) \simeq \mathcal{F}$, and similarly for $\widetilde{\pi}$. The holonomy leaf space $\widetilde{X /\!\!/ \mathcal{F}}$ is only a pro-Deligne Mumford stack étale over $X /\!\!/ \mathcal{F}$ and which is relatively connected.

Definition 6.4.0.1. *With the notations and assumptions above, the* ∞-category of sheaves locally constant along $\mathcal{F}$ *(also called* relative local systems along $\mathcal{F}$ *is defined by*

$$\mathcal{D}_{lc}(\mathcal{F}) := \Gamma(X, \pi^{-1}(\mathbf{Perf}))$$

where $\mathbf{Perf}$ *is the stack of perfect complexes of* $\mathcal{O}$*-modules on* $X /\!\!/ \mathcal{F}$*.*

Suppose that $\mathcal{F} = *_X$ is the tautological foliation on X, then $X /\!\!/ \mathcal{F} = *$ and therefore $\pi^{-1}(\mathbf{Perf}) = \pi^{-1}(\mathbf{Perf}(k))$ is the stack of locally constant complexes of sheaves of k-modules on X. Therefore, $\mathcal{D}_{lc}(\mathcal{F})$ can be identified in this case with the ∞-category of complexes of sheaves on X with locally constant cohomology sheaves and perfect fibers.

Definition 6.4.0.1 possesses an interesting interpretation in terms of the monodromic leaf space. Indeed, by definition we have and equivalence of ∞-categories

$$\mathcal{D}_{ct}(\mathcal{F}) \simeq \mathbf{Map}_{Pro(St(X/\!\!/\mathcal{F}))}(\widetilde{X /\!\!/ \mathcal{F}}, \mathbf{Perf}).$$

Therefore, objects in $\mathcal{D}_{ct}(\mathcal{F})$ can be, and probably should be, understood as representations of the connected pro-stack $\widetilde{X /\!\!/ \mathcal{F}}$ in the stack of perfect complexes $\mathbf{Perf}$. Morally speaking these are analytic families, parameterised by $X /\!\!/ \mathcal{F}$, of locally constant perfect complexes on the maximal leaves $\mathcal{L}_x^{max}(\mathcal{F})$, where x varies in $X /\!\!/ \mathcal{F}$. The possible pathologies related to singularities of $\mathcal{F}$ are taken into account in the pro-structure of the object $\widetilde{X /\!\!/ \mathcal{F}}$.

We let X and $\mathcal{F} \in \mathcal{F}ol(X)$ as in Theorem 6.2.2.1. We define a sheaf of cdga's on X as

$$\mathcal{A}_{\mathcal{F}} := |\mathbf{DR}_{\mathcal{F}}| \in \mathbf{cdga}(X),$$

where $|-|$ is the non-Tate realization functor of Remark 1.3.4.5. In other words $\mathcal{A}_{\mathcal{F}}$ is the sheaf sending $U \subset X$ to the Hodge completed derived de Rham cohomology along the leaves $\widehat{C}^*_{DR}(\mathcal{F}_U)$. This is a non-negatively graded sheaf of cdga's (because $\mathcal{F}$ is quasi-smooth) and moreover we know that our conditions imply that $H^0(\mathcal{A}_{\mathcal{F}})$ is naturally isomorphic to $\mathcal{O}_{\mathcal{F}} = \pi^{-1}(\mathcal{O}_{X/\!\!/\mathcal{F}})$ (because of Corollary 2.4.2.5 and Lemma 6.2.2.3), the sheaf of locally constant holomorphic functions along $\mathcal{F}$.

Definition 6.4.0.2. *With the notations and assumptions above, the* ∞-category of sheaves almost locally constant along $\mathcal{F}$ *is defined by*

$$\mathcal{D}_{alc}(\mathcal{F}) := \mathbf{Perf}(\mathcal{A}_{\mathcal{F}})$$

the ∞-category of sheaves of perfect $\mathcal{A}_{\mathcal{F}}$-modules on X.

Finally, we also have the perfect crystals along $\mathcal{F}$ which, by definition, are sheaves E of graded mixed $\mathbf{DR}_{\mathcal{F}}$-modules on X satisfying the following two conditions.

(1) As a sheaf of graded $\mathbf{DR}_{\mathcal{F}}$-module E is free of weight 0

$$\mathbf{DR}_{\mathcal{F}} \otimes_{\mathcal{O}_X} E^{(0)} \simeq E.$$

(2) The $\mathcal{O}_X$-module $E^{(0)}$ is perfect.

This ∞-category will be denoted by $\mathbf{Perf}(\mathcal{F})$ as in the algebraic situation.

We can now consider two natural ∞-functors

$$\mathbf{Perf}(\mathcal{F}) \xrightarrow{Sol} \mathcal{A}_{\mathcal{F}} - \mathbf{dg}(X) \xleftarrow{\mathcal{A}_{\mathcal{F}} \otimes \mathcal{O}_{\mathcal{F}}} \mathcal{D}_{lc}(\mathcal{F}).$$

The ∞-functor Sol (for *solutions*) sends a crystal E to the sheaf of morphisms $\mathbb{R}\underline{\mathcal{H}om}_{\mathbf{Perf}(\mathcal{F})}(\mathcal{O}_X, E)$, where $\mathcal{O}_X$ is the usual unit crystal along $\mathcal{F}$. It is obviously equipped with a left $\mathcal{A}_{\mathcal{F}}$-module structure, as $\mathcal{A}_\mathcal{F} \simeq \mathbb{R}\underline{\mathcal{H}om}_{\mathbf{Perf}(\mathcal{F})}(\mathcal{O}_X, \mathcal{O}_X)$. The ∞-functor on the right hand side simply is the base change from $\mathcal{O}_\mathcal{F}$-module to $\mathcal{A}_\mathcal{F}$ via the canonical morphism

$$\mathcal{O}_\mathcal{F} = H^0(\mathcal{A}_\mathcal{F}) \to \mathcal{A}_\mathcal{F}$$

whose existence is insured by the fact that $\mathcal{A}_\mathcal{F}$ is cohomologically concentrated in non-negative degrees. By construction, this second ∞-functor factors throught the full sub-∞-category $\mathbf{Perf}(\mathcal{A}_\mathcal{F}) \subset \mathcal{A}_\mathcal{F} - \mathbf{dg}(X)$. However, we de not if Sol has its image contained in perfect $\mathcal{A}_\mathcal{F}$-modules as well, and we are thus forced to introduce the notion of unipotent objects as follows.

Definition 6.4.0.3. *An object $E \in \mathbf{Perf}(\mathcal{F})$ is* unipotent*, if locally on X, E belongs to the thick triangulated sub-∞-category of $\mathbf{Perf}(\mathcal{F})$ generated by the unit crystal $\mathcal{O}_X$. The full sub-∞-category of unipotent objects will be denoted by*

$$\mathbf{Perf}^{uni}(\mathcal{F}) \subset \mathbf{Perf}(\mathcal{F}).$$

By the definition above, it is now clear that if E is unipotent, then $Sol(E)$ is itself locally in the thick triangulated sub-∞-category of $\mathcal{A}_\mathcal{F} - \mathbf{dg}(X)$ generated by $\mathcal{A}_\mathcal{F}$. In other words, Sol sends unipotent crystals to perfect $\mathcal{A}_\mathcal{F}$-modules.

Theorem 6.4.0.4. *The ∞-functors constructed above*

$$\mathbf{Perf}^{uni}(\mathcal{F}) \xrightarrow{Sol} \mathbf{Perf}(\mathcal{A}_\mathcal{F}) \xleftarrow{\mathcal{A}_\mathcal{F}\otimes_{\mathcal{O}_\mathcal{F}}} \mathcal{D}_{lc}(\mathcal{F})$$

satisfy the following assertions.

(1) The ∞-functor Sol induces an equivalence $\mathbf{Perf}^{uni}(\mathcal{F}) \simeq \mathbf{Perf}(\mathcal{A}_\mathcal{F})$.
(2) The ∞-functor $\mathcal{A}_\mathcal{F}\otimes_{\mathcal{O}_\mathcal{F}}$ induces an equivalence when restricted to the genuine sub-category $\mathbf{Vect}(\mathcal{O}_\mathcal{F}) \subset \mathcal{D}_{lc}(\mathcal{F})$, consisting of locally free $\mathcal{O}_\mathcal{F}$-modules of finite rank.
(3) Any object $E \in \mathbf{Perf}^{uni}(\mathcal{F})$ which is a vector bundle as an $\mathcal{O}_X$-module is unipotent.

Proof. (1) By the definition of unipotent objects Sol is clearly fully faithful. It is also locally essentially surjective, as perfect $\mathcal{A}_\mathcal{F}$-modules are locally build out from $\mathcal{A}_\mathcal{F}$. As both ∞-categories are global sections of stacks over X, this shows that in fact Sol is an equivalence of the ∞-categories as wanted.

(2) This is formally implied by the fact that $\mathcal{A}_\mathcal{F}$ is cohomologically non-negatively graded, and the fact that the morphism $\mathcal{O}_\mathcal{F} \to \mathcal{A}_\mathcal{F}$ identifies $\mathcal{O}_F$ with $H^0(\mathcal{A}_\mathcal{F})$. The image of $\mathbf{Vect}(\mathcal{O}_F)$ by $\mathcal{A}_\mathcal{F}\otimes_{\mathcal{O}_\mathcal{F}}$ clearly consists of all locally free $\mathcal{A}_\mathcal{F}$-modules, which we can denote by $\mathbf{Vect}(\mathcal{A}_\mathcal{F})$.

(3) This is a consequence of [Mal77] and is proved in [TV23b, Thm. 3.2.3]. We do not reproduce the poof here. ∎

Remark 6.4.0.5. Part (1) of the theorem above is true under much weaker assumptions on $\mathcal{F}$. In particular it also applies to some situations where $\mathcal{F}$ is not rigid. We refer to [TV23b] for more on this.

An important consequence of Theorem 6.4.0.4 is the following corollary, which is the most interesting version of our Riemann-Hilbert correspondence.

Corollary 6.4.0.6. *Under the same assumptions as in Theorem 6.4.0.6, the functor Sol induces an equivalence of categories*

$$\mathbf{Vect}(\mathcal{F}) \simeq \mathbf{Vect}(\mathcal{O}_{\mathcal{F}}) \simeq \Gamma(X, \pi^{-1}(\mathbf{Vect})).$$

Combine with GAGA, this gives the following corollary.

Corollary 6.4.0.7. *Let X be a smooth and proper algebraic complex variety and $\mathcal{F} \in \mathcal{F}ol(X)$. We assume that $\mathcal{F}^h \in \mathcal{F}ol(X^h)$ satisfies the conditions of Theorem 6.2.2.1. Then, Sol induces an equivalences of categories*

$$\mathbf{Vect}(\mathcal{F}) \simeq \mathbf{Vect}(\mathcal{O}_{\mathcal{F}^h}) \simeq \Gamma(X^h, \pi^{-1}(\mathbf{Vect})).$$

Tracking back the various functors, it is easy to see that the equivalences of the above corollary are equivalence of tensor categories, and are further functorial in the pair $(X, \mathcal{F})$. In particular, fixing a point $x \in X(\mathbb{C})$ provides a fiber functor

$$\omega_x : \mathbf{Vect}(\mathcal{F}) \to \mathbf{Vect}(\mathbb{C})$$

as explained in 3.4. The abstract algebraic holonomy group $\widehat{Hol}^{alg}_{\mathcal{F}}(x)$ is defined in Definition 3.4.0.2 as the group of tensor automorphisms of ω_x restricted to the tensor sub-category generated by the conormal bundle $\mathcal{N}^*_{\mathcal{F}}$. By corollary 6.4.0.7, when X is smooth proper and $\mathcal{F}$ satisfies the assumptions of 6.2.2.1, this can also be interpreted as the group of tensor automorphisms of the fiber functor

$$\omega_x : \mathbf{Vect}(\mathcal{O}_{\mathcal{F}^h}) \to \mathbf{Vect}(\mathbb{C})$$

now restricted to the object $\pi^{-1}(\Omega^1_{X/\!\!/\mathcal{F}}) \in \mathbf{Vect}(\mathcal{O}_{\mathcal{F}^h})$. Now, the holonomy group H_x, in the sense of Definition 6.2.2.2, obviously acts on the fiber of $\Omega^1_{X/\!\!/\mathcal{F}}$ at x, and thus there is a natural morphism

$$H_x \to \widehat{Hol}^{alg}_{\mathcal{F}}(x)$$

from the holonomy in the sense of 6.2.2.2 to the abstract algebraic holonomy in the sense of 3.4.0.2. Both of these groups acts faithfully on the formal completion of $X /\!\!/ \mathcal{F}$ at $\pi(x)$ and thus the morphism $H_x \to \widehat{Hol}^{alg}_{\mathcal{F}}(x)$ is injective. We believe it can also be Zariski dense in many situations but we do not adress this question here. Finally, we can now compare the two notions of holonomy we have defined in this book.

Corollary 6.4.0.8. *With the notations and assumptions of Corollary 6.4.0.7, the holonomy group H_x of 6.2.2.2 identifies with a sub-group of the algebraic holonomy $Hol^{alg}_{\mathcal{F}}(x)$ of 3.4.*

An important aspect of the corollary above is that the group $Hol^{alg}_{\mathcal{F}}(x)$ admits a purely algebraic description whereas the group H_x is of transcendental nature.

6.5. Reeb stability and algebraic integrability

In this section we give a second application of the existence of a leaf space, and we prove a fully algebraic integrability result. This theorem is on one side an algebraic version of [**Per01**] proved in the Kähler situation and for smooth foliations, but on the other side also a generalization to the quasi-smooth and thus singular setting. Our proof is mainly inspired by [**Per01**] with Gromov compactness being replaced by the properness of Hilbert schemes.

THEOREM 6.5.0.1. *Let X be a smooth, proper and connected complex algebraic variety and $\mathcal{F} \in \mathcal{F}ol(X/\mathbb{C})$ be a transversely smooth derived foliation on X. We suppose that $\mathcal{F}$ is flat, smooth in codimension 2, and locally connected. We denote by X^h and $\mathcal{F}^h$ the analytifications of X and $\mathcal{F}$. Then, the following three conditions are equivalent.*

(1) The analytic Deligne-Mumford stack $X^h /\!\!/ \mathcal{F}^h$ is algebraizable and proper.
(2) There exists a smooth and proper effective Deligne-Mumford stack M, and a flat surjective morphism $p : X \to M$ with connected fibers such that $p^*(0) \simeq \mathcal{F}$.
(3) There exists one point $x \in X^h$, such that $\mathcal{L}_x^{max}(\mathcal{F}^h)$, the maximal leaf passing through x, is compact, and its holonomy group $Hol_{\mathcal{F}^h}(x)$ is finite (or equivalently the holonomy covering $\widetilde{\mathcal{L}_x^{max}(\mathcal{F}^h)}$ is compact).

Proof. (1) $\Rightarrow$ (2) Let M be a smooth and proper Deligne-Mumford stack with $M^h \simeq X^h /\!\!/ \mathcal{F}^h$. By GAGA, the morphism $\pi : X^h \to M^h$ algebraizes to a morphism $p : X \to M$. Moreover, the GAGA theorem for perfect derived foliations (Theorem 5.5.0.4), we know that the analytification ∞-functor $\mathcal{F}ol(X) \to \mathcal{F}ol(X^h)$ is an equivalence of categories. From $\pi^*(0) \simeq \mathcal{F}^h$, and from the compatibility of the analytification with pull-backs (see §5.5), we deduce that $p^*(0) \simeq \mathcal{F}$. Moreover, as $\pi = p^h$ has connected fibers, so does p.

(2) $\Rightarrow$ (3) By the universal property of the leaf spaces (Proposition 6.2.2.9) we have a canonical identification $M^h \simeq X^h /\!\!/ \mathcal{F}^h$. In particular, the maximal leaves of $\mathcal{F}^h$ are automatically the analytification of the fibers of p, and thus, by hypothesis, they are compact. In the same way, the holonomy groups of $\mathcal{F}^h$ are the isotropy groups of M, and so they are finite because M is proper.

(3) $\Rightarrow$ (1) This implication is the main content of the theorem, and its proof will be somehow long. Let $x \in X$ be a closed point such that $\mathcal{L}_x^{max}(\mathcal{F}^h)$ is compact. The monomorphism $\mathcal{L}_x^{max}(\mathcal{F}^h) \hookrightarrow X^h$ is thus a closed immersion, and by GAGA $\mathcal{L}_x^{max}(\mathcal{F}^h)$ arises as the analytification of a closed subscheme $X_x \subset X$. We denote by $\widehat{X_x}$ the formal completion of X along X_x.

In the same manner, the monomorphism $BH_x \hookrightarrow X^h /\!\!/ /\mathcal{F}^h$ is a closed immersion. Indeed, $X^h \longrightarrow X^h /\!\!/ \mathcal{F}^h$ being flat and surjective, it is enough (by flat descent) to show that the pull-back $BH_x \times_{X^h /\!\!/ \mathcal{F}^h} X^h \hookrightarrow X^h$ is a closed immersion. But $BH_x \times_{X^h /\!\!/ \mathcal{F}^h} X^h \simeq X_x^h \simeq \mathcal{L}_x^{max}(\mathcal{F}^h)$, and we have seen that it is closed in X^h, by assumption. We can therefore consider $\widehat{BH_x}$, the formal completion of the stack $X^h /\!\!/ \mathcal{F}^h$ along the closed substack BH_x. We thus get a diagram of formal stacks

$$\begin{tikzcd} \widehat{X_x} \arrow[r] \arrow[d] & X \\ \widehat{BH_x}. & \end{tikzcd} \tag{25}$$

The formal stack $\widehat{BH_x}$ is moreover the form $[\widehat{A}^d/H_x]$, and as H_x is finite we can even assume that the action of H_x is induced by a linear action on $\mathbb{A}^d$. The above diagram can be consider as a formal $\widehat{BH_x}$-point of the stack $Fin(X)$, of finite schemes over X, whose reduced BH_x-point is given by $X_x \to X \times BH_x$. Since the stack $Fin(X)$ is an algebraic stack locally of finite presentation, this formal point can be algebraized by Artin algebraization theorem ([**Art69**]). Therefore we can find a pointed smooth variety (S, s), which is an étale neighborhood of 0 in $\mathbb{A}^d$, with a compatible H_x-action (fixing s), a scheme Z with a flat and proper map $Z \to [S/H_x]$, and a diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "q"] & X \\ {[S/H_x].} & \end{tikzcd} \tag{26}$$

The diagram (26) is such that the fiber Z at $BH_x = [s/H_x] \hookrightarrow [S/H_x]$ is isomorphic to X_x, as a subscheme in X. Moreover, the formal completion of (26) at $X_x \subset Z$ recovers the diagram (25) above.

Lemma 6.5.0.2. *By possibly shrinking (S, s) to a smaller étale neighborhood of* 0 *in $\mathbb{A}^d$, we have $f^*(\mathcal{F}^h) \simeq q^*(0)$ as derived analytic foliations on Z (i.e. in $\mathcal{F}ol(Z^h)$).*

Proof of the lemma. To prove this lemma we use the uniqueness of derived enhancement of Proposition 5.4.0.3. For this, we need to prove that $f^*(\mathcal{F}^h)$ and $q^*(0)$ are both smooth in codimension 2, and moreover that the corresponding differential ideals in $\Omega^1_{Z^h}$ coincide.

We first notice that, by construction, the morpshim $f : Z \to X$ is formally étale around $X_x \hookrightarrow Z$, and thus, by shrinking S if necessary, we may assume that f is an étale morphism. Therefore, it is clear that $f^*(\mathcal{F}^h)$ is smooth in codimension 2, as it is locally isomorphic to $\mathcal{F}$. Concerning $q^*(0)$, in order to prove that it is smooth in codimension 2 it is enough to show that $\mathbb{L}_{Z/[S/H_x]}$ is a vector bundle outside of a codimension 2 closed subset. However, the perfect complexes $\mathbb{L}_{Z/[S/H_x]}$ and $f^*(\mathbb{L}_{\mathcal{F}})$ are quasi-isomorphic on the formal completion $\widehat{X_x}$. Because the stack of quasi-isomorphisms between two perfect complex is an Artin stack of finite presentation (see [**TV07**]), we deduce that these two perfect complexes must be quasi-isomorphic locally around $X_x \subset Z$. Therefore, by shrinking S if necessary, we can assume that they are quasi-isomorphic on Z. This implies that also $q^*(0)$ is smooth in codimension 2. Finally, we can use the same argument but now for the stack of quasi-isomorphisms between $\mathbb{L}_{Z/[S/H_x]}$ and $f^*(\mathbb{L}_{\mathcal{F}})$ compatible with the canonical morphism

$$\mathbb{L}_{Z/[S/H_x]} \longleftarrow \Omega^1_Z \longrightarrow f^*(\mathbb{L}_{\mathcal{F}}).$$

This shows that the two quotients of coherent sheaves

$$\Omega^1_Z \to H^0(\mathbb{L}_{Z/[S/H_x]}) \qquad \Omega^1_Z \to H^0(f^*(\mathbb{L}_{\mathcal{F}}))$$

are isomorphic, and thus their kernel must be equal. These kernels being precisely the differential ideals $K_{q^*(0)}$ and $K_{f^*(\mathcal{F})}$, we see that these must coincide. Therefore, all the conditions are met to apply Proposition 5.4.0.3 and we deduce the lemma. $\square$

Since the morphism $f : Z \to X$ is étale, it is easy to see that $f^*(\mathcal{F})^h$ is not only smooth in codimension 2, but it is also flat and locally connected. We can thus apply functoriality of leaf spaces (Proposition 6.2.2.9) in order to get a commutative diagram in the analytic category

$$\begin{tikzcd} Z^h \arrow[r, "f"] \arrow[d, "p"'] & X^h \arrow[d, "\pi"] \\ Z^h /\!\!/ f^*(\mathcal{F})^h \arrow[r, "g"'] \arrow[d, "r"'] & X^g /\!\!/ \mathcal{F}^h \\ {[S^h/H_x].} & \end{tikzcd}$$

The morphism r is here a finite étale cover, which can be identified with the relative connected components of $Z \to [S/H_x]$. Since X_x is connected, we see that this finite étale cover must be trivial, and thus r is an equivalence. We therefore conclude the existence of commutative diagram

$$\begin{tikzcd} Z^h \arrow[r, "f"] \arrow[d, "p"'] & X^h \arrow[d, "\pi"] \\ {[S^h/H_x]} \arrow[r, "g"'] & X^g /\!\!/ \mathcal{F}^h. \end{tikzcd}$$

Because f is étale, and $q^*(0) \simeq \pi^*(0)$, a comparison of the cotangent complexes shows that g must be étale, too.

As a result, for any $y \in S^h$, we have an étale morphism of connected analytic spaces

$$\{y\} \times_{[S^h/H_x]} Z^h \longrightarrow \widetilde{\mathcal{L}}^{max}_{f(y)}(\mathcal{F}).$$

However, because p is proper we deduce that the left hand side is a proper analytic space, and that the above morphism is a finite etale cover, and thus a proper surjective morphism.

In particular, $\widetilde{\mathcal{L}}^{max}_{f(y)}(\mathcal{F})$ must be compact. We conclude that for all $y \in X$ in the image of f, the maximal leaf $\mathcal{L}^{max}_y(\mathcal{F}) \subset X^h$ is compact and its holonomy H_y is finite. As f is an étale morphism, its image is a dense Zariski open subset in X, and we denote it by $W \subset X$. Let $y \in X - W$ be a closed point, and let y_i be a sequence of closed points in X converging to y in the analytic topology. We can moreover assume that the holonomy groups of the y_i are all trivial, as these form a Zariski dense open substack in $[S/H_x]$. We then consider the corresponding sequence of closed subschemes $\mathcal{L}^{max}_{y_i}(\mathcal{F}^h) \subset X$. By compactness of the Hilbert scheme of X, this sequence of subschemes has a limit $L \subset X$. Clearly π sends L to $BH_y \subset X^h /\!\!/ \mathcal{F}^h$, and thus L is contained in the maximal leaf $\mathcal{L}^{max}_y(\mathcal{F}^h)$. Moreover, y being the limit of y_i, we have $y \in L$. More generally, any point $y' \in \mathcal{L}^{max}_y(\mathcal{F})$ is a limit of points $y'_i \in \mathcal{L}^{max}_{y_i}(\mathcal{F}^h)$, so that $y' \in L$. Therefore L and $\mathcal{L}^{max}_y(\mathcal{F}^h)$ coincide set theoretically. This implies in particular that $\mathcal{L}^{max}_y(\mathcal{F}^h)$ is compact, that the monomorphism $\mathcal{L}^{max}_y(\mathcal{F}^h) \hookrightarrow X^h$ is a closed immersion, and by GAGA that $\mathcal{L}^{max}_y(\mathcal{F}^h)$ is algebraizable. We thus conclude that the maximal leaves $\mathcal{L}^{max}_y(\mathcal{F}^h)$ are compact and algebraizable for all $y \in X^h$.

It remains to show that the holonomy group $H_y = Hol_y(\mathcal{F}^h)$ is finite for all y. For this, we use the same techniques. We write the maximal leaf $\mathcal{L}^{max}_y(\mathcal{F}^h)$ as the limit, in the Hilbert scheme of X, of a sequence $\mathcal{L}^{max}_{y_i}(\mathcal{F}^h)$ of compact leaves with trivial holonomy. We can fit this limit in a diagram of analytic spaces

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \\ S, & \end{tikzcd}$$

where S is a small holomorphic disk, Z is flat and proper with connected fibers over S, $Z \to X \times S$ is a closed immersion, and $Z_0 = \mathcal{L}^{max}_y(\mathcal{F}^h) \subset X$ is the central fiber at the point $0 \in S$. We let $Z_t \subset X$ denotes the "generic" or Milnor fiber. The nearby cycles in dimension 0 form a constructible sheaf on the central fiber X_0, and therefore any open cover U_i of X_0 in X can be refined to a cover such that each $U_i \cap X_t$ has a number of connected components which is uniformly bounded. By covering X_0 with U_i belonging to our basis $\mathcal{B}$ for the topology (see the construction of $X /\!\!/ \mathcal{F}$ in §6.2.2), we deduce the existence of an étale morphism $V \to X^h /\!\!/ \mathcal{F}^h$ covering the point y, and such that $\pi^{-1}(V) \cap \mathcal{L}^{max}_{y_i}(\mathcal{F}^h)$ possesses at most m connected components for some fixed integer m. Equivalently, let $G \rightrightarrows V$ be the étale groupoid induced on V, so that $[V/G]$ is an open substack in $X^h /\!\!/ \mathcal{F}^h$. Then, all the orbits of the points y_i are finite of order at most m in V. As a result, if $h \in G_y$ is a point in the stabilizer of $y \in V$, since all the y_i's have trivial holonomies, we must have that h^m is the identity near y_i, for all i (for all i big enough such that h^m is defined at y_i). By analytic continuation, we have that $h^m = id$. Now, let H_y be the holonomy group at a point y. We have seen that H_y is of finite exponent m. Moreover, as it is a subgroup of the group of germs of holonomorphic isomorphisms of $\mathbb{C}^d$ at 0, it must be finite (see [**Per01**, Lem. 2]).

To summarize, we have seen that all the maximal leaves are compact and all the holonomy groups are finite. We still need to show that this implies that $X^h /\!\!/ \mathcal{F}^h$ is *proper* and *algebraizable*. We start by proving that $X^h /\!\!/ \mathcal{F}^h$ is a separated Deligne-Mumford stack, i.e. that its diagonal is a finite morphism. For this, we come back to the very beginning of the proof of (3) $\Rightarrow$ (1). Now that we know thar all the maximal leaves are compact and all holonomy groups are finite, we can run exactly the same argument starting with any point $x \in X$. We have seen that there exists a commutative diagram

$$\begin{tikzcd} Z^h \arrow[r, "f"] \arrow[d, "p"'] & X^h \arrow[d, "\pi"] \\ {[S^h/H_x]} \arrow[r, "g"'] & X^h /\!\!/ \mathcal{F}^h. \end{tikzcd}$$

In this diagram p is a proper morphism, and moreover the natural morphism $Z^h \to [S^h/H_x] \times_{X^h /\!\!/ \mathcal{F}^h} X^h$ is étale. Properness of p implies that this étale morphism is moreover a finite covering. As a consequence $[S^h/H_x] \times_{X^h /\!\!/ \mathcal{F}^h} X^h \to [S^h/H_x]$ is proper. As the étale morphisms of the form $[S^h/H_x] \to X^h /\!\!/ \mathcal{F}^h$ cover the whole stack $X^h /\!\!/ \mathcal{F}^h$, we deduce that the morphism $\pi : X^h \to X^h /\!\!/ \mathcal{F}^h$ is a proper morphism.

We now consider the nerve of π, which is a proper and flat groupoid $G \rightrightarrows X^h$. The diagonal morphism $G \to X^h \times X^h$ is thus a proper and unramified morphism, hence a finite morphism. As $X^h \to X^h /\!\!/ \mathcal{F}^h$ is a flat covering, this implies that the diagonal of the stack $X^h /\!\!/ \mathcal{F}^h$ is a finite morphism, and thus that $X^h /\!\!/ \mathcal{F}^h$ is a separated Deligne-Mumford stack. Finally, as $X^h \to X^h /\!\!/ \mathcal{F}^h$ is proper and surjective we deduce that $X^h /\!\!/ \mathcal{F}^h$ is also proper.

To finish the proof, we need to show that $X^h /\!\!/ \mathcal{F}^h$ is algebraizable. But this follows by considering the groupoid $G \rightrightarrows X^h$ above (i.e. the nerve of the morphism π). Since $G \to X^h \times X^h$ is finite, and X^h and G are both proper algebraic spaces, we know by GAGA that the groupoid $G \rightrightarrows X^h$ is the analytification of a proper flat algebraic groupoid $G' \rightrightarrows X$. We can then consider the quotient stack $[X/G]$ of this flat groupoid: it is an (algebraic) Deligne-Mumford stack whose analytification is $X^h /\!\!/ \mathcal{F}^h$. $\square$

REMARK 6.5.0.3. We can combine the above theorem with Corollary 6.4.0.8 which shows that finiteness of H_x is implied by the finiteness of the algebraic holonomy group $Hol^{alg}_{\mathcal{F}}(x)$, which can be described purely in algebraic terms with using the transcendent topology.

CHAPTER 7

Derived foliations on schemes and stacks in non-zero characteristic

In this last chapter we introduce derived foliations over an arbitrary ground ring. We present two different non-equivalent notion; derived foliations and infinitesimal derived foliations. We show in particular how derived foliations can be useful to study characteristic classes, and prove in that direction vanishing theorem in crystalline cohomology. Finally, we show that infinitesimal derived foliations have remarkable formal integrability properties, and explain how they are related to infinitesimal cohomology, as derived foliations are relatedd to de Rham cohomology.

All along this chapter we denote by k an *arbitrary* commutative ring.

7.1. The graded circle and graded derived loop spaces

We start by defining a group stack called the *graded circle* that we will use in order to define derived foliations in non-zero characteristic. Let $k{<}x{>}$ the free commutative k-algebra with divided powers over one generator. It can be presented as the quotient of the free commutative k-algebra over generators $x^{(n)}$, for $n \geq 0$, with relations

$$x^{(n)}.x^{(m)} = \binom{n+m}{n} x^{(n+m)}$$

for all $n \geq 0$ (in particular $x^{(0)} = 1$). The commutative ring $k{<}x{>}$ is endowed with a natural comultiplication

$$\Delta : k{<}x{>} \longrightarrow k{<}x{>} \otimes_k k{<}x{>}$$

which is the morphism of k-algebras defined on generators by

$$\Delta(x^{(n)}) = \sum_{i+j=n} x^{(i)} \otimes x^{(j)}.$$

This makes $k{<}x{>}$ into a commutative and cocommutative Hopf k-algebra, whose corresponding commutative group scheme $\mathbf{Spec}\, k{<}x{>}$ can be naturally identified with the PD-hull at 0 of the additive group $\mathbb{G}_a$. Using the notations from [**Dri24**], this group scheme will be denoted by $\mathbb{G}_a^\sharp$ (it was denoted by Ker in [**MRT22**]). As the group $\mathbb{G}_a^\sharp$ is commutative, its classifying stack $B\mathbb{G}_a^\sharp$ inherits a canonical group structure and will be considered as a group stack.

Definition 7.1.0.1. *The* graded circle *(over k) is the group stack*

$$S^1_{gr} := B\mathbb{G}_a^\sharp$$

REMARK 7.1.0.2. The terminology *graded circle* suggests that S^1_{gr} is the associated graded of a filtered object. This *filtered circle* is introduced in [MRT22] where it is identified with the universal Hochschild-Kostant-Rosenberg filtration.

We can give several interpretations of the group $\mathbb{G}_a^\sharp$ and of the group stack S^1_{gr}. We will mainly use the following cohomological interpretation, showing that S^1_{gr} behaves like a *formal circle*.

The natural morphism $\mathbb{G}_a^\sharp \to \mathbb{G}_a$ induces a canonical morphism on the corresponding classifying stacks $S^1_{gr} \to B\mathbb{G}_a$. This defines a canonical degree 1 cohomology class

$$\epsilon^{(1)} \in H^1(S^1_{gr}, \mathcal{O}).$$

This class can be shown to be a generator for the cohomology ring and thus provides a natural graded ring isomorphism

$$H^*(S^1_{gr}, \mathcal{O}) \simeq k \oplus k.\epsilon^{(1)}.$$

We can say more by considering the following commutative diagram of stacks

$$\begin{tikzcd}
\mathbf{Spec}\, k[\epsilon] \arrow[r] \arrow[d] & Spec\, k \arrow[d] \\
\mathbf{Spec}\, k \arrow[r] & S^1_{gr},
\end{tikzcd}$$

for which the natural isomorphism making this diagram commutative is given by the element ϵ in $\mathbb{G}_a^\sharp(k[\epsilon])$: this is the element given by the morphism $k < x > \to k[\epsilon]$ sending $x^{(1)}$ to ϵ and all the $x^{(n)}$ for $n > 1$ to zero. The commutative diagram above then provides a canonical morphism on $\mathcal{O}$-cohomology complexes

$$C^*(S^1_{gr}, \mathcal{O}) \to k \times_{k[\epsilon]} k$$

where the fiber product on the right hand side is taken in the ∞-category $\mathbf{dg}_k$.

PROPOSITON 7.1.0.3. *The morphism above induces an equivalence in* $\mathbf{dg}_k$

$$C^*(S^1_{gr}, \mathcal{O}) \to k \times_{k[\epsilon]} k \simeq k \oplus k[-1].$$

We will not provide the proof of the Proposition above and will refer to (see [MRT22, Prop. 3.4.3] for details) for the details. This equivalence lives in $\mathbf{dg}_k$ but, by construction, can be lifted to include algebra structures. For us it is important to realize it as an equivalence of commutative cosimplicial algebras (see [MRT22, Thm. 3.4.17]).

REMARK 7.1.0.4. Proposition 7.1.0.3 is one of the main reason to introduce the group stack S^1_{gr}. We warn the reader that it badly fails if one tries to use $B\mathbb{G}_a$ as in the characteristic zero case, as already $H^2(B\mathbb{G}_a, \mathbb{G}_a) \neq 0$ (as length 2 Witt vectors provides a non-trivial extension of $\mathbb{G}_a$ by itself).

The multiplicative group $\mathbb{G}_m$ acts on the group scheme $\mathbb{G}_a^\sharp$ by dilations. Functorially, this action is described as follows. For a commutative k-algebra A, $\mathbb{G}_m(A)$ is the group of invertible elements λ in A and $\mathbb{G}_a^\sharp(A)$ is the additive group of sequences $\{a^{(n)}\}$ of elements in A satisfying $a^{(n)}.a^{(m)} = \binom{n+m}{n} a^{(n+m)}$. The action of $\lambda \in A^*$ is then given by

$$\lambda\{a^{(n)}\} = \{\lambda^n a^{(n)}\}$$

This provides an action of $\mathbb{G}_m$ on the group stack S^1_{gr}. The multiplicative group $\mathbb{G}_m$ also acts naturally on $\mathbf{Spec}\, k[\epsilon]$ by dilations, and the equivalence of Proposition 7.1.0.3 is then naturally compatible with the $\mathbb{G}_m$-actions on both sides.

The importance of Proposition 7.1.0.3 (when taking into account the algebras' structures) is that the graded circle acts as the spectrum of a square zero trivial extension by $k[-1]$. An important consequence of this fact is the following corollary.

Corollary 7.1.0.5. *For any derived affine k-scheme $\mathbf{Spec}\, A$, we have a canonical $\mathbb{G}_m$-equivariant identification of derived stacks*

$$\underline{\mathbf{Map}}(S^1_{gr}, X) \simeq \mathbb{V}(\mathbb{L}_X[1]) = \mathbf{Spec}\,(\mathbf{Sym}_A(\mathbb{L}_{A/k}[1])).$$

Sketch of proof. The commutative diagram

$$\begin{tikzcd} \mathbf{Spec}\, k[\epsilon] \arrow[r] \arrow[d] & Spec\, k \arrow[d] \\ \mathbf{Spec}\, k \arrow[r] & S^1_{gr}, \end{tikzcd}$$

defines a commutative diagram of derived stacks

$$\underline{\mathbf{Map}}(S^1_{gr}, X) \longrightarrow X \times_{\mathbb{V}(\mathbb{L}_X)} X \simeq \mathbb{V}(\mathbb{L}_X[1]).$$

The fact that this morphism is an equivalence can be easily proven by the Whitehead theorem: a morphism of derived Artin stacks is an equivalence if and only if it is formally étale and an equivalence on the classical truncations (see [**TV08**, 2.2.2]). The formally étale assumption can be checked using Proposition 7.1.0.3 as the tangent complex of the right hand side can be understood as a cohomology complex of S^1_{gr}. We refer to [**MRT22**, Thm. 5.1.3] for more details. □

As a consequence of Corollary 7.1.0.5, we see that, for any derived affine k-scheme X, $\mathbb{V}(\mathbb{L}_X[1])$ carries a canonical action of the semi-direct product group stack

$$\mathcal{H} := \mathbb{G}_m \ltimes S^1_{gr}.$$

This semi-direct group stack $\mathcal{H}$ is the pertinent generalization of our group $\mathcal{H}_0 = \mathbb{G}_m \ltimes B\mathbb{G}_a$ (Definition 1.5.0.1) outside of characteristic zero situations. The representations of $\mathcal{H}$ are indeed graded mixed complexes, as we will see in the next sections. For the moment we use $\mathcal{H}$ in order to introduce the notion of graded loop space.

DEFINITION 7.1.0.6. *For a derived affine k-scheme X, the* graded loop space *is the derived affine scheme*

$$\mathcal{L}^{gr}(X/k) := \underline{\mathbf{Map}}(S^1_{gr}, X).$$

It comes naturally equipped with a canonical action of the group $\mathcal{H}$.

For non-affine derived stacks, we use the left Kan extension of the functor $X \mapsto \mathcal{L}^{gr}(X/k)$. To avoid any confusion we will denote by $\widehat{\mathcal{L}}^{gr}(-/k)$ this extension.

DEFINITION 7.1.0.7. *The derived graded loop space of a derived stack X is $\widehat{\mathcal{L}}^{gr}(X/k)$ defined above. It comes with a canonical action of the group $\mathcal{H}$.*

Note that for general derived Artin stack X, Corollary 7.1.0.5 fails. There is always a natural descent morphism $\widehat{\mathcal{L}}^{gr}(X) \to \mathbb{V}(\mathbb{L}_X[1])$ but this is not an equivalence in general. For a general derived Artin stack things get even more complicated and $\widehat{\mathcal{L}}^{gr}(X)$ involves some form of formal completion of $\mathbb{V}(\mathbb{L}_X[1])$ along the zero section.

The induced morphism $\widehat{\mathcal{L}}^{gr}(X/k) \to \mathbb{V}(\mathbb{L}_X[1])$ is however an equivalence when X is more specifically a derived Deligne-Mumford stack.

COROLLARY 7.1.0.8. *For a derived Deligne-Mumford stack X the natural morphism*

$$\widehat{\mathcal{L}}^{gr}(X/k) \longrightarrow \mathbb{V}(\mathbb{L}_X[1])$$

is an equivalence.

Proof. Indeed, both functors $\widehat{\mathcal{L}}^{gr}(-/k)$ and $\mathbb{V}(\mathbb{L}_{-}[1])$ are stacks on the small étale site of X, and Corollary 7.1.0.5 implies that the morphism in consideration defines a local equivalence between these stacks. □

REMARK 7.1.0.9. For a general Artin stack X there is a close connection between $\widehat{\mathcal{L}}^{gr}(X/k)$ and $\mathbb{V}(\mathbb{L}_X[1])$. It is for instance possible to prove that these two derived stacks have equivalent algebras of $\mathbb{G}_m$-equivariant functions. This can be proved using some standard descent property as well as extensions of the results of §A.5 to the case of arbitrary characteristics (replacing Λ-powers with their left derived versions). We will not pursue this direction.

The geometric interpretation of graded mixed complexes in characteristic zero mentioned in §1.5 remains valid in any characteristic using the group stack $\mathcal{H}$ defined above when working over general base rings. The construction of §1.5 remains similar, and produces a canonical ∞-functor

$$\phi : \epsilon - \mathbf{dg}_k \longrightarrow \mathbf{QCoh}(B\mathcal{H}).$$

PROPOSITON 7.1.0.10. *The ∞-functor ϕ induces an equivalence of symmetric monoidal ∞-categories*

$$\phi : \epsilon - \mathbf{dg}_k \simeq \mathbf{QCoh}(B\mathcal{H}).$$

Proof. This is proven in [**MRT22**, Prop. 4.2.3 (ii)]. □

7.2. Derived foliations via graded derived loop spaces

The geometric interpretation of derived foliations given in Section 2.5.2 (see Definition 2.5.2.1) can be generalized in arbitrary characteristic by replacing $\mathcal{H}_0$ with the group stack $\mathcal{H} = \mathbb{G}_m \ltimes S^1_{gr}$.

DEFINITION 7.2.0.1. *Let X be a derived stack over k. A* derived foliation $\mathcal{F}$ on X (relative to k) *consists of the data of a $\mathcal{H}$-equivariant derived stack $\mathbb{V}(\mathcal{F})$ over $\widehat{\mathcal{L}}^{gr}(X/k)$, such the following condition is satisfied: for any derived affine scheme S and morphism $S \to X$, the $\mathbb{G}_m$-equivariant morphism*

$$\mathbb{V}(\mathcal{F}) \times_{\widehat{\mathcal{L}}^{gr}(X/k)} \widehat{\mathcal{L}}^{gr}(S/k) \to S$$

makes $\mathbb{V}(\mathcal{F})$ into a perfect derived linear stack over S.

By definition, derived foliations $\mathcal{F}$ on X form a full sub-∞-category of $\mathbf{dSt}^{\mathcal{H}}_{/\widehat{\mathcal{L}}^{gr}(X/k)}$, the ∞-category of $\mathcal{H}$-equivariant derived stacks over $\widehat{\mathcal{L}}^{gr}(X/k)$.

REMARK 7.2.0.2. Outside of the case where k is a $\mathbb{Q}$-algebra, the above definition does not have an easy purely algebraic description. Indeed graded mixed cdga's must be replaced by the more complex notion of *graded mixed LSym-algebra*, where *LSym* is the ∞-monad given by the left derived symmetric powers. We refer to [**Rak20**] for such a purely algebraic approach, and warn the reader that the comparison with our definition above is not straightforward. In what follows we will thus avoid any tentative to provide an algebraic version of derived foliations over general rings and will stick to the geometric definition above.

The derived foliations defined above share the same formal properties that we have already seen in characteristic zero, such as pull-backs and existence of finite limits. For a morphism of derived stacks $X \to Y$ we have an ∞-category $\mathcal{F}ol(X/Y)$ of relative derived foliations over X relative to Y. This ∞-category is functorial by pull-back for commutative diagrams of derived stacks

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, "s"'] & Y. \end{tikzcd}$$

Explicitly, an object $\mathcal{F} \in \mathcal{F}ol(X/Y)$ is represented by an $\mathcal{H}$-equivariant derived stack $\mathbb{V}(\mathcal{F}) \to \widehat{\mathcal{L}}^{gr}(X/Y)$. Its pull-back along the commutative diagram above is given by

$$(f,s)^*(\mathbb{V}(\mathcal{F})) = \mathbb{V}(\mathcal{F}) \times_{\widehat{\mathcal{L}}^{gr}(X/Y)} \widehat{\mathcal{L}}^{gr}(X'/Y').$$

Cotangent complexes of derived foliations are defined as in §4.3.1. Namely, for $\mathbb{V}(\mathcal{F}) \to \widehat{\mathcal{L}}^{gr}(X)$ a derived foliation over X, and a derived affine $U = \mathbf{Spec}\, A$ with a morphism $U \to X$, the $\mathbb{G}_m$-equivariant derived affine stack object $\mathbb{V}(\mathcal{F}) \times_{\widehat{\mathcal{L}}^{gr}(X)} U \to U$ corresponds, by [**Mon21**], to a well defined A-dg-module $\mathbb{L}^{big}_{\mathcal{F}}(U)$ such that $\mathbb{V}(\mathcal{F}) \times_{\widehat{\mathcal{L}}^{gr}(X)} U$ and $\mathbb{V}(\mathbb{L}^{big}_{\mathcal{F}}(U))$ are equivalent in $\mathbf{dSt}_k/U$. When U varies in $\mathbf{dAff}_k/X$, these glue to a (non-quasi-coherent) $\mathcal{O}_X$-dg-module

$\mathbb{L}^{big}_{\mathcal{F}}$ called the *big cotangent complex of* $\mathcal{F}$. Its associated quasi-coherent complex (see §A.4) is called *the cotangent complex of* $\mathcal{F}$, and is denoted by

$$\mathbb{L}_{\mathcal{F}} := (\mathbb{L}^{big}_{\mathcal{F}})^{qcoh} \in \mathbf{QCoh}(X).$$

A derived foliation $\mathcal{F}$ on X is called *perfect* if its cotangent complex $\mathbb{L}_{\mathcal{F}}$ is perfect.

For any derived stack X over k, we have a final object $*_X \in \mathcal{F}ol(X/k)$ which is given by $\mathbb{V}(\mathcal{F}) = \widehat{\mathcal{L}}^{gr}(X/k)$. Similarly, when X is a derived Artin stack locally of finite presentation over k, the ∞-category $\mathcal{F}ol(X/k)$ has an initial object $0_X \in \mathcal{F}ol(X/k)$. It can be described as an explicit derived stack $\mathbb{V}(\mathcal{F}) \to \widehat{\mathcal{L}}^{gr}(X)$ derived stack as follows. We write X as a colimit of derived affine opschemes $colim_i\, U_i$, so that $\widehat{\mathcal{L}}^{gr}(X) \simeq colim_i\, \mathcal{L}^{gr}(U_i)$ ($\widehat{\mathcal{L}}^{gr}$ commutes with colimits has it is defined by a left Kan extension). For each index i, we form the relative graded loop stack

$$\widehat{\mathcal{L}}^{gr}(U_i/X) := \mathcal{L}^{gr}(U_i) \times_{\widehat{\mathcal{L}}^{gr}(X)} X \to \mathcal{L}^{gr}(U_i),$$

which comes equipped with its natural $\mathcal{H}$-action. The $\mathcal{H}$-equivariant derived stacks $\widehat{\mathcal{L}}^{gr}(U_i/X)$ certainly glue to a global object over $\widehat{\mathcal{L}}^{gr}(X)$, however in general $\widehat{\mathcal{L}}^{gr}(U_i/X)$ is not a derived affine stack over U_i. We thus replace $\widehat{\mathcal{L}}^{gr}(U_i/F)$ by its graded affinization (see §A.9.2)

$$\widehat{\mathcal{L}}^{gr}(U_i/F)^{aff} := \mathbb{R}\mathbf{Spec}^{\Delta,gr}\left(\mathcal{O}(\widehat{\mathcal{L}}^{gr}(U_i/F)^{aff})\right) \to U_i.$$

The group stack $\mathcal{H}$ being affine over $B\mathbb{G}_m$, it continues to act on $\widehat{\mathcal{L}}^{gr}(U_i/F)^{aff}$. Finally, by our descent result for cotangent complexes (see §A.5) we have that $\widehat{\mathcal{L}}^{gr}(U_i/F)^{aff} \simeq \mathbb{V}(\mathbb{L}_{U_i/F}[1])$ as graded derived affine stacks. The family of objects $\widehat{\mathcal{L}}^{gr}(U_i/F)^{aff}$ then glue as a global $\mathcal{H}$-equivariant derived stack $\mathbb{V}(\mathcal{F}) \to \widehat{\mathcal{L}}^{gr}(F)$, which thus defines an object in $\mathcal{F}ol(F)$. It can be checked, as in §2.1.5.1, that this defines an initial object in $\mathcal{F}ol(F)$.

We finish this general part by stating that integrable sub-bundles of the tangent bundle of a smooth scheme canonically define derived foliations in the sense of Definition 7.2.0.1. This result is taken from [**Mon22**], whose proof is reproduced below.

Propositon 7.2.0.3. *Let $f : X \to S$ be a smooth morphism of derived Deligne-Mumford stacks. Let $E \subset \mathbb{T}_{X/S}$ be a sub-bundle of the relative tangent bundle of X over S such that E is stable by the Lie bracket of relative vector fields. Then, there exists a unique action of $\mathcal{H}$ on $\mathbb{V}(E^\vee[1])$, covering the $\mathcal{H}$-action on $\mathbb{V}(\mathbb{L}_{X/S}[1]) \simeq \mathcal{L}^{gr}(X/S)$.*

Proof. The purpose of this proof is to show that the space of $\mathcal{H}$-action on $\mathbb{V}(E^\vee[-1])$ compatible, via the map $\mathbb{V}(\mathbb{L}_{X/S}[1]) \simeq \mathcal{L}^{gr}(X/S) = \mathbb{V}(\mathbb{L}_X[-1])$, with the $\mathcal{H}$-action on $\mathbb{V}(\mathbb{L}_X[1])$, is contractible. This is a local statement on the small étale site of X and thus we can assume that X is an affine smooth scheme over k.

We first consider $\mathbb{V}(E^\vee[1])$ as a $\mathbb{G}_m$-equivariant derived stack, and we study the $\mathbb{G}_m$-equivariant group stack G of self equivalence of $\mathbb{V}(E^\vee[1])$. More precisely, G is the group stack over $B\mathbb{G}_m$ of self-equivalences of $\mathbb{V}(E^\vee[1]) \to B\mathbb{G}_m$. The group stack G a priory comes equipped with a natural derived structure, but we will only be interested in the underlying underived stack $G \to B\mathbb{G}_m$. Also, we will not need the full stack G but only $G_0 \subset G$ the full sub-stack of the connected component of the identity, defined as the pull-back of stacks relative to $B\mathbb{G}_m$

$$\begin{tikzcd} G_0 \arrow[r] \arrow[d] & G \arrow[d] \\ \star \arrow[r, "id"'] & \pi_0(G). \end{tikzcd}$$

The group stack G_0 is connected, $\pi_0(G_0) \simeq *$ as a sheaf over $B\mathbb{G}_m$.

Because $\mathbb{V}(E^\vee[1]) = \mathbf{Spec}\, Sym_A(E^\vee[1])$, it is easy to compute the homotopy sheaves $\pi_i(G_0)$, as sheaves over $B\mathbb{G}_m$, or equivalently as $\mathbb{G}_m$-equivariant sheaves of groups on $\mathbf{Spec}\, k$. We have short exact sequence of sheaves on the big fpqc site of affine k-schemes

$$0 \longrightarrow Hom_A(E^\vee, \wedge^i(E^\vee))^\sim \longrightarrow \pi_i(G_0) \longrightarrow Der_k(A, \wedge^{i-1}E^\vee)^\sim \longrightarrow 0.$$

The left hand side is the quasi-coherent sheaf on $\mathbf{Spec}\, k$ associated with the k-module of A-linear maps $E^\vee \to \wedge^i(E^\vee)$, whereas the right hand side is the quasi-coherent sheaf on $\mathbf{Spec}\, k$ associated with the k-module of k-linear derivations from A to the A-module $\wedge^{i-1}E^\vee$. The $\mathbb{G}_m$-action on both of these quasi-coherent sheaves is induced by the canonical homothetic action on $E^\vee$ and is thus pure of weight $i-1$ on both side.

Now, an $\mathcal{H}$-action on $\mathbb{V}(E^\vee[-1])$, compatible with the already existing $\mathbb{G}_m$-action, is given by a $\mathbb{G}_m$-equivariant morphism

$$\rho : B\mathcal{H} \to BG_0.$$

The stack $B\mathcal{H}$ is an affine stack (see Appendix A.9) of the form $\mathbf{Spec}^\Delta\, \mathbb{Z}[u]$, where u is a free variable in degree 2, and $\mathbb{Z}[u]$ is the free commutative cosimplicial-simplicial ring freely generated by u, as this follows easily from the fact that $\mathcal{H} = \mathbf{Spec}^\Delta\, H^*(S^1)$ (see 7.1.0.3 and [**MRT22**, Thm. 3.4.17]). Moreover, the $\mathbb{G}_m$-action on $B\mathcal{H}$ is such that u is of weight 1. We can thus understand the space of $\mathbb{G}_m$-equivariant morphisms $\rho : B\mathcal{H} \to BG_0$ by Postnikov induction on BG_0, with in mind the above description of the homotopy sheaves $\pi_i(BG_0) \simeq \pi_{i-1}(G_0)$. For this we use the standard cellular decomposition, inside the ∞-category of the $\mathbb{G}_m$-equivariant affine stacks, $B\mathcal{H} \simeq colim_k(B\mathcal{H})_{\leq 2k}$

$$\begin{tikzcd} B\mathcal{H}_{\leq 2k} \arrow[r] & B\mathcal{H}_{\leq 2k+2} \\ S^{2k+1}_{gr}(k+1) \arrow[r] \arrow[u, "h_k"] & \star \arrow[u] \end{tikzcd}$$

where $S^p_{gr}(q)$ denotes the *affine graded sphere* $\mathbf{Spec}^\Delta\, \mathbb{Z} \oplus \mathbb{Z}[-p](q)$, and h_k is induced by the graded versions of the standard Hopf maps $S^{2k+1}_{gr}(k+1) \to B\mathcal{H}_{\leq 2k}$.

Using this cellular decomposition, and by weight considerations, we see in particular that the projection on the first non-trivial Postnikov layer

$$BG_0 \to \tau_{\leq 2}(BG_0) = K(\pi_2, 2)$$

induces an equivalences of mapping spaces between $\mathbb{G}_m$-equivariant stacks

$$\mathbf{Map}_{\mathbb{G}_m-\mathbf{St}}(B\mathcal{H}, BG_0) \simeq \mathbf{Map}_{-\mathbb{G}_m-\mathbf{St}}(B\mathcal{H}, K(\pi_2, 2)) \simeq Hom_{\mathbb{G}_m-Gp}(\mathbb{G}_a^\sharp, \pi_2),$$

where $\pi_2 = \pi_1(G_0)$. Using the fact that group morphisms $\mathbb{G}_a^\sharp \to \mathbb{G}_a$ are in one-to-one correspondence with elements in k (because $H^1(B\mathbb{G}_a^\sharp, \mathcal{O}) \simeq k$, see Proposition 7.1.0.3), we see that $Hom_{\mathbb{G}_m-Gp}(\mathbb{G}_a^\sharp, \pi_2) \simeq \pi_2(\mathbf{Spec}\, k)$ can be identified with the global sections of the sheaf π_2.

We have thus seen that the space of $\mathbb{G}_m$-compatible actions on $\mathbb{V}(E^\vee[1])$ is discrete and equivalent to the underlying set of $\pi_2(\mathbf{Spec}\, k)$, which sits in a short exact sequence

$$0 \longrightarrow Hom_A(E^\vee, \wedge^2(E^\vee)) \longrightarrow \pi_2(\mathbf{Spec}\, k) \longrightarrow Der_k(A, E^\vee) \longrightarrow 0.$$

The elements in the set $\pi_2(\mathbf{Spec}\, k)$ can be described as pairs (d, u), where $d : A \to E^\vee$ is a k-linear derivation, and $u : E^\vee \to \wedge^2(E^\vee)$ is a d-semi-linear morphism

$$u(am) = d(a) \wedge m + a.u(m).$$

We can therefore conclude the proof of the proposition, as we can apply this result on both $\mathbb{V}(\mathbb{L}_X[-1])$ and $\mathbb{V}(E^\vee[1])$ and see that the only possible compatible $\mathcal{H}$-action on $\mathbb{V}(E^\vee[1]))$ must be induced by the de Rham differential on the quotient

$$Sym_A(\Omega_A^1[1]) \twoheadrightarrow Sym_A(E^\vee)$$

and is therefore given by the pair (d, u), where $d : A \to E^\vee$ is the dual of the anchor map $a : E \to \mathbb{T}_X$, and u is dual to the restriction of the Lie bracket on E (i.e. the Chevalley Eilenberg differential)

$$u(\phi)(x \wedge y) = a(x)(\phi(y)) - a(y)(\phi(x)) - \phi([a(x), a(y)])$$

for $\phi : E \to A$ an A-linear form and $x, y \in E$. □

7.3. Foliated de Rham theory

Recall that any morphism of derived stacks $f : X \to Y$ induces and adjunction given by push-forward and pull-back on quasi-coherent sheaves

$$f^* : \mathbf{QCoh}(X) \leftrightarrows \mathbf{QCoh}(Y) : f_*.$$

The left adjoint p^* is a symmetric monoidal ∞-functor, and therefore this adjunction induces an adjunction on the corresponding ∞-categories of commutative algebras in the sense of [Lur22a, Def. 2.1.3.1]

$$p^* : CAlg(X) \leftrightarrows CAlg(Y) : p_*,$$

and this adjunction covers the adjunction on quasi-coherent sheaves (i.e. p_* and p^* commute with the forgetful ∞-functor forgetting the commutative algebra structure). To void confusions objects in $CAlg(\mathbf{QCoh}(X)))$ will be called *quasi-coherent E_∞-algebras on X*, and we will use the general notation

$$E_\infty - \mathbf{Alg}(\mathcal{C}) := CAlg(\mathcal{C}),$$

for any symmetric monoidal ∞-category $\mathcal{C}$. In characteristic zero these coincide with quasi-coherent sheaves of $\mathcal{O}_X$-cdga's over X.

Let $X \in \mathbf{dSt}_k$ be a derived Artin stack over k, and $\mathcal{F} \in \mathcal{F}ol(X/k)$ be a derived foliation (relative to k). By definition, $\mathcal{F}$ is given by an $\mathcal{H}$-equivariant derived stack $\mathbb{V}(\mathcal{F})$ together with an $\mathcal{H}$-equivariant morphism

$$\mathbb{V}(\mathcal{F}) \to \widehat{\mathcal{L}}^{gr}(X).$$

We can pass to the corresponding quotient stacks, and obtain this way a morphism of derived stacks over $B\mathcal{H}$

$$\begin{tikzcd} {[\mathbb{V}(\mathcal{F})/\mathcal{H}]} \arrow[r] \arrow[dr, "p"'] & {[\widehat{\mathcal{L}}^{gr}(X)/\mathcal{H}]} \arrow[d, "q"] \\ & B\mathcal{H}. \end{tikzcd}$$

By considering the adjunctions induced by this morphisms we get a canonical morphism of quasi-coherent E_∞-algebras $q_*(\mathcal{O}) \to p_*(\mathcal{O})$ on $B\mathcal{H}$. We can compose with the symmetric monoidal equivalence given by the Tate realization (see Corollary 1.3.4.4)

$$| - |^t : \mathbf{QCoh}(B\mathcal{H}) \simeq \widehat{\mathbf{dg}}_k^{fil},$$

and get this way a morphism of filtered and complete E_∞-algebras $|q_*(\mathcal{O})|^t \to |p_*(\mathcal{O})|^t$.

Definition 7.3.0.1. *Let Let $X \in \mathbf{dSt}_k$, and $\mathcal{F} \in \mathcal{F}ol(X/k)$. With the above notations, we define*

(1) the completed derived de Rham cohomology of X (relative to k) *is the complete filtered E_∞-algebra*

$$\widehat{C}_{DR}(X, \mathcal{O}) := |q_*(\mathcal{O})|^t \in E_\infty - \mathbf{Alg}(\widehat{\mathbf{dg}}_k^{fil}).$$

(2) The completed foliated derived de Rham cohomology of the derived foliation $\mathcal{F}$ (relative to k) *is the complete filtered E_∞-algebra*

$$\widehat{C}_{DR}(\mathcal{F}, \mathcal{O}) := |p_*(\mathcal{O})|^t \in E_\infty - \mathbf{Alg}(\widehat{\mathbf{dg}}_k^{fil}).$$

The corresponding cohomology groups will be denoted by

$$\widehat{H}^*_{DR}(X) := H^i(\widehat{C}^*_{DR}(X, \mathcal{O})) \qquad \widehat{H}^*_{DR}(\mathcal{F}) := H^i(\widehat{C}^*_{DR}(\mathcal{F}, \mathcal{O})).$$

We also have a definition *with coefficients in crystals*, as in Definition 3.2.0.1. To get this, we introduce the ∞-category of (quasi-coherent) crystals along $\mathcal{F}$ as a certain full sub-∞-category $\mathbf{QCoh}(\mathcal{F})$ of $\mathbf{QCoh}([\mathbb{V}(\mathcal{F})/\mathcal{H}])$. To explain the definition of $\mathbf{QCoh}(\mathcal{F})$, we start with a general derived stack X with an action of $\mathbb{G}_m$ together with a $\mathbb{G}_m$-equivariant morphism $X \to Y$, considered as a morphism $p : [X/\mathbb{G}_m] \to Y$. A quasi-coherent complex $E \in \mathbf{QCoh}([X/\mathbb{G}_m])$ is said to be *graded free relative to* Y, if the canonical morphism

$$p^*(p_*(E)) \to E$$

is an equivalence in $\mathbf{QCoh}([X/\mathbb{G}_m])$. With this notion, by definition an object $E \in \mathbf{QCoh}([\mathbb{V}(\mathcal{F})/\mathcal{H}])$ lies in $\mathbf{QCoh}(\mathcal{F})$ if its pull-back to $\mathbf{QCoh}([\mathbb{V}(\mathcal{F})/\mathbb{G}_m])$ is graded free with respect to the $\mathbb{G}_m$-equivariant projection $\mathbb{V}(\mathcal{F}) \to X$.

For any crystal $E \in \mathbf{QCoh}(\mathcal{F})$ on $\mathcal{F}$, we thus have a *complete foliated derived de Rham cohomology with coefficients in* E, defined by

$$\widehat{C}_{DR}(\mathcal{F}, E) := |p_*(E)|^t.$$

The object $\widehat{C}_{DR}(\mathcal{F}, E)$ is canonically a complete filtered module over the complete filtered E_∞-algebra $\widehat{C}_{DR}(\mathcal{F}, \mathcal{O})$.

One can prove, using the descent results of §A.5, that the underlying graded complex of $p_*(\mathcal{O})$ is equivalent to

$$p_*(\mathcal{O}) \simeq \bigoplus_p \mathbb{H}(F, \mathbb{L}\Lambda^p \mathbb{L}_{\mathcal{F}})[p].$$

The Hodge filtration discussed in §3.3 also makes sense in the present setting. The only difference is here that the *left derived* functors of the exterior product must be used. We gather the results in the following Proposition, and leave the details to the reader.

PROPOSITON 7.3.0.2. *For a perfect derived foliation $\mathcal{F}$ on a derived Artin stack X, there exists a canonical filtration $F^*_{\mathcal{F}}$ on the E_∞-algebra $\widehat{C}_{DR}(X/k)$ of completed de Rham cohomology on X, with the following properties.*

(1) The filtration $F^*_{\mathcal{F}}$ is functorial in X and $\mathcal{F}$.
(2) The associated graded, as a graded E_∞-algebra, is given by

$$Gr^i_{F^*_{\mathcal{F}}}(\widehat{C}_{DR}(X/k)) \simeq \widehat{C}_{DR}(\mathcal{F}, \mathbb{L}\Lambda^{-i}\mathcal{N}^*_{\mathcal{F}})[i].$$

In the above proposition, $\mathbb{L}\Lambda^i\mathcal{N}^*_{\mathcal{F}}$ is the i-th derived exterior product of $\mathcal{N}^*_{\mathcal{F}}$ (defined as usual as being the fiber of $\mathbb{L}_{X/k} \to \mathbb{L}_{\mathcal{F}/k}$), equipped with its natural structure of a perfect crystal along $\mathcal{F}$ (see Proposition 3.3.0.2).

7.4. Characteristic classes

In this section we construct *characteristic classes* of derived foliations as cohomology classes in the foliated de Rham coholomogy introduced in the previous section. When the base is a perfect field, we then use the comparison between derived de Rham cohomology and crystalline cohomology to prove the existence of residues for foliations, in the sense of [**BB72b**], in the positive characteristic setting.

We start by general results and constructions concerning the de Rham cohomology of the classifying stacks $B\mathbf{Gl}_n$. The completed derived de Rham complex $\widehat{C}^*_{DR}(B\mathbf{Gl}_n, \mathcal{O})$ is a complete filtered E_∞-algebra whose associated graded is equivalent to

$$\bigoplus_p H^*(B\mathbf{Gl}_n, S^p(\mathfrak{g}^*))[-2p],$$

where $\mathfrak{g} = Lie(\mathbf{Gl}_n)$ is the Lie algebra of the group scheme $\mathbf{Gl}_n$ and $S^p(\mathfrak{g}^*)$ denotes the (underived) p-th symmetric power of its linear dual (as a k-module). The group $\mathbf{Gl}_n$ acts by the adjoint action on $\mathfrak{g}$ and thus on $S^p(\mathfrak{g}^*)$, this action defines $S^p(\mathfrak{g}^*)$ as a quasi-coherent sheaf on the stack $B\mathbf{Gl}_n$.

By construction the complete filtered E_∞-algebra $\widehat{C}^*_{DR}(B\mathbf{Gl}_n)$ can be written as $|q_*(\mathcal{O})|^t$ where $q_*(\mathcal{O})$ is a quasi-coherent E_∞-algebra over $B\mathcal{H}$. The symmetric monoidal ∞-category $\mathbf{QCoh}(B\mathcal{H})$ can be endowed with its canonical t-structure, for which the pull-back along the canonical point $\mathbf{Spec}\, k \to \mathbf{QCoh}(\mathcal{BH})$ is t-exact and conservative. Therefore, $q_*(\mathcal{O})$ possesses a connective cover $B \to q_*(\mathcal{O})$ inside $E_\infty - \mathbf{Alg}(\mathbf{QCoh}(B\mathcal{H}))$, which is simply given by the non-derived direct image of $\mathcal{O}$ by q. The underlying graded complex of this connective cover B is concentrated in degree 0 and is isomorphic to $\oplus_p (S^p(\mathfrak{g}^*))^{\mathbf{Gl}_n}$. As B is concentrated in cohomological degree 0 the $B\mathcal{H}$-action is automatically trivial, and thus the induced morphism on the Tate realizations provides a morphism of complete filtered E_∞-algebras We thus obtain a natural morphism of filtered E_∞-algebras

$$|B|^t := \mathcal{A}_n \to \widehat{C}^*_{DR}(B\mathbf{Gl}_n, \mathcal{O}) = |q_*(\mathcal{O})|^t.$$

By the definition of the Tate realization and because the mixed structure is trivial on B, we have

$$\mathcal{A}_n \simeq \oplus_p S^p(\mathfrak{g}^*)^{\mathbf{Gl}_n}[-2p]$$

where the filtration is the split filtration associated with the graduation for which $(\mathfrak{g}^*)^{\mathbf{Gl}_n}[-2]$ sits in weight -1.

The inclusion of the maximal torus in $\mathbf{Gl}_n$ induces, by restrictions, a morphism of graded E_∞-algebras

$$\mathcal{A}_n \to k[x_1, \dots, x_n]^{\Sigma_n}$$

which is an isomorphism (and each x_i sits in degree 2 and weight -1). We thus have an isomorphism of graded E_∞-algebras

$$k[c_1, \dots, c_n] \simeq \mathcal{A}_n$$

where c_i corresponds to the i-th elementary symmetric polynomial in the $x_1, \dots, x_n$, and is therefore of cohomological degree $2i$ and weight $-i$. Via the map $\mathcal{A}_n \to \widehat{C}^*_{DR}(B\mathbf{Gl}_n, \mathcal{O})$ these classes defines natural morphism of complexes of k-modules

$$c_i : k \to F^{-i}\widehat{C}^*_{DR}(B\mathbf{Gl}_n)[2i].$$

REMARK 7.4.0.1. In [Tot18, Thm. 10.2] it is proven that the higher cohomology groups of $\mathbb{H}(B\mathbf{Gl}_n, S^p(\mathfrak{g}^*))$ all vanish, and therefore the natural morphism induces an isomorphism

$$\mathcal{A}_n \to \widehat{C}^*_{DR}(B\mathbf{Gl}_n)$$

However, the construction of the Chern classes c_i in $\widehat{C}^*_{DR}(B\mathbf{Gl}_n)$ does not require the results of [Tot18], as the mere existence of the morphism from $\mathcal{A}_n$ to $\widehat{C}^*_{DR}(B\mathbf{Gl}_n)$ is sufficient.

We now extend the universal Chern class to a class in derived completed de Rham cohomology of the derived stack of perfect complexes, using the standard methods of [Gil81].

Each Chern class c_i can be considered as a mophism of underived stacks on the big etale site of affine k-schemes

$$c_i : \mathbb{Z} \times B\mathbf{Gl}_\infty \to F^{-i}\widehat{C}^*_{DR}(-, \mathcal{O})[2i],$$

where $\mathbf{Gl}_\infty$ is defined as the colimit of the $\mathbf{Gl}_n$ for the natural inclusion $\mathbf{Gl}_n \hookrightarrow \mathbf{Gl}_{n+1}$ given by the direct sum wuth a trivial line bundle. By the universal property of Quillen "+ construction", this factors as a morphism of stacks $c_i : (\mathbb{Z} \times B\mathbf{Gl}_\infty)^+ \to F^{-i}\widehat{C}^*_{DR}(-, \mathcal{O})[2i]$, or in other words as a morphism

$$c_i : \underline{K}^{\mathrm{cl}} \to F^{-i}\widehat{C}^*_{DR}(-, \mathcal{O})[2i]$$

where $\underline{K}^{\mathrm{cl}}$ is the K-theory stack (i.e. the étale stackification of the K-theory functor sending a commutative k-algebra A to its K-theory space $K(A)$). If we denote by $i^* : \mathbf{dSt}_k \to \mathbf{St}_k$ the restriction functor from the ∞-category of derived stacks to the ∞-category of stacks, the above morphism can be written as

$$c_i : i^*(\underline{K}) \to i^*(F^{-i}\widehat{C}^*_{DR}(-, \mathcal{O})[2i])$$

where now $\underline{K}$ is the derived stack sending a simplicial commutative k-algebra A to the Waldhausen K-theory space of the ∞-category of perfect A-modules. The functor i^* has a left adjoint $i_! : \mathbf{St}_k \to \mathbf{dSt}_k$ obtained by the left Kan extension induced by the embedding of k-algebras into simplicial k-algebras. By adjunction, the morphism c_i induces a morphism of derived stacks

$$c_i : i_!i^*(\underline{K}) \to F^{-i}\widehat{C}^*_{DR}(-, \mathcal{O})[2i],$$

However, it is well known that the adjunction morphism $i_!i^*(\underline{K}) \to \underline{K}$ is an equivalence, and this can be seen easily from the fact that the derived stack of vector bundles $\mathbf{Vect}$ is such that the ajdunction morphism $i^*_!(\mathbf{Vect}) \to \mathbf{Vect}$ is an equivalence (simply because $B\mathbf{Gl}_n$ is a smooth underived stack). We therefore obtain a morphism of derived stacks

$$c_i : \underline{K} \to F^{-i}\widehat{C}^*_{DR}(-, \mathcal{O})[2i].$$

Composed with the canonical morphism $\mathbf{Perf} \to \underline{K}$ we obtain the desired extensions

$$c_i : \mathbf{Perf} \to F^{-i}\widehat{C}^*_{DR}(-, \mathcal{O})[2i],$$

and thus canonical classes

$$c_i \in H^{2i}(F^{-i}\widehat{C}^*_{DR}(\mathbf{Perf}, \mathcal{O}))).$$

Definition 7.4.0.2. *The* universal i-th Chern class *is*

$$c_i \in H^{2i}(F^{-i}\widehat{C}^*_{DR}(\mathbf{Perf}, \mathcal{O}))$$

constructed above.

We are now ready to define characteristic classes of perfect crystals over derived foliations. Let X be a derived Artin stack (locally of finite presentation over k) and $\mathcal{F} \in \mathcal{F}ol(X/k)$ a derived foliation on X (relative to k).

Let $E \in \mathbf{Perf}(\mathcal{F})$ be a perfect crystal along $\mathcal{F}$. The object E can be represented by a morphism between pairs, consisting of derived stacks with derived foliations

$$\phi_E : (X, \mathcal{F}) \to (\mathbf{Perf}, 0_{\mathbf{Perf}})$$

where $0_{\mathbf{Perf}}$ is the initial foliation on $\mathbf{Perf}$. Therefore, the morphism ϕ_E induces a morphism on completed derived de Rham cohomologies

$$\widehat{C}_{DR}(\mathbf{Perf}, \mathcal{O}) \to \widehat{C}_{DR}(X, \mathcal{O})$$

which is a morphism of filtered E_∞-algebras for the Hodge filtrations on both sides (Proposition 7.3.0.2). The Hodge filtration $F^*_{0_{\mathbf{Perf}}}$ on the left hand side is the usual Hodge filtration F as the derived foliation on $\mathbf{Perf}$ is here the initial foliation. We therefore obtain for any integer i an induced morphism

$$F^i\widehat{C}^*_{DR}(\mathbf{Perf}, \mathcal{O}) \longrightarrow F^i_{\mathcal{F}}\widehat{C}^*_{DR}(X, \mathcal{O}).$$

As a consequence, we see that the image of the universal Chern classes c_i in $\widehat{H}^{2i}_{DR}(X)$ has natural lifts through $\widehat{H}^{2i}(F^{-i}_{\mathcal{F}}\widehat{C}^*_{DR}(X, \mathcal{O})) \to \widehat{H}^{2i}_{DR}(X)$.

A first direct consequence of the previous considerations is the following vanishing result of Chern classes for perfect crystals along derived foliations.

Corollary 7.4.0.3. *Let X be a derived Artin stack locally of finite presentation over k, and $\mathcal{F} \in \mathcal{F}ol(X/k)$ be a perfect derived foliation. Then, for any $E \in \mathbf{Perf}(\mathcal{F})$, the chern classes $c_i(E) \in \widehat{H}^{2i}_{DR}(X)$ of the underlying perfect complex E on X possess canonical lifts to $H^{2i}(F^{-i}_{\mathcal{F}}\widehat{C}^*_{DR}(X, \mathcal{O}))$. It particular, the image of $c_i(E)$ by the natural moprhism*

$$H^{2i}_{DR}(X) \to H^{2i}_{DR}(\mathcal{F}, \mathcal{O}) \simeq H^{2i}(Gr^0_{F_{\mathcal{F}}}\widehat{C}^*_{DR}(X, \mathcal{O}))$$

is zero as soon as $i > 0$.

A slightly more sophisticated consequence is the following result on characteristic classes of the conormal complex of a derived foliation. Recall from Proposition 7.3.0.2 that, for X and $\mathcal{F}$ as above, the Hodge filtration $F^*_{\mathcal{F}}$ is a filtration on $\widehat{C}^*_{DR}(X, \mathcal{O})$ whose associated graded is given, as a graded E_∞-algebra, by foliated de Rham cohomology with coefficients in the derived external products of the conormal complex

$$Gr^i_{F^*_{\mathcal{F}}}\widehat{C}^*_{DR}(X, \mathcal{O}) \simeq \widehat{C}^*_{DR}(\mathcal{F}, \mathbb{L}\Lambda^{-i}\mathcal{N}^*_{\mathcal{F}})[i]$$

where $\mathbb{L}\Lambda^i\mathcal{N}^*_{\mathcal{F}}$ is the i-th derived exterior product of $\mathcal{N}^*_{\mathcal{F}}$, equipped with its natural structure of a perfect crystal along $\mathcal{F}$. In particular, when $\mathcal{N}_{\mathcal{F}}$ is a vector bundle of rank r, $\mathbb{L}\Lambda^i\mathcal{N}^*_{\mathcal{F}} \simeq \Lambda^i\mathcal{N}^*_{\mathcal{F}} \simeq 0$ for all $i > r$. As a consequence, using the multiplicative nature of the Hodge filtration, we obtain the following important corollary.

Corollary 7.4.0.4. *Let X be a derived Artin stack locally of finite presentation over k, and $\mathcal{F} \in \mathcal{F}ol(X)$ be a perfect derived foliation whose conormal complex $\mathcal{N}^*_{\mathcal{F}}$ is a vector bundle of rank r. Then, for any homogenuous polynomial $P(x_1, \dots, x_r)$, with $deg(x_i) = -i$, of total degree $deg(P) = q < -r$, the element*

$$P(c_1(\mathcal{N}^*_{\mathcal{F}}), \dots, c_r(\mathcal{N}^*_{\mathcal{F}})) : k \to \widehat{C}^*_{DR}(X, \mathcal{O})$$

is canonically homotopic to zero.

Proof. The vector bundle $\mathcal{N}^*_{\mathcal{F}}$ has canonical structure of a crystal along $\mathcal{F}$, and thus, by Corollary 7.4.0.3, all the morphisms of dg-modules over k

$$c_i(\mathcal{N}^*_{\mathcal{F}}) : k \to \widehat{C}^*_{DR}(X, \mathcal{O})$$

possess canonical factorisations

$$c_i(\mathcal{N}^*_{\mathcal{F}}) \to F^{-i}_{\mathcal{F}}\widehat{C}^*_{DR}(X, \mathcal{O}).$$

As the Hodge filtration is a filtration of E_∞-algebras, we deduce that the induced morphism $P(c_1(\mathcal{N}^*_{\mathcal{F}}), \dots, c_r(\mathcal{N}^*_{\mathcal{F}}))$ has a canonical factorisation as

$$P(c_1(\mathcal{N}^*_{\mathcal{F}}), \dots, c_r(\mathcal{N}^*_{\mathcal{F}})) : k \to F^{-r-1}_{\mathcal{F}}\widehat{C}^*_{DR}(X, \mathcal{O}) \to \widehat{C}^*_{DR}(X, \mathcal{O}).$$

But, by assumption $\mathbb{L}\Lambda^i\mathcal{N}^*_{\mathcal{F}} = 0$ for $i > r$, and thus $F^{-r-1}_{\mathcal{F}}\widehat{C}^*_{DR}(X, \mathcal{O}) \simeq 0$. $\square$

7.5. Residues of singular foliations in positive characteristics

In this section we exploit the results on characteristic classes defined and studied in the previous section, in the specific case of a smooth variety over a perfect field of characteristic $p \neq 2$. Our objective is to prove a crystalline version of the Baum-Bott residue theorem of [**BB72a**].

We let k be a perfect field of characteristic $p > 0$, and we assume $p \neq 2$. We start by explaining the well-known relations between the crystalline cohomology over $\mathbb{W}(k)$ and derived de Rham cohomology. We let $S_n = \mathbf{Spec}\,\mathbb{W}_n(k)$ be the spectrum of the ring of Witt vectors of length n over k, $\mathbb{W}(k) = lim_n \mathbb{W}_n(k)$ the corresponding filtered complete algebra, and $S := \mathsf{Spf}\,\mathbb{W}(k) = "colim"_n S_n$ its formal spectrum. In the sequel, in order to avoid confusions with our notation $\mathbf{DR}(A/\mathbb{W}_n(k))$, we will write $dR(A/\mathbb{W}_n(k))$ to denote the usual (underived) de Rham complex of a commutative $\mathbb{W}_n(k)$-algebra A

$$dR(A/\mathbb{W}_n(k)) := \left(A \to \Omega^1_{A/\mathbb{W}_n(k)} \to \Omega^2_{A/\mathbb{W}_n(k)} \to \dots\right)$$

where A sits in degree 0.

For a commutative simplicial $\mathbb{W}_n(k)$-algebra A, which we assume to be a finite cell $\mathbb{W}_n(k)$-algebra (so each A_m is a polynomial algebra over $\mathbb{W}_n(k)$), we consider the projection $A \to \pi_0(A)$ and the corresponding augmentation simplicial ideal $I \subset A$. We let $D(I)$ be the divided power envelope of A along I relative to $\mathbb{W}_n(k)$ (endowed with its canonical divided power structure), so we have a commutative diagram of commutative simplicial $\mathbb{W}_n(k)$-algebras

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & D(I) \arrow[d] \\ & \mathbb{W}_n(k). \end{tikzcd}$$

The commutative A-algebra $D(I)$ carries a canonical flat connection, and thus we can form its usual de Rham complex, which is the simplicial object in $\mathbf{dg}_k^{fil}$

$$[m] \mapsto D(I) \otimes_{A_m} dR(A_m/\mathbb{W}_n(k)),$$

where the filtrations are here given by the PD-Hodge filtration of [**Ber74**, §V-2.3.1] (convolution of the standard Hodge filtration with the PD-adic filtration). We remind that the $(-i)$-layer of this filtration is the subcomplex (for a fixed given m)

$$I_m^{[i]} \to I_m^{[i-1]} \otimes \Omega^1_{A_m/\mathbb{W}_n(k)} \to I_m^{[i-2]} \otimes \Omega^2_{A_m/\mathbb{W}_n(k)} \to \dots,$$

where $I_m^{[q]}$ denotes the q-th PD-powers of the ideal I_m in $D(I_m)$. By taking the geometric realization of this simplicial object (i.e. its colimit in the ∞-category $\mathbf{dg}_k^{fil}$) we obtain a well defined object $|dR(D(I)/\mathbb{W}_n(k))| \in \mathbf{dg}_{WW(k)}^{fil}$ which is functorial in A. It defines an ∞-functor $A \mapsto |dR(D(I))|$, which by [**Ber74**, §V-2.3.2] is naturally equivalent to the functor sending A to the complex of crystalline cohomology of the $\mathbb{W}_n(k)$-algebra $\pi_0(A)$.

For any A as above, the morphism $A \to D(I)$ induces a morphism on the de Rham complexes $|dR(A)| \to |dR(D(I))|$ where the left hand side is the geometric realization of the simplicial object $[m] \mapsto dR(A_m/\mathbb{W}_n(k))$. We can complete this morphism with respect to the filtrations on both sides, and thus get an induced morphism

$$|\widehat{dR(A)}| \to |\widehat{dR(D(I))}|,$$

which is functorial in A. The left hand side is naturally equivalent to the complex $\widehat{C}^*_{DR}(X/S_n, \mathcal{O})$ computing derived de Rham cohomology of $X = \mathbf{Spec}\, A$ relative to $\mathbb{W}_n(k)$, whereas the right hand side is the PD-completed crystalline cohomology of $t_0(X) = \mathbf{Spec}\, \pi_0(A)$ relative to $\mathbb{W}_n(k)$ denoted by

$$\widehat{\mathcal{C}}^*_{crys}(t_0(X)/\mathbb{W}_n(k)) := |\widehat{dR(D(I))}|.$$

We thus have constructed a functorial diagram of objects in $\mathbf{dg}_{\mathbb{W}(k)}^{fil}$

$$\widehat{C}^*_{DR}(X/S_n, \mathcal{O}) \longrightarrow \widehat{\mathcal{C}}^*_{crys}(t_0(X)/S_n) \longleftarrow \mathcal{C}^*_{crys}(t_0(X)/S_n)$$

where the map on the right hand side is the completion morphism with respect to the PD-Hodge filtration. This diagram of morphisms is not only functorial in X, but also functorial with respect to the natural projections $\mathbb{W}_{n+1}(k) \to \mathbb{W}_n(k)$, so that we can pass to the limit over n

$$\widehat{C}^*_{DR}(X/S) \xrightarrow{\phi} \widehat{\mathcal{C}}^*_{crys}(t_0(X)/S) \longleftarrow \mathcal{C}^*_{crys}(t_0(X)/S).$$

Finally, the functoriality in X can be used to glue these constructions to get a global version valid for any derived stack X which is locally of finite presentation over S (i.e. a compatible system of derived Artin stacks X_n of finite presentation over S_n). Moreover, we could also keep track of the multiplicative structures involved and contemplate the natural lift of the previous diagram to the ∞-category of filtered E_∞-algebras over $\mathbb{W}(k)$. Note also that, as $\widehat{\mathcal{C}}^*_{crys}(s/S) \simeq \mathbb{W}(k)$, the morphism ϕ enters in a commutative diagram of complete filtered E_∞-algebras

$$\begin{tikzcd}
\widehat{C}^*_{DR}(X/S) \arrow[r] & \widehat{\mathcal{C}}^*_{crys}(X/S) \\
\widehat{C}^*_{DR}(s/S) \arrow[r] \arrow[u] & \mathbb{W}(k) \arrow[u]
\end{tikzcd}$$

and thus produces a morphism

$$\widehat{C}^*_{DR}(X/S) \otimes_{\widehat{C}^*_{DR}(s/S)} \mathbb{W}(k) \to \mathcal{C}^*_{crys}(X/S)$$

(here we have noted $s = \mathbf{Spec}\, k$).

Propositon 7.5.0.1. *Let X be a smooth variety over k, considered as a derived scheme of finite presentation over S via the natural morphism $s = \mathbf{Spec}\, k \to S$.*

(1) The completion morphism with respect to the PD-Hodge filtration

$$\mathcal{C}^*_{crys}(X/S) \to \widehat{\mathcal{C}}^*_{crys}(X/S)$$

is an equivalence.

(2) The morphism ϕ constructed above

$$\widehat{C}^*_{DR}(X/S) \otimes_{\widehat{C}^*_{DR}(s/S)} \mathbb{W}(k) \to \widehat{\mathcal{C}}^*_{crys}(X/S)$$

is an equivalence.

Proof. Assertion (1) follows from the fact that the PD-Hodge filtration is already complete (in fact it is finite) when X is smooth over k. Indeed, by descent we can assume that X is affine to prove (1). We can then compute the crystalline cohomology of X over S_n by choosing a smooth lift X_n over S_n. The scheme X_n is the PD-hull of X relative to S_n, and the PD-adic filtration is then finite as this can be seen using [**Ber74**, §I-3.2.4] (we use here that $p \neq 2$). We thus have that the PD-Hodge filtration is finite and thus complete.

It remains to prove (2). By construction

$$\widehat{C}^*_{DR}(X/S) \otimes_{\widehat{C}^*_{DR}(s/S)} \mathbb{W}(k) \to \widehat{\mathcal{C}}^*_{crys}(X/S)$$

is a morphism of complete $\mathbb{W}(k)$-dg-modules, and thus to check that it is an equivalence it is enough to check that it is so after base change along the augmentation morphism $\mathbb{W}(k) \to k$. After such a base change, the morphism ψ can be written as

$$\widehat{C}^*_{DR}(X \times s \times_S s/s) \otimes_{\widehat{C}^*_{DR}(s \times_S s/s)} k \longrightarrow \widehat{C}^*_{DR}(X/s).$$

By Künneth, this is also written as the canonical morphism

$$\widehat{C}^*_{DR}(X/s) \otimes_k \widehat{C}^*_{DR}(s \times_S s/s) \otimes_{\widehat{C}^*_{DR}(s \times_S s/s)} k \longrightarrow \widehat{C}^*_{DR}(X/s)$$

which is obviously an equivalence. □

By combining Corollary 7.4.0.4 and Proposition 7.5.0.1 we finally find the following corollary, which is our version of the Baum-Bott's residue theorem in characteristic $p > 0$. We let X be a smooth variety over k. For a derived foliation $\mathcal{F}$ on X relative to $s = \mathbf{Spec}\, k$, we will call an *S-structure on $\mathcal{F}$* the data of a derived foliation $\mathcal{F}' \in \mathcal{F}ol(X/S)$ lifting $\mathcal{F}$ via the pull-back $i : \mathcal{F}ol(X/S) \to \mathcal{F}ol(X/s)$ along $i : s \hookrightarrow S$.

COROLLARY 7.5.0.2. *Let X be a smooth variety over k, and $D \subset \Omega^1_{X/k}$ be a coherent subsheaf which is a differential ideal: $dR : D \to \Omega^2_{X/k}$ factors through the image of $D \otimes \Omega^1_{X/k} \to \Omega^2_{X/k}$. Let $U \subset X$ be a dense open on which D is a sub-bundle of rank q of $\Omega^1_{X/k}$, and Z its reduced closed complement. Let us fix an S-structure on the derived foliation defined by $D_{|U} \subset \Omega^1_{U/k}$ (by Proposition 7.2.0.3).*

Then, for any homogeneous polynomial $P \in \mathbb{Z}[c_1, \dots, c_q]$ of degree $d < -q$ (with $deg(c_i) = -i$), there exists a canonical element in local cohomology supported on Z

$$Res_{D,P} \in H^{-d}_{Z,crys}(X/S)$$

whose image in $H^{-d}_{crys}(X/S)$ coincide with $P(c_1(D), \dots, c_q(D))$.

Note that the residue $Res_{D,P}$ does depend on the choice of the S-structure on the foliation defined by the differential ideal $D_{|U}$. Note also that when $U = X$, that is when D is a sub-bundle of rank q in $\Omega^1_{X/k}$ then we find that $P(c_1(D), \dots, c_q(D))$ vanishes in crystalline cohomology $H^d_{crys}(X/S)$. Without the condition on the existence of an S-structure, we would only have that this class is divisible by p.

Finally, we note that if both X and D lifts to characteristic zero, then the existence of the S-structure is automatic (and the lift provides a canonical one). However, S-structures can exist without X being liftable to S. For instance, the tautological foliation $*_{X/s} \in \mathcal{F}ol(X/s)$ always possesses a canonical lift, namely $*_{X/S}$, even if X does not itself lift to S. In a way, the existence of an S-structure should be thought of as the existence of a lift of an hypothetical leaf space of the foliation from s to S.

7.6. Infinitesimal derived foliations

In this section we present an alternative notion of derived foliations in arbitrary characteristics. In characteristic zero this notion is equivalent to our original notion of derived foliations, by means of the famous *red shift self equivalence* of the ∞-category of graded mixed complexes. However, outside of characteristic zero this notion is different from the one presented previously in this chapter (Definition 7.2.0.1). We call this alternative notion *infinitesimal derived foliations*, as its relation with infinitesimal cohomology is similar to the relation between derived foliation and de Rham cohomlogy. It is also very closely related to the notion of *partition Lie algebras and Lie algebras* and we refer to [**Fu12**] for a precise comparison statement.

We will show that any infinitesimal derived foliation possesses an underlying derived foliation in the sense of 7.2.0.1. It is important to understand that infinitesimal derived foliations are somehow more restricted and could be thought of as derived foliations together with some extra structures/properties. For instance, infinitesimal derived foliations should also be closely related to p-restricted foliations of [**Eke87**].

We start by considering a derived group stack $\mathcal{H}_\pi$, which will be playing a similar role than the group stack $\mathcal{H}$ considered in Section 7.1. For this we consider $\mathbb{G}_a[-1] := \Omega_0\mathbb{G}_a$, the derived group scheme (over k) defined as the derived loop space of the additive group $\mathbb{G}_a$ taken at 0. By definition, we have

$$\mathbb{G}_a[-1] \simeq \mathbf{Spec}\, k \times_{\mathbb{G}_a} \mathbf{Spec}\, k \simeq \mathbf{Spec}\, k \oplus k[1],$$

where $k \oplus k[1]$ is the simplicial commutative ring obtained as the trivial square zero extension of k by $k[1]$. The natural $\mathbb{G}_m$-action on $\mathbb{G}_a$, which by our convention is of weight -1 (see right before Definition 1.5.0.1), induces an action of $\mathbb{G}_m$ on the derived group scheme $\mathbb{G}_a[-1]$. Algebraically, this $\mathbb{G}_m$ action on $\mathcal{O}(\mathbb{G}_a[-1])) = k \oplus k[1]$ is such that $k[1]$ is pure of weight 1.

Definition 7.6.0.1. *The* derived group scheme $\mathcal{H}_\pi$ *is defined to be the semi-direct product of* $\mathbb{G}_a[-1]$ *by* $\mathbb{G}_m$

$$\mathcal{H}_\pi := \mathbb{G}_m \ltimes \mathbb{G}_a[-1].$$

According to the above definition, the underlying derived scheme of $\mathcal{H}_\pi$ is $\mathbf{Spec}\,(k \oplus k[1])[t, t^{-1}]$, i.e. the spectrum of the Laurent polynomials ring with coefficients in $k \oplus k[1]$. The group structure on $\mathcal{H}_\pi$ provides $(k \oplus k[1])[t, t^{-1}]$ with a structure of a cogroup object inside the ∞-category of simplicial commutative k-algebras, or in other words as a k-linear commutative Hopf simplicial algebra. If we write $k \oplus k[1]$ as $k[\epsilon]$, with ϵ a free generator in homotopical degree 1 (so cohomological degree -1), this makes $k[\epsilon][t, t^{-1}]$ into a a simplicial commutative Hopf algebra for the usual *twisted* comultiplication

$$\Delta(\epsilon) = \epsilon \otimes 1 + t \otimes \epsilon \qquad \Delta(t) = t \otimes t.$$

Similarly to Proposition 7.1.0.10, we have the following result, identifying graded mixed complexes with representations of the group $\mathcal{H}_\pi$.

PROPOSITON 7.6.0.2. *There exists a natural equivalence of k-linear symmetric monoidal ∞-categories*

$$\epsilon-\mathbf{dg}_k^{gr} \simeq \mathbf{QCoh}(B\mathcal{H}_\pi).$$

Proof. The proof is similar to [**MRT22**, Prop. 4.2.3 (ii)], and in fact easier as the group $\mathcal{H}_\pi$ is an affine derived group scheme, which simplifies some of the arguments. We start by writing $B\mathcal{H}_\pi$ as the standard geometric realization of the simplicial object $[n] \mapsto \mathcal{H}_\pi^n$. By descent we have a canonical equivalence of symmetric monoidal ∞-categories

$$\mathbf{QCoh}(B\mathcal{H}_\pi) \simeq \lim_n \mathcal{O}(\mathcal{H}_\pi)^{\otimes n} - \mathbf{dg}_k.$$

We observe that the simplicial object $[n] \to \mathcal{H}_\pi^n$ is the nerve of the zero section $\mathbf{Spec}\, k \to [\mathbb{A}^1/\mathbb{G}_m]$, so we have, by pull-back, a symmetric monoidal ∞-functor

$$\mathbf{QCoh}([\mathbb{A}^1/\mathbb{G}_m]) \longrightarrow \lim_n \mathcal{O}(\mathcal{H}_\pi)^{\otimes n} - \mathbf{dg}_k \simeq \mathbf{QCoh}(B\mathcal{H}_\pi).$$

By the comparison between graded mixed complexes and complete filtered complexes (see Corollary 1.3.4.4) the Tate realization provides a lax symmetric monoidal full embedding

$$\epsilon-\mathbf{dg}_k^{gr} \hookrightarrow \mathbf{QCoh}([\mathbb{A}^1/\mathbb{G}_m])$$

and thus by composition a lax symmetric monoidal ∞-functor

$$\psi : \epsilon-\mathbf{dg}_k^{gr} \longrightarrow \mathbf{QCoh}(B\mathcal{H}_\pi).$$

We now prove that ψ is a symmetric equivalence. For this, we start by noticing that the lax symmetric monoidal structure of ψ is in fact strict. For this we consider the natural projection $\mathcal{H}_\pi \to \mathbb{G}_m$, and the induced morphism $p : B\mathcal{H}_\pi \to B\mathbb{G}_m$ on classifying derived stacks. The composition of ψ by the push-forward $p_* : \epsilon-\mathbf{dg}_k^{gr} \to \mathbf{QCoh}(B\mathbb{G}_m) \simeq \mathbf{dg}_k^{gr}$, is easily seen to be equivalent to the ∞-functor sending a graded mixed objects E to the graded object $E^{(*)}[-2*]$. In particular, its lax symmetric monoidal structure is strict. Moreover, as $p_* : \mathbf{QCoh}(B\mathcal{H}_\pi) \to \mathbf{QCoh}(B\mathbb{G}_m)$ is clearly conservative (because p is induced by an affine morphism), we deduce that the lax symmetric monoidal structure on ψ is a symmetric monoidal structure.

It remains to show that the ∞-functor underlying ψ is an equivalence of ∞-categories. For this we come back to the situation

$$\epsilon-\mathbf{dg}_k^{gr} \hookrightarrow \mathbf{QCoh}([\mathbb{A}^1/\mathbb{G}_m]) \longrightarrow \mathbf{QCoh}(B\mathcal{H}_\pi).$$

By Corollary 1.3.4.4, the image of the first full embedding is the full sub-∞-category of objects which are complete at 0. We thus have to show that $\mathbf{QCoh}([\mathbb{A}^1/\mathbb{G}_m]) \longrightarrow \mathbf{QCoh}(B\mathcal{H}_\pi)$ identifies the full sub-∞-category of complete modules with $\mathbf{QCoh}(B\mathcal{H}_\pi)$. This follows easily from the fact that $B\mathcal{H}_\pi$ is the realization of the nerve of the zero section $B\mathbb{G}_m \hookrightarrow [\mathbb{A}^1/\mathbb{G}_m]$ as shown by the following lemma.

LEMMA 7.6.0.3. *Let A be a k-algebra and $f \in A$ be a non-zero divisor. Let B^* be the cosimplicial E_∞-algebra given by the derived conerve of $A \to A/f$*

$$B^n := A/f \otimes_A A/f \otimes_A \cdots \otimes_A A/f.$$

Then the descent ∞-functor

$$\mathbf{dg}_A \to \lim_n \mathbf{dg}_{B^n}$$

induces an equivalence

$$\hat{\mathbf{dg}}_{A,f} \simeq \lim_n \mathbf{dg}_{B^n}$$

where $\hat{\mathbf{dg}}A, f$ is the ∞-category of f-complete A-dg-modules (E such that $E \simeq \lim_n(E/f^n)$).

Proof of the lemma. The descent ∞-functor $\phi : \mathbf{dg}_A \to \lim_n \mathbf{dg}_{B^n}$ sends an A-dg-module to the pull-back $E \otimes_A B^*$. It has a right adjoint ψ sending $E^* \in \lim_n \mathbf{dg}_{B^n}$ to $\lim_n E^n \in \mathbf{dg}_A$.

We can form the cocartesian square of commutative cosimplicial E_∞-algebras

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A/f \arrow[d] \\ B^* \arrow[r] & C^*. \end{tikzcd}$$

Then, $A/f \to C^*$ is the derived conerve of the morphism $A/f \to A/f \otimes_A A/f$ which possesses a retraction given by the multiplication in A/f. Therefore, the cosimplicial object C^* is simplicially contractible. This implies that the descent ∞-functor

$$\mathbf{dg}_{A/f} \to \lim_n(\mathbf{dg}_{C^n})$$

is in fact an equivalence of ∞-categories. We consider now the commutative diagram of ∞-categories obtained by base change from the above diagram

$$\begin{tikzcd} \mathbf{dg}_A \arrow[r] \arrow[d] & \lim_n(\mathbf{dg}_{B^n}) \arrow[d] \\ \mathbf{dg}_{A/f} \arrow[r] & \lim_n(\mathbf{dg}_{C^n}). \end{tikzcd}$$

In this diagram the vertical ∞-functors are conservative and commutes with limits. Indeed, A/f is a perfect A-dg-module and thus $(A/f) \otimes_A -$ commutes with limits. Similarly, each C^n is perfect over B^n. As a result, the diagram is right adjointable in the sense of [**Lur22a**, Def. 4.7.4.13], because of the explicit description of the right adjoints of the horizontal ∞-functors in terms of limits. As the bottom horizontal ∞-functor is an equivalence, so is the top horizontal ∞-functor, because the unit of counit of the top adjunction can be checked to be equivalences after applying the vertical conservative ∞-functors (so that they become the unit and counit of the bottom adjunction). □

We can apply the previous lemma to the natural simplicial objects representing $B\mathbb{G}_m \hookrightarrow [\mathbb{A}^1/\mathbb{G}_m]$. This finishes the proof of the proposition. □

In parallel to the notion of graded loop space we introduce the *infinitesimal* graded loop space as follows.

DEFINITION 7.6.0.4. *Let X be a derived stack. The* infinitesimal graded loop space of X *is defined by*

$$\mathcal{L}^{gr}_{\pi}(X) := \underline{\mathbf{Map}}(\mathbb{G}_a[-1], X) \in \mathbf{dSt}.$$

If $X \to Y$ is a morphism of derived stacks, the relative infinitesimal graded loop space of X over Y *is defined by*

$$\mathcal{L}^{gr}_{\pi}(X/Y) := \mathcal{L}^{gr}_{\pi}(X) \times_{\mathcal{L}^{gr}_{\pi}(Y)} Y,$$

for the natural morphism $Y \to \mathcal{L}^{gr}_{\pi}(Y)$ corresponding to the projection $Y \times \mathbb{G}_a[-1] \to Y$.

The group $\mathbb{G}_m$ acts on $\mathbb{G}_a[-1]$, and the group $\mathbb{G}_a[-1]$ acts on itself by translation. Therefore we get a natural action of $\mathcal{H}_\pi = \mathbb{G}_m \ltimes \mathbb{G}_a[-1]$ on the stack $\mathbb{G}_a[-1]$, and thus on the infinitesimal graded loop space of any derived stack. The object $\mathcal{L}^{gr}_{\pi}(X)$ will always be considered as an object in the ∞-topos $\mathbf{dSt}^{\mathcal{H}_\pi}$ of derived stacks endowed with a $\mathcal{H}_\pi$-action (and $\mathcal{H}_\pi$-equivariant maps).

The following proposition identifies $\mathcal{L}^{gr}_{\pi}(X)$ with its $\mathbb{G}_m$-action when X is a derived Artin stack. For this we consider the canonical point $* \to \mathbb{G}_a[-1]$ given by the unit, and its induced projection $\mathcal{L}^{gr}_{\pi}(X) \to X$. Note that this projection is $\mathbb{G}_m$-equivariant, but is not $\mathbb{G}_a[-1]$ equivariant as the $\mathbb{G}_a[-1]$-action by translation does not fix the base point. The derived stack $\mathcal{L}^{gr}_{\pi}(X) \to X$ thus lives in the ∞-topos $\mathbf{dSt}^{\mathbb{G}_m}_X$ of $\mathbb{G}_m$-equivariant objects over X. Recall from [**Mon21**] that this category contains as a full sub-∞-category the ∞-category of derived linear stacks over X, which is equivalent to $\mathbf{QCoh}^-(X)$ of eventually coconnective quasi-coherent complexes on X. The full-embedding $\mathbf{QCoh}^-(X) \hookrightarrow \mathbf{dSt}^{\mathbb{G}_m}_X$ is given by the construction $\mathbb{V}$, sending $E \in \mathbf{QCoh}^-(X)$ to $\mathbb{V}(E) \to X$ to the ∞-functor sending an affine $u : \mathbf{Spec}\, A \to X$ to the space $Map_{\mathbf{dg}_A}(u^*(E), A)$ (see Definition A.9.2.4).

PROPOSITON 7.6.0.5. *Let X be a derived Artin stack. The $\mathbb{G}_m$-equivariant derived stack $\mathcal{L}^{gr}_{\pi}(X) \to X$ is canonically equivalent to the linear stack $\mathbb{V}(\mathbb{L}_X[-1]) \to X$, associated to the quasi-coherent complex $\mathbb{L}_X[-1]$.*

Proof. As $\mathbb{G}_a[-1] = \mathbf{Spec}\, k \oplus k[1]$, this is simply the definition of $\mathbb{L}_X$ in terms of points with values in square zero extensions (see [**TV08**, Def. 1.4.1.5]). □

The terminology *infinitesimal graded loop space* has been chosen because of the theorem below, expressing that $\mathcal{H}_\pi$-equivariant functions on $\mathcal{L}^{gr}_{\pi}(X)$ computes infinitesimal cohomology of X, at least in the smooth case. Recall that if $X \to S = \mathbf{Spec}\, k$ is a smooth morphism of schemes, we can define $(X/S)_{inf}$ as the functor sending an affine $\mathbf{Spec}\, A \to S$ to $X(A_{red})$, where A_{red} is defined to be the discrete ring $\pi_0(A)_{red}$. The functor $(X/S)_{inf}$ is a derived stack over S, and by definition infinitesimal cohomology of X relative to S is defined by

$$C^*_{inf}(X/S) := C^*((X/S)_{inf}, \mathcal{O}),$$

which will be considered as a k-linear E_∞-algebra. Because X is smooth over S, the canonical morphism $X \to (X/S)_{inf}$ is an epimorphism of derived stacks (because of the infinitesimal lifting property). Therefore, $H^*_{inf}(X/S)$ can also be computed as the limit

$$C^*_{inf}(X/S) \simeq \lim_n (C^*(X^{(n)}, \mathcal{O})),$$

where $X^{(*)}$ is the the nerve of X over $(X/S)_{inf}$, which can also be described as the nerve of the formal groupoid $X^{(2)} \leftleftarrows X$ where $X^{(2)}$ is the formal completion of $X \times_S X$ along the diagonal.

In order to state the theorem below, let us consider the quotient map

$$p : [\mathcal{L}^{gr}_\pi(X/S)/\mathcal{H}_\pi] \to S$$

The direct image of the structure sheaf $p_*(\mathcal{O})$ is by definition the $\mathcal{H}_\pi$-equivariant cohomology of $\mathcal{L}^{gr}_\pi(X/S)$ with coefficients in the structure sheaf $\mathcal{O}$. It will be symbolicaly denoted by $C^*(\mathcal{L}^{gr}_\pi(X/S))^{\mathcal{H}_\pi}$.

We produce a natural morphism $C^*_{inf}(X/S) \to C^*(\mathcal{L}^{gr}_\pi(X/S))^{\mathcal{H}_\pi}$ as follows. The natural morphism $X \to (X/S)_{inf}$ induces a morphism on infinitesimal graded loop spaces $\mathcal{L}^{gr}_\pi(X/S) \to \mathcal{L}^{gr}_\pi((X/S)_{inf}/S)$. By the definition of $(X/S)_{inf}$, and from the fact that the reduced group $\mathcal{H}_\pi$ is trivial, it is easy to see that the constant map morphism $(X/S)_{inf} \to \mathcal{L}^{gr}_\pi((X/S)_{inf}/S)$ is an equivalence. In particular, we get this way a natural morphism of derived stacks with $\mathcal{H}_\pi$-actions $\mathcal{L}^{gr}_\pi(X/S) \to (X/S)_{inf}$. By considering cohomology with coefficients in $\mathcal{O}$ we finally get the required morphism

$$\phi_{X/S} : C^*_{inf}(X/S) \longrightarrow C^*(\mathcal{L}^{gr}_\pi(X/S))^{\mathcal{H}_\pi}.$$

By construction it is a morphism of E_∞-algebras functorial in X/S. Note that the above morphism always exists, for a general morphism of derived stacks $X \to S$ without assuming any extra conditions.

THEOREM 7.6.0.6. *Let $X \to S = \mathbf{Spec}\, k$ be a smooth scheme. Then, the above morphism*

$$\phi_{X/S} : C^*_{inf}(X/S) \simeq C^*(\mathcal{L}^{gr}_\pi(X/S))^{\mathcal{H}_\pi}$$

is an equivalence.

Proof. We consider the commutative diagram of derived stacks with $\mathcal{H}_\pi$-actions

$$\begin{tikzcd} X \arrow[r, "id"] \arrow[d] & X \arrow[d] \\ \mathcal{L}^{gr}_\pi(X/S) \arrow[r, "\phi_{X/S}"'] & (X/S)_{inf}. \end{tikzcd}$$

This diagram induces a morphism of simplicial objects on the corresponding nerves of the vertical morphisms. The nerve of the morphism $X \to \mathcal{L}^{gr}_{\pi}(X/S)$ can be described by the simplicial object $[n] \mapsto \mathcal{L}^{gr}_{\pi}(X/X^n)$, where X sits inside X^n by the diagonal embedding. At the simplicial level n, the morpshim induced on the nerve is thus an $\mathcal{H}_{\pi}$-equivariant morphism of stacks over S

$$\psi_{X,n} : \mathcal{L}^{gr}_{\pi}(X/X^n) \longrightarrow X^{(n)},$$

where $X^{(n)}$ is the formal completion of X^n along its diagonal.

LEMMA 7.6.0.7. *The morphism $\psi_{X,n}$ induces by pull-backs an equivalence of E_∞-algebras*

$$C^*(X^{(n)}, \mathcal{O}) \simeq C^*(\mathcal{L}^{gr}_{\pi}(X/X^n))^{\mathcal{H}_{\pi}}.$$

Proof Lemma 7.6.0.7. By descent we can easily restrict to the case where $X = \mathbf{Spec}\, A$ is a smooth affine scheme over $S = \mathbf{Spec}\, k$. In this case, $C^*(X^{(n)}, \mathcal{O})$ identifies canonically with $\hat{A}^{\otimes_k n}$, the formal completion of $A^{\otimes_k n}$ along the canonical augmentation $A^{\otimes_k n} \to A$. Similarly, $\mathcal{L}^{gr}_{\pi}(X/X^n)$ is canonically identified with the smooth affine scheme $\mathbf{Spec}\, Sym_A((\Omega^1_{A/k})^{n-1})$. Therefore, the morphism $\psi_{X,n}$ is a morphism of discrete commutative algebras

$$\psi_{A,n} : \hat{A}^{\otimes_k n} \longrightarrow |Sym_A((\Omega^1_{A/k})^{n-1})|^t,$$

where the right hand side is the Tate realization of the graded mixed object $Sym_A((\Omega^1_{A/k})^{n-1})$ induced by the $\mathcal{H}_{\pi}$-action on $\mathcal{L}^{gr}_{\pi}(X/X^n)$. The morphism $\psi_{A,n}$ is compatible with the canonical augmentations to A, and thus is a morphism of complete filtered commutative algebras. Therefore, in order to prove it is an isomorphism it is enough to show that it induces an isomorphism on the associated graded objects. This induced morphism is a morphism

$$gr(\psi_{A,n}) : Gr^*(\hat{A}^{\otimes_k n}) \longrightarrow Sym_A((\Omega^1_{A/k})^{n-1}).$$

Moreover, because A is a smooth k-algebra, the left hand side is canonically isomorphic to $Sym_A(I_n/I_n^2)$, where I_n is the augmentation ideal. Finally, we have the usual natural isomorphism of A-modules $I_n/I_n^2 \simeq (\Omega^1_{A/k})^{n-1}$, and thus the morphism $gr\psi_{A,n}$ can be identified with a graded endomorphism of the graded algebra $Sym_A((\Omega^1_{A/k})^{n-1})$. As it is compatible with augmentations to A it is thus determined by an A-linear endomorphism of $(\Omega^1_{A/k})^{n-1}$. By functoriality in A and in the simplicial direction n, we see moreover that it is determined by a functorial endormorphism the A-module $\Omega^1_{A/k}$, and thus must be the multiplication by an element $\lambda \in k$. To conclude the lemma, we have to show that $\lambda = 1$.

By functoriality, it is easy to reduce to the case $A = k[T]$ is a polynomial ring, and by base change we can even assume that $k = \mathbb{Z}$. In order to check that $\lambda = 1$ we can base change from $\mathbb{Z}$ to $\mathbb{Q}$, and thus reduce to the case $A = \mathbb{Q}[T]$ and $k = \mathbb{Q}$. As we are now in characteristic zero, $C^*(\mathcal{L}^{gr}_{\pi}(X/X^n))^{\mathcal{H}_{\pi}}$ identifies canonically with the completed de Rham complex $\widehat{C}^*_{DR}(X/X^n)$. Indeed, the red shift equivalence sends the graded mixed cdga $\mathcal{O}(\mathcal{L}^{gr}_{\pi}(X/X^n))$ to $\mathcal{O}(\mathcal{L}^{gr}(X/X^n))$. Finally, the morphism $\psi_{X,n}$ is now the natural isomorphism relating derived de Rham cohomology of a closed embedding with functions on the formal completion (see for instance Proposition 1.4.2.3). □

The lemma implies the theorem by descent along the epimorphisms $X \to \mathcal{L}^{gr}_\pi(X/S)$ and $X \to (X/S)_{inf}$. □

7.6.1. Infinitesimal derived foliations. As before, we fix k a base simplicial commutative ring.

DEFINITION 7.6.1.1. *Let $X = \mathbf{Spec}\, A$ be a derived affine scheme over k. A (perfect)* infinitesimal derived foliation $\mathcal{F}$ on X (relative to k) *consists of a $\mathcal{H}_\pi$-equivariant derived stack $\mathbb{V}(\mathcal{F})$ together with a $\mathcal{H}_\pi$-equivariant morphism $p : \mathbb{V}(\mathcal{F}) \to \mathcal{L}^{gr}_\pi(X/k)$, such that, as a $\mathbb{G}_m$-equivariant stack over X, $\mathbb{V}(\mathcal{F})$ is a perfect linear derived affine stack (i.e. of the form $\mathbb{V}(E)$ for E a perfect complex on X).*

The infinitesimal derived foliations on X (relative to k) form an ∞-category $\mathcal{F}ol_\pi(X/k)$, which by definition is a full sub-∞-category of $(\mathbf{dSt}_k^{\mathcal{H}_\pi})/\mathcal{L}^{gr}_\pi(X/k)$, the $\mathcal{H}_\pi$-equivariant derived stacks over $\mathcal{L}^{gr}_\pi(X/k)$. The construction $X \mapsto \mathcal{F}ol_\pi(X/k)$ can be promoted to an ∞-functor $\mathcal{F}ol_\pi(-/k) : \mathbf{dAff}_k^{op} \to \mathbf{Cat}_\infty$, by defining pull-backs as follows: for a morphism $f : X \to Y$ of affine derived k-schemes, we define f^* to be induced by the base change

$$(\mathbf{dSt}_k^{\mathcal{H}_\pi})/\mathcal{L}^{gr}_\pi(Y/k) \longrightarrow (\mathbf{dSt}_k^{\mathcal{H}_\pi})/\mathcal{L}^{gr}_\pi(X/k),$$

along the induced morphism $\mathcal{L}^{gr}_\pi(f/k) : \mathcal{L}^{gr}_\pi(X/k) \to \mathcal{L}^{gr}_\pi(Y/k)$. As the ∞-functor $\mathcal{F}ol_\pi(-/k)$ clearly is a hyperstack for the fpqc topology, it can be uniquely extended to a colimit preserving ∞-functor

$$\mathcal{F}ol_\pi(-/k) : \mathbf{dSt}_k \to \mathbf{Cat}_\infty^{op}.$$

Let X be a derived affine k-scheme, and $\mathcal{F} \in \mathcal{F}ol_\pi(X/k)$ be an infinitesimal derived foliation on X. By [**Mon21**], the linear stack $\mathbb{V}(\mathcal{F}) \to X$ is of the form $\mathbb{V}(\mathbb{L}_{\mathcal{F}/k}[-1])$, for some uniquely defined perfect complex $\mathbb{L}_{\mathcal{F}/k}$ on X.

DEFINITION 7.6.1.2. *With the notations above, the complex $\mathbb{L}_{\mathcal{F}/k}$ is called the* cotangent complex of $\mathcal{F}$ (relative to k). *It is also simply denoted by $\mathbb{L}_{\mathcal{F}}$ if k is clear from the context.*

When X is a general derived stack and $\mathcal{F} \in \mathcal{F}ol_\pi(X/k)$, it is also possible to define the cotangent complex $\mathbb{L}_{\mathcal{F}/k}$, as a quasi-coherent complex over X following the same lines as Definition 2.1.3.4. We have first a big cotangent complex $\mathbb{L}^{big}_{\mathcal{F}}$, which is an $\mathcal{O}_X$-dg-module over the big site of derived affine schemes over X: it sends $u : \mathbf{Spec}\, A \to X$ to the A-dg-module $\mathbb{L}_{u^*(\mathcal{F})/k}$. This $\mathcal{O}_X$-dg-module is not quasi-coherent and we thus apply the quasi-coherator construction of §A.4 to get the desired cotangent complex

$$\mathbb{L}_{\mathcal{F}/k} := (\mathbb{L}^{big}_{\mathcal{F}/k})^{qcoh}.$$

The special case where X is a derived *Deligne-Mumford* stack simplifies, as the cotangent complexes of infinitesimal foliations are stable by étale pull-backs, and thus will glue globally on X. This applies in particular to the case where X is any derived scheme. In these cases, we have the natural morphism $\mathbb{L}^{big}_{\mathcal{F}/k} \to \mathbb{L}_{\mathcal{F}/k}$ restricts to an equivalence on the small étale site of X.

Let X be a derived affine scheme of finite presentation over k. The ∞-category $\mathcal{F}ol_\pi(X/k)$ admits a final object, namely the identity morphism $\mathcal{L}^{gr}_\pi(X/k) \to \mathcal{L}^{gr}_\pi(X/k)$. It is denoted by $*_{X/k}$, and its cotangent complex is naturally equivalent to $\mathbb{L}_{X/k}$, the cotangent complex of X. Indeed, it is easy to see, simply by contemplating the functors of points and using the very definition of the cotangent complex, that $\underline{\mathbf{Map}}(G_0, X) \simeq \mathbb{V}(\mathbb{L}_{X/k}[-1])$. Similarly, the initial object in $\mathcal{F}ol_\pi(X/k)$ exists, is denoted by $0_{X/k}$, and its cotangent complex is 0. It corresponds to the constant-map morphism $\mathbb{V}(0_{X/k}) = X \longrightarrow \mathcal{L}^{gr}_\pi(X/k)$.

7.6.2. Infinitesimal cohomology of infinitesimal derived foliations. Let X be a derived Artin stack of finite presentation over k, and $\mathcal{F} \in \mathcal{F}ol_\pi(X/k)$ be an infinitesimal derived foliation on X. We define the ∞-category of quasi-coherent complexes along $\mathcal{F}$ as follows. We first consider $\mathbf{QCoh}([\mathbb{V}(\mathcal{F})/\mathcal{H}_\pi])$, the ∞-category of quasi-coherent complexes on the quotient stack $[\mathbb{V}(\mathcal{F})/\mathcal{H}_\pi]$ (which is also, by definition, the ∞-category of $\mathcal{H}_\pi$-equivariant quasi-coherent complexes on $\mathbb{V}(\mathcal{F})$). The ∞-category $\mathbf{QCoh}(\mathcal{F})$ is defined as the full sub-∞-category of $\mathbf{QCoh}([\mathbb{V}(\mathcal{F})/\mathcal{H}_\pi])$ whose objects E are graded free: the pull-back of E on $[\mathbb{V}(\mathcal{F})/\mathbb{G}_m]$ is of the form $p^*(E_0)$ where $p : [\mathbb{V}(\mathcal{F})/\mathbb{G}_m] \to X$ is the natural projection.

In the definition above we use the identification $\mathbf{QCoh}(B\mathcal{H}_\pi)$ with the ∞-category of graded mixed complexes (see Proposition 7.6.0.2).

Definition 7.6.2.1. *With the notations above, the* ∞-category of quasi-coherent complexes along $\mathcal{F}$ *is* $\mathbf{QCoh}(\mathcal{F})$*. For* $E \in \mathbf{QCoh}^{\mathcal{F}}(X)$*, the* (derived and Hodge completed) the infinitesimal cohomology of X along $\mathcal{F}$ with coefficients in E *is defined to be*

$$\widehat{C}^*_{\mathrm{inf}}(\mathcal{F}, E) := |q_*(E)|^t \in \widehat{\mathbf{dg}}^{fil}_k$$

where $q : [\mathbb{V}(\mathcal{F})/\mathcal{H}_\pi] \longrightarrow B\mathcal{H}_\pi$ *is the structural map.*

The definition above is justified by our Theorem 7.6.0.6, stating that when X is smooth over k, and $\mathcal{F} = *_{X/k}$ and $E = \mathcal{O}_X$, $\widehat{C}^*_{\mathrm{inf}}(\mathcal{F}, E)$ coincides with infinitesimal cohomology defined using the usual infinitesimal site. For a general derived Artin stack X, $\widehat{C}_{\mathrm{inf}}(*_{X/k}, \mathcal{O})$ computes the *completed derived infinitesimal cohomology of* X (relative to k). Indeed, it is easy to see the ∞-functor $X \mapsto \mathcal{O}(\mathcal{L}^{gr}_\pi(X/k))$, from affine derived schemes to graded mixed complexes, is equivalent to its extension by sifted colimits from smooth X's. For $A \in \mathbf{sCR}$, where each A_n is a smooth algebra over k, we thus have

$$\widehat{C}_{\mathrm{inf}}(*_X, \mathcal{O}_X) \simeq \mathrm{colim}_{[n]} C^*((Spec\, A_n)_{\mathrm{inf}}, \mathcal{O}),$$

where the colimit is computed in the ∞-category of *complete* filtered complexes. The right hand side of this equivalence is, by definition, the completed derived infinitesimal cohomology of X, and is denoted by $\mathbb{R}\hat{C}^*(X_{\inf}, \mathcal{O})$. This notion is extended to any derived Artin stack locally of finite presentation over k by the usual gluing formula

$$\mathbb{R}\hat{C}^*(X_{\inf}, \mathcal{O}) := \lim_{U \to X} \mathbb{R}\hat{C}^*(U_{\inf}, \mathcal{O})$$

where the limit runs over affine U locally of finite presentation.

COROLLARY 7.6.2.2. *For a general derived Artin stack X of finite presentation over k, $\widehat{C}_{\inf}(*_{X/k}, \mathcal{O})$ recovers the Hodge completed derived infinitesimal cohomology of X relative to k*

$$\widehat{C}_{\inf}(*_{X/k}, \mathcal{O}) \simeq \mathbb{R}\hat{C}^*(X_{\inf}, \mathcal{O}).$$

7.6.3. Integrability by formal groupoids. As opposed to derived foliations, infinitesimal derived foliations turn out to be *formally integrable* in a very strong sense. The most general integrability statements can be found in [**BMN25**]. Below, we prove a special easy case by elementary means, showing that the ∞-category of smooth infinitesimal derived foliations over a k-scheme X is in fact *equivalent* to the category of *formally smooth formal groupoids* over X.

Let X be a smooth k-scheme and $\mathcal{F} \in \mathcal{F}ol_\pi(X/k)$. We say that $\mathcal{F}$ is *smooth* if $\mathbb{L}_\mathcal{F}$ is a vector bundle over X. In this case, we consider $X \times_{\mathbb{V}(\mathcal{F})} X$, endowed with its natural $\mathcal{H}_\pi$-action and with its first projection to X. Note that, as a graded derived stack, $X \times_{\mathbb{V}(\mathcal{F})} X$ is of the form $\mathbb{V}(\mathbb{L}_F)$, and thus is a the total space of the tangent bundle $\mathbb{T}_\mathcal{F}$ of $\mathcal{F}$. Also, the natural projection $X \times_{\mathbb{V}(\mathcal{F})} X \to X$ is $\mathcal{H}_\pi$-equivariant. As a result, $X \times_{\mathbb{V}(\mathcal{F})} X$ is a new object in $\mathcal{F}ol_\pi(X)$, that will be simply denoted by $\Omega_X\mathcal{F}$, and called the *loop space* of $\mathcal{F}$. Being the fiber product of the canonical morphism $0_X \to \mathcal{F}$, the loop space $\Omega_X\mathcal{F}$ comes equipped with a natural structure of a groupoid object in $\mathcal{F}ol_\pi(X)$, whose object of objects is $0_X \in \mathcal{F}ol_\pi(X/k)$. Note that as 0_X is not the final object $\Omega_X\mathcal{F}$ is not a group object and merely a groupoid.

PROPOSITON 7.6.3.1. *The full sub-∞-category of $\mathcal{F}ol_\pi(X/k)$ formed by objects $\mathcal{F}$ whose cotangent complex is of the form $V[1]$ for V a vector bundle on X, is naturally equivalent to the category of formal schemes Z with an identification $Z_{red} \simeq X$, which are locally equivalent on X to $X \times \widehat{\mathbb{A}}^d$ where $d = rank(V)$.*

Proof. To each $\mathcal{F}$ as in the Proposition, we associate the sheaf of infinitesimal cohomology $\widehat{C}_{\inf}(\mathcal{F})$ on X. By assumptions on the cotangent complex of $\mathcal{F}$, $\widehat{C}_{\inf}(\mathcal{F})$ is a sheaf of discrete complete filtered commutative algebras, augmented to $\mathcal{O}_X$ and its associated graded object is isomorphic to $\mathcal{O}_X[[t_1, \dots, t_d]]$. Therefore, $Z_\mathcal{F} := \mathsf{Spf}(\widehat{C}_{\inf}(\mathcal{F}))$ is a formal scheme with a canonical identification $Z_{red} \simeq X$ which is locally equivalent to $X \times \widehat{\mathbb{A}}^d$.

The fact that the construction $\mathcal{F} \mapsto Z_\mathcal{F}$ induces the equivalence of the Proposition simply follows from the fact that the Tate realization induces an equivalence between graded mixed complexes and complete filtered complexes (see [**TV20**, Prop. 1.3.1]). $\square$

By Proposition A.9.2.1, for any smooth infinitesimal derived foliation $\mathcal{F}$, its loop space $\Omega_X\mathcal{F}$ can be realized as a groupoid object in formal schemes acting on X

$$Z_{\Omega_X\mathcal{F}} \longrightarrow X \times X.$$

The construction $\mathcal{F} \mapsto Z_{\Omega_X\mathcal{F}}$ provides an functor from the category of smooth infinitesimal derived foliations on X, and the category of formally smooth formal groupoids over X. We can produce a functor in the other direction, as follows. Starting from a formally smooth formal groupoid $G \to X \times X$, we can consider its nerve $G_* \to X$, which is a simplicial object in formal schemes. Taking functions on G_* provides a cosimplicial complete filtered commutative algebra $\mathcal{O}(G_*)$ whose associated graded is the cosimplicial algebra of functions on the simplicial scheme BV, where $V = e^*(\Omega^1_{G/X})$ is the vector bundle of invariant relative forms on G. Using the equivalence between graded mixed complexes and complete filtered complexes of [**TV20**, Prop. 1.3.1], we find that $\mathcal{O}(G_*)$ can be realized as a cosimplicial object in the category of graded commutative algebras endowed with a compatible action on G_0. Passing then to *Spec*, we get a simplicial diagram of affine schemes endowed with a $\mathcal{H}_\pi$-action, whose underlying simplicial diagram (obtained by forgetting the action) is the usual simplicial scheme BV. Taking geometric realization we get a $\mathcal{H}_\pi$-action on the derived stack $\mathbb{V}(V[1])$, which defines a smooth infinitesimal derived foliations $\mathcal{F}$ on X.

We can then subsume the previous discussion in the following corollary.

COROLLARY 7.6.3.2. *The category of smooth infinitesimal derived foliations on X is equivalent to the category of formally smooth formal groupoids on X.*

We note that the above corollary marks a major difference between derived foliations of Section §7.2 and infinitesimal derived foliations. Indeed, any finite dimensional Lie algebra L over a field k, defines a smooth derived foliation over k, by considering $Sym(L^\vee[1])$ endowed with the mixed structure coming from the Chevalley-Eilenberg differential. However, it is not true that any Lie algebra can be integrated to a smooth formal scheme in general. On the contrary, the above corollary, when applied to $X = \mathbf{Spec}\, k$ with k a field, states that smooth infinitesimal derived foliations over k form a category equivalent to smooth formal groups over k. Any such smooth infinitesimal derived foliation is of the form $\mathbb{V}(V[-1])$ for V a finite dimensional k-vector space V. Therefore, the data of an infinitesimal derived foliation structure on $\mathbb{V}(V[-1])$ seems to be related to that of a *partition Lie algebra structure* in the sense of [**BCN21**]. The precise comparison is out of the scope of this note, but has been carried out in [**Fu12**].

7.6.4. Comparison between infinitesimal derived foliations and derived foliations: the "de Rhamification" functor. In this § we explain how to construct a derived foliation from any infinitesimal derived foliation. This construction will be called the *de Rhamification* functor, as it is related to the natural "forgetful" morphism from (Hodge-completed and derived) infinitesimal cohomology to de Rham cohomology.

We start by $X = \mathbf{Spec}\, A$ a derived affine scheme of finite presentation over k and $\mathcal{F} \in \mathcal{F}ol_\pi(X)$ an infinitesimal derived foliation. The natural projection $\mathbb{V}(\mathcal{F}) \to X$ possesses a zero section $s : X \to \mathcal{F}$ which is moreover naturally $\mathcal{H}_\pi$-equivariant. We can then consider the relative derived loop space of s $\widehat{\mathcal{L}}^{gr}(X/\mathbb{V}(\mathcal{F}))$, which fits in a Cartesian square of $\mathcal{H}_\pi$-equivariant derived stacks

$$\begin{tikzcd} \widehat{\mathcal{L}}^{gr}(X/\mathbb{V}(\mathcal{F})) \arrow[r] \arrow[d] & \mathcal{L}^{gr}(X/k) \arrow[d] \\ X \arrow[r] & \widehat{\mathcal{L}}^{gr}(\mathbb{V}(\mathcal{F})/k). \end{tikzcd}$$

The group stack $\mathcal{H}$ acts by its standard action on $\widehat{\mathcal{L}}^{gr}(X/\mathbb{V}(\mathcal{F}))$, and this action is itself $\mathcal{H}_\pi$-equivariant, or in other words is combined into a natural action of $\mathcal{H} \times \mathcal{H}_\pi$. The derived stack $\widehat{\mathcal{L}}^{gr}(X/\mathbb{V}(\mathcal{F}))$ is a priori not affine, but we can consider its affinization (see §A.9)

$$\widehat{\mathcal{L}}^{gr}(X/\mathbb{V}(\mathcal{F}))^{aff} := \mathbb{R}\mathbf{Spec}^{\Delta}(\mathcal{O}(\mathcal{L}^{gr}(X/\mathbb{V}(\mathcal{F})))).$$

Because the group stack $\mathcal{H} \times \mathcal{H}_\pi$ is affine and $\mathcal{O}(\mathcal{H} \times \mathcal{H}_\pi)$ is perfect as a k-dg-module, it is easy to check that $\mathcal{H} \times \mathcal{H}_\pi$ continues to act on $\widehat{\mathcal{L}}^{gr}(X/\mathbb{V}(\mathcal{F}))^{aff}$ in a natural manner (because the affinization preserves taking products with $\mathcal{H} \times \mathcal{H}_\pi$). Moreover, using the descent theorem 2.1.3.8 we have a canonical identification

$$\widehat{\mathcal{L}}^{gr}(X/\mathbb{V}(\mathcal{F}))^{aff} \simeq \mathbb{R}\mathbf{Spec}^{\Delta}(Sym_A(\mathbb{L}_{\mathcal{F}}[1])) = \mathbb{V}(\mathbb{L}_{\mathcal{F}}[1]).$$

We have therefore constructed a natural action of $\mathcal{H} \times \mathcal{H}_\pi$ on $\mathbb{V}(\mathbb{L}_{\mathcal{F}}[1])$. By construction, the induced action of $B\mathbb{G}_m \times B\mathbb{G}_m$ is the diagonal action, making $\mathbb{L}_{\mathcal{F}}[1]$ pure of weight $(1, 1)$.

LEMMA 7.6.4.1. *With the above notations, the action of $\mathbb{G}_a[-1]$ on $\mathbb{V}(\mathbb{L}_{\mathcal{F}}[1])$ is canonically trivial.*

Proof of the lemma. This is formally deduced from the fact that the action of $\mathbb{G}_a[-1]$, or more generally of $\mathcal{H}_\pi$, respects the action of $\mathcal{H}$. In particular, $\mathbb{G}_a[-1]$ acts on $\mathbb{V}(\mathbb{L}_{\mathcal{F}}[1])$ by respecting the grading. Using [**Mon21**] this action is then determined by a linear action of $\mathbb{G}_a[-1]$ on the perfect complex $\mathbb{L}_{\mathcal{F}}$, compatible with the gradings, and thus by a graded mixed structure on $\mathbb{L}_{\mathcal{F}}$, which is pure of weight 1. Such a graded mixed A-dg-module is represented by an object $E \in \epsilon - \mathbf{dg}_k^{gr}$ with a fixed quasi-isomorphism $E^{(1)} \to \mathbb{L}_{\mathcal{F}}$ and $E^{(i)}$ being acyclic for all $i \neq 1$. Such an object is equivalent E' with $(E')^{(1)} = \mathbb{L}_{\mathcal{F}}$ and $(E')^{(i)} = 0$ for $i \neq 1$ (and thus $\epsilon = 0$). □

We now form the morphism $K := B\mathcal{H}_\pi \times_{B\mathbb{G}_m} B\mathcal{H} \to B\mathcal{H}_\pi \times B\mathcal{H}$, the corresponding quotient derived stack $[\mathbb{V}(\mathbb{L}_{\mathcal{F}}[1])/K] \to B\mathcal{H}$, and its affinization

$$F := [\mathbb{V}(\mathbb{L}_{\mathcal{F}}[1])/K]^{aff} \to B\mathcal{H}.$$

Lemma 7.6.4.1 implies that $F \times_{B\mathcal{H}} B\mathbb{G}_m$ is naturally equivalent to $\mathbb{V}(\mathbb{L}_{\mathcal{F}}[1])$, and the natural projection $\mathbb{V}(\mathbb{L}_{\mathcal{F}}[1]) \to X$ makes F into a derived foliation on X whose cotangent complex is identified with $\mathbb{L}_{\mathcal{F}}$.

The above construction is clearly functorial in $\mathcal{F}$ and X, and thus defines for any derived Artin stack X a well defined ∞-functor

$$(-)^{DR} : \mathcal{F}ol_{\pi}(X/k) \to \mathcal{F}ol(X/k),$$

sending an infinitesimal derived foliation $\mathcal{F}$ to its de Rhamification $\mathcal{F}^{DR}$.

Definition 7.6.4.2. *The above ∞-functor $(-)^{DR}$ is called the* de Rhamification *construction. For an infinitesimal derived foliation $\mathcal{F} \in \mathcal{F}ol_{\pi}(X/k)$, the derived foliation $\mathcal{F}^{DR}$ is called the* underlying derived foliation *of $\mathcal{F}$.*

We note in particular that the cotangent complexes are preserved by de Rhamification

$$\mathbb{L}_{\mathcal{F}} \simeq \mathbb{L}_{\mathcal{F}^{DR}}.$$

The behaviour of the de Rhamification construction is rather subtle. For instance, already the question of which derived foliations arise as the de Rhamification of infinitesimal derived foliations is a non-trivial question. Derived foliations are not formally integrable, so our formal integrability result Corollary 7.6.3.2 provides a first obstruction for a derived foliation to come from an infinitesimal derived foliation. This can be seen already at the level of classical smooth foliations, as formally integrable foliations on a smooth varieties in characteristic $p > 0$ must always be stable by the p-th power operation on the tangent sheaf (see [**Eke87**]). The precise relation between infinitesimal derived foliations and an hypothetical notion p-restricted derived foliations is also not totally obvious. By the results of [**Fu12**] the question translate, for derived foliations over a perfect field, to the comparison between p-restricted dg-Lie algebras and partition Lie algebras of [**BCN21**]. As far as we know this comparison has not been clarified yet.

Nevertheless the existence of $\mathcal{F} \mapsto \mathcal{F}^{DR}$ is already helpful in the following sense. Starting with an infinitesimal derived foliation $\mathcal{F}$ we can exploit its formal integrability, and use $\mathcal{F}^{DR}$ when we want to study its de Rham cohomology, which is much better behaved than infinitesimal cohomology. For instance, we believe that the ∞-category of crystals on $\mathcal{F}^{DR}$ is more interesting for the study of cohomological properties than the ∞-category of crystals on $\mathcal{F}$ (defined naturally using $\mathcal{H}_{\pi}$-equivairant quasi-coherent complexes on $\mathbb{V}(\mathcal{F})$).

APPENDIX A

A.1. Model categories and ∞-categories

A.1.1. Model categories. The notion of model category, due to D. Quillen [**Qui67**], is a very convenient axiomatization of a homotopy theoretical context. Modern standard references include [**Hov99**, **Hir03**].

Definition A.1.1.1. *Let $\mathcal{C}$ be a category. A* model structure *on $\mathcal{C}$ consists of the choice of three classes $\mathsf{W}(\mathcal{C})$ (weak equivalences), $\mathsf{Fib}(\mathcal{C})$ (fibrations), and $\mathsf{Cofib}(\mathcal{C})$ (cofibrations) of morphisms in $\mathcal{C}$, subject to the following properties.*

1. *If (f, g) is a pair of composeable morphisms in $\mathcal{C}$ (i.e. $f \circ g$ exists in $\mathcal{C}$), if two among the morphisms $f, g, f \circ g$ is in $\mathsf{W}(\mathcal{C})$, then the same is true for the remaining one.*
2. *Each class $\mathsf{W}(\mathcal{C})$, $\mathsf{Fib}(\mathcal{C})$, and $\mathsf{Cofib}(\mathcal{C})$ is stable under composition and retracts*[1] *in $\mathcal{C}$.*
3. *Let*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "p"] \\ Z \arrow[r, "g"'] & T \end{tikzcd}$$

be a commutative diagram in $\mathcal{C}$, with $i \in \mathsf{Cofib}(\mathcal{C})$ and $p \in \mathsf{Fib}(\mathcal{C})$. If either i or p also belongs to $\mathsf{W}(\mathcal{C})$, then there exists a morphism $h : Z \to Y$ such that $p \circ h = g$ and $h \circ i = f$.

4. *Any morphism $f : X \to Y$ in $\mathcal{C}$ can be factored as $f = p \circ i$ and as $f = q \circ j$ where $i \in \mathsf{Cofib}(\mathcal{C})$, $p \in \mathsf{Fib}(\mathcal{C}) \cap \mathsf{W}(\mathcal{C})$, $j \in \mathsf{Cofib}(\mathcal{C}) \cap \mathsf{W}(\mathcal{C})$, $q \in \mathsf{Fib}(\mathcal{C})$. Moreover, these factorizations are both required to be functorial in f.*

A model category *consists of a complete and cocomplete category $\mathcal{C}$ endowed with a model structure.*

Model categories' vocabulary. Let $(\mathcal{C}, \mathsf{W}(\mathcal{C}), \mathsf{Fib}(\mathcal{C}), \mathsf{Cofib}(\mathcal{C}))$ be a model category. Morphisms in $\mathsf{Fib}(\mathcal{C}) \cap \mathsf{W}(\mathcal{C})$ are called *trivial fibrations*, while morphisms in $\mathsf{Cofib}(\mathcal{C}) \cap \mathsf{W}(\mathcal{C})$ are called *trivial cofibrations*. An object P (respectively, Q) in $\mathcal{C}$ is said to be *fibrant* (respectively, *cofibrant*) if the canonical map from P to the final object $*_{\mathcal{C}}$ (resp., from the intial object $\emptyset_{\mathcal{C}}$

[1] See [**Lur09**, Def. A.2.1.1 (3)] for a precise statement of stability under retracts

to Q) is a fibration (resp., a cofibration). By (4), any object $x \in \mathcal{C}$ admits a diagram, which is moreover *functorial* in x,

$$Qx \xrightarrow{p} x \xrightarrow{i} Rx$$

where Qx is cofbrant, p is a trivial fibration (by applying (4) to $\emptyset_{\mathcal{C}} \to x$), Rx is fibrant and i is a trivial cofibration (by applying (4) to $x \to *_{\mathcal{C}}$). The *homotopy category* $\mathsf{ho}(\mathcal{C})$ of the model category $\mathcal{C}$ is the Gabriel-Zisman localization $\mathcal{C}[\mathsf{W}(\mathcal{C})^{-1}]$ of $\mathcal{C}$ at weak equivalences. It admits also an equivalent description as the category whose objects are those objects in $\mathcal{C}$ that are both fibrant and cofibrant, with morphisms sets given by homotopy classes of morphisms between such objects ([**Hov99**, §1.2]).

If $(\mathcal{C}, \mathsf{W}(\mathcal{C}), \mathsf{Fib}(\mathcal{C}), \mathsf{Cofib}(\mathcal{C}))$ and $(\mathcal{D}, \mathsf{W}(\mathcal{D}), \mathsf{Fib}(\mathcal{D}), \mathsf{Cofib}(\mathcal{D}))$ are model categories, an adjoint pair (F, G), with left adjoint $F : \mathcal{C} \to \mathcal{D}$ is called a *Quillen adjunction* or *Quillen pair* (and F, respectively G, is called a *left*, resp. *right*, *Quillen functor*) if F preserves cofibrations and trivial cofibrations or if, equivalently, G preserves fibrations and trivial fibrations. A Quillen adjunction induces an adjunction at the level of the corresponding homotopy categories ([**Hov99**, §1.3]), and it is said to be a Quillen equivalence if the induced adjoints are (mutually inverse) equivalences of categories.

A.1.2. ∞-categories. There are several, a priori different, definitions of ∞-categories: ∞-categories as simplicial (i.e. categories enriched over the category of simplicial sets) categories ([**Lur09**, Definition 1.1.4.1]), ∞-categories as quasi-categories ([**Lur09**, Definition 1.1.2.4]), ∞-categories as Segal categories ([**Sim11**]), ∞-categories as complete Segal spaces ([**Rez01**, **JT07**]), ∞-categories as Segal spaces ([**MN24**]), ∞-categories as topological (i.e. categories enriched in the category of compactly generated weakly Hausdorff topological spaces) ([**Lur09**, Definition 1.1.1.6]). The common idea behind these definitions is that a ∞-category should be a category (maybe weakly) enriched in topological spaces, or in any equivalent homotopical model for topological spaces (e.g. simplicial sets). For a precise comparison between these different models, the interested reader may consult [**Lur09**], [**Ber10**], and [**Ber18**].

For example, any usual 1-category gives rise to an ∞-category, to which it is usually identified. If for ∞-categories we take the model of quasi-categories, then the ∞-category associated to a 1-category $\mathcal{C}$ is its *nerve* $\mathsf{N}(\mathcal{C})$, i.e. the simplicial set such that

$$\mathsf{N}(\mathcal{C})_n := \mathsf{Fun}([n], \mathcal{C}).$$

All the standard concepts in usual (1-)category theory have appropriate extensions to ∞-categories. For example, we have a notion of ∞-*functor* $\mathbf{F} : \mathbf{C} \to \mathbf{D}$: in the model of ∞-categories given by quasi-categories (respectively, simplicial categories), an ∞-functor is just a morphism between the corresponding simplicial sets (respectively, a simplicially enriched functor between the corresponding simplicially enriched categories). Given two ∞-categories $\mathbf{C}$ and $\mathbf{D}$, there is, actually, an ∞-category $\mathbf{Fun}_{\infty}(\mathbf{C}, \mathbf{D})$ of ∞-functors from $\mathbf{C}$ to $\mathbf{D}$ (see [**Lur09**,

1.2.7.2, 1.2.7.3] for quasi-categories). There is a notion of *adjoint ∞-functors* between ∞-categories (see [**Lur09**, Definition 5.2.2.1] for quasi-categories), notions of *limit* and *colimit* for diagrams in an ∞-category (see [**Lur09**, §1.2.13] for these concepts inside a quasi-category), Kan extensions (see [**Lur09**, §4.3] for quasi-categories), etc. For any pair (x, y) of objects inside an ∞-category $\mathbf{C}$, there is a *mapping space* $Map_{\mathbf{C}}(x, y)$ between them, that can be identified with a simplicial set, and is defined up to homotopy (i.e. it is well defined in $ho(\mathsf{sSets})$). The *homotopy category* of an ∞-category $\mathbf{C}$, is the 1-category $\mathsf{ho}(\mathbf{C})$ with the same objects as $\mathbf{C}$ and morphism from x to y given by $\pi_0(Map_{\mathbf{C}}(x, y))$. There is an obvious ∞-functor $\mathbf{C} \to \mathsf{ho}(\mathbf{C})$, and a morphism in $\mathbf{C}$ will be called an equivalence if it becomes an isomorphism in $\mathsf{ho}(\mathbf{C})$.
Moreover, (small) ∞-categories and ∞-functors between them, organize themselves[2] again into a ∞-category $\mathbf{Cat}_\infty$ (see [**Lur09**, Chapter 3] for quasi-categories). More is true: $\mathbf{Cat}_\infty$ is moreover a $(\infty, 2)$-category, i.e. an ∞-categorical version of a bicategory, but we will mainly view it simply as an ∞-category. The nerve construction from 1-categories to ∞-categories, recalled above, provides a fully faithful functor.

For an ∞-category $\mathbf{C}$ and a set $\mathbf{W}$ of morphisms, one can construct another ∞-category denoted by either $L_{\mathbf{W}}(\mathbf{C})$ or $\mathbf{C}[\mathbf{W}^{-1}]$ where, informally speaking, the morphisms in $\mathbf{W}$ has been inverted. The construction also provides a canonical ∞-functor $L : \mathbf{C} \to L_{\mathbf{W}}(\mathbf{C})$ such that, for any ∞-category $\mathbf{D}$, the induced ∞-functor

$$\mathbf{Fun}_\infty(L_{\mathbf{W}}(\mathbf{C}), \mathbf{D}) \longrightarrow \mathbf{Fun}_\infty(\mathbf{C}, \mathbf{D})$$

is fully faithful with essential image consisting of those ∞-functors $\mathbf{C} \to \mathbf{D}$ sending morphisms in $\mathbf{W}$ into equivalences in $\mathbf{D}$. The ∞-category $L_{\mathbf{W}}(\mathbf{C})$ is called the *localization of* $\mathbf{C}$ *at (or along)* $\mathbf{W}$.

Definition A.1.2.1. *Given a model category* $(C, \mathsf{W}(\mathcal{C}), \mathsf{Fib}(\mathcal{C}), \mathsf{Cofib}(\mathcal{C}))$*, its* underlying or associated ∞-category *will be the localization* $L_{\mathsf{W}(\mathcal{C})}(\mathsf{N}(\mathcal{C}^{\mathrm{cof}}))$*, where* $\mathcal{C}^{\mathrm{cof}}$ *denotes the full subcategory of* $\mathcal{C}$ *consisting of cofibrant objects.*

Remark A.1.2.2. In the main text, if $\mathcal{C}$ is a model category, we have simply denoted by $\mathbf{C}$ its underlying ∞-category.

A Quillen adjunction between model categories (A.1.1) induces an adjunction between ∞-functors on the underlying ∞-categories ([**Lur09**, §5.2.4]), which is an equivalence of ∞-categories if the Quillen pair is a Quillen equivalence.

A.2. Derived schemes and stacks: basic facts and notations

In this § we will mainly fix the notations used in the main text, we do *not* mean to give a short introduction to derived algebraic geometry, for which we refer to [**Toë09**,**Toe14**,**PV21**].

[2]With appropriate set-theoretic or universes's caveats.

Let k be a base discrete commutative ring (e.g. $k = \mathbb{Z}$), we denote by sCAlg_k the model category[3] of simplicial commutative k-algebras (where both weak equivalences and fibrations are detected by the forgetful functor $\mathsf{sCAlg}_k \to \mathsf{sSets}$), and by $\mathbf{sCAlg}_k$ the underlying ∞-category (see A.1.1). Note that, for $A \in \mathsf{sCAlg}_k$, $\pi_*(A) = \oplus_{i\geq 0}\pi_i(A)$ has a natural induced structure of a commutative graded k-algebra. In particular, $\pi_0(A)$ has an induced structure of usual commutative k-algebra, and each $\pi_i(A)$ is a $\pi_0(A)$-module. We say that A is *Noetherian* if $\pi_0(A)$ is a Noetherian ring, and each $\pi_i(A)$ is a finitely generated (discrete) $\pi_0(A)$-module. We will say that A is *discrete* if $\pi_i(A) = 0$ for any $i > 0$ (i.e. A is weakly equivalent to the constant simplicial k-algebra $\pi_0(A)$).

For a simplicial commutative k-algebra A, the associated *affine derived scheme* is the ∞-functor

$$\mathbf{Spec}\, A := \mathsf{Map}_{\mathbf{sCAlg}_k}(A, -) : \mathbf{sCAlg}_k \longrightarrow \mathbf{sSets} = \mathbf{Top}$$

We let $\mathbf{dAff}_k$, the ∞-category of *affine derived schemes* over k, be the full ∞-subcategory of $\mathbf{Fun}_\infty(\mathbf{sCAlg}_k, \mathbf{Top})$ essential image of the ∞-functor $\mathbf{Spec}$. The ∞-category $\mathbf{dAff}_k$ has arbitrary pullbacks: for $B \longleftarrow A \longrightarrow C$ a diagram in $\mathbf{sCAlg}_k$, the corresponding pullback in $\mathbf{dAff}_k$ is given by $\mathbf{Spec}(B \otimes^{\mathbb{L}}_A C)$, where $B \otimes^{\mathbb{L}}_A C$ is the derived tensor product of B and C in A/sCAlg_k, obtained by replacing either B or C by a cofibrant replacement in the undercategory A/sCAlg_k, before taking the usual (strict) tensor product.

The inclusion of usual discrete commutative k-algebras in sCAlg_k, as constant simplicial object, induces a fully faithful functor $j : \mathbf{Aff}_k \to \mathbf{dAff}_k$ having as right adjoint $\mathbf{t}_0 : \mathbf{Spec}\, A \mapsto \mathrm{Spec}(\pi_0(A))$, called the *truncation functor*. An important property of j is that it does *not* preserve general pullbacks: if $S \longleftarrow R \longrightarrow T$ is a diagram in CAlg_k, then $S \otimes^{\mathbb{L}}_R T \neq S \otimes_R T$ unless S or T are flat over R.

If, as often in the main text, the fixed base ring k is a commutative $\mathbb{Q}$-algebra, then we may replace sCAlg_k with the model category cdga_k of commutative differential non-positively graded (aka *connective*) k-algebras, since we have an equivalence of the underlying ∞-categories $\mathbf{sCAlg}_k \simeq \mathbf{cdga}_k$. If, under this equivalence, $A \in \mathbf{sCAlg}_k$ corresponds to $A' \in \mathbf{cdga}_k$, we have $\pi_i(A) \simeq H^{-i}(A')$ for any $i \geq 0$. For this reason, we will sometimes abuse notations and write $\pi_i(B)$ for $H^{-i}(B)$ even when $B \in \mathbf{cdga}_k$.

For $A \in \mathsf{sCAlg}_k$, its normalization $\mathsf{N}(A)$ is a commutative differential graded k-algebras, and we denote by Mod_A the model category of *not necessarily bounded* $\mathsf{N}(A)$-dg modules (see [**TV08**, Ch. 2.2] for more details). As usual, $\mathbf{Mod}_A$ will be the underlying ∞-category of Mod_A. The same notation is used for $A \in \mathsf{cdga}_k$. For $M \in \mathsf{Mod}_A$ and $i \in \mathbb{Z}$, we will write

[3]This is, moreover, a cofibrantly generated simplicial model structure; see [**Hir03**] for the meaning of these words.

$\pi_i(M)$ or $H^{-i}(M)$ interchangeably.

If $\mathbf{C}$ denotes either $\mathbf{sCAlg}_k$ or $\mathbf{cdga}_k$, a morphism $A \to B$ in $\mathbf{C}$ is called *locally of finite presentation* if the functor $\mathbf{Map}_{A/\mathbf{C}}(B, -)$ commutes with filtered colimits (i.e. B is a compact object inside $A/\mathbf{C}$). Since the forgetful ∞-functor $A/\mathbf{C} \to \mathbf{Mod}_A$ preserves and reflects filtered colimits, this condition is equivalent to $\mathbf{Map}_{\mathbf{Mod}_A}(B, -) : \mathbf{Mod}_A \to \mathbf{Top}$ preserving filtered colimits. A morphism $A \to B$ in $\mathbf{C}$ is called *locally almost of finite presentation* (sometimes abbreviated to *lafp*) $\mathbf{Map}_{\mathbf{Mod}_A}(B, -)$ preserves filtered colimits when restricted to each full ∞-subcategory $\mathbf{Mod}_A^{\leq n}$ of *truncated* A-modules (i.e. A-modules M such that $\pi_i(M) = 0$ for $i > n$. For example, if A is Noetherian, then $A \to B$ is lafp iff B is Noetherian, and $\pi_0(B)$ is a finitely presented $\pi_0(A)$-algebra in the usual sense of commutative algebra.

One can consider different ∞-topologies ([**Lur09**, 6.2.2], [**TV05**, 3.1]) on the ∞-category $\mathbf{dAff}_k$. For example, the *étale* ∞-topology is generated by the ∞-pre-topology defined as follows: for $X = \mathbf{Spec}\, A \in \mathbf{dAff}_k$ where $A \in \mathsf{sCAlg}_k$ or $A \in \mathsf{cdga}_k$, étale covering families of X are given by arbitrary families $\{A \to A_i\}_{i \in \mathrm{I}}$ such that the induced family $\pi_0(A) \to \pi_0(A_i)_{i \in \mathrm{I}}$ is a usual étale covering family of $\mathbf{t}_0(\mathbf{Spec}\, A) = \mathrm{Spec}(\pi_0(A))$, and moreover, the canonical maps

$$\pi_i(A) \otimes_{\pi_0(A)} \pi_0(A_i) \to \pi_i(A_i)$$

are isomorphisms[4] for all $i \geq 0$ (and we say, then, that each $A \to A_i$ is *strongly étale*). Once an ∞-topology τ on $\mathbf{dAff}_k$ is fixed, we define the ∞-category of *derived τ-stacks*

$$\mathbf{dSt}_k^\tau := \mathbf{Sh}((\mathbf{dAff}_k, \tau), \mathbf{Top})$$

the category of $\mathbf{Top}$-valued ∞-sheaves on the ∞-site $(\mathbf{dAff}_k, \tau)$ ([**Lur09**, Def. 6.2.2.6], [**TV05**, Def. 3.4.9]). Therefore, a derived τ-stack is an ∞-functor $F : \mathbf{dAff}_k^{\mathrm{op}} \to \mathbf{Top}$ (i.e. a ∞-preseheaf on $\mathbf{dAff}_k$) satisfying τ-hyperdescent, i.e. the canonical morphism $F(X) \to F(U_\bullet)$ is an equivalence in $\mathbf{Top}$, for any τ-hypercover $U_\bullet \to X$.

In order to select geometrically meaningful and interesting derived stacks, one imposes recursively defined *geometricity* conditions on atlases and representable morphisms (see [**TV08**, Ch. 1.3], [**Toe14**, 3.3] and [**PV21**, §4]). In particular, a derived *Deligne-Mumford* stack is a derived étale stack admitting an étale atlas (by derived affine schemes), while a derived *Artin* (or *algebraic*) stack is a derived étale stack admitting a smooth atlas.

Any connective k-linear cdga A possesses a cotangent complex $\mathbb{L}_{A/k}$ relative to k. By definition $\mathbb{L}_{A/k}$ is an A-dg-module such that derivations $A \to M$ are equivalent to A-dg-modules maps $\mathbb{L}_{A/k} \to M$ (see [**TV08**, §1.4.1]). More precisely, for $M \in \mathbf{dg}_A$ an A-dg-module we can form the trivial square zero extension $A \oplus M$, which is a new cdga augmented to A. We then

[4]Recall our convention $\pi_i(A) := \mathrm{H}^{-i}(A)$ for $A \in \mathsf{cdga}_k$, and $i \geq 0$.

have functorial equivalences of mapping spaces

$$\mathbf{Map}_{\mathbf{dg}_A}(\mathbb{L}_{A/k}, M) \simeq \mathbf{Map}_{\mathbf{cdga}_k/A}(A, A \oplus M),$$

and the right hand side is, by definition, the space of derivations on A with coefficients in M. In particular, the identity of $\mathbb{L}_{A/k}$ defines a universal derivation $dR : A \to \mathbb{L}_{A/k}$. The cotangent complex of non-affine derived schemes and more generally of derived Artin stacks is discussed below in §A.5.

A.3. Formal completion

Let A be a connective cdga over a field k of characteristic zero endowed with au augmentation $x : A \to k$. We can define a pro-Artinian cdga $\underline{\hat{A}}_x$ (or simply $\underline{\hat{A}}_x$), called the *formal completion of A at x*, as follows. We consider $A/\mathbf{dgart}_k^*$ the full sub-∞-category of the ∞-category of augmented connective cdga under A consisting of all morphisms $A \to A'$ where A' is a local Artinian cdga. Recall that Artinian here means that $H^0(A')$ is a local Artinian k-algebra and moreover that $H^*(A')$ is finite dimensional (so all $H^i(A')$ are finite dimensional and vanish for $i << 0$). We note that $A/\mathbf{dgart}_k^*$ is a filtered ∞-category as it admits finite limits. The following definition therefore makes sense.

Definition A.3.0.1. *With the above notations,* the formal completion of A at x *is the pro-object in* $\mathbf{dgart}_k^*$ *defined by*

$$\underline{\hat{A}}_x := \text{" lim"}_{A' \in (A/\mathbf{dgart}_k^*)} A'.$$

The pro-object $\underline{\hat{A}}$ has a realization $\hat{A} = \lim_{A' \in (A/\mathbf{dgart}_k^*)} A'$ which is a cdga. We will see below that $\hat{A}$ is connective when A is almost of finite presentation. This does not follow formally from the definition (as connective cdga's are not stable by limits inside the bigger ∞-category of all cdga's). This will be a consequence of a more explicit description of the pro-object $\underline{\hat{A}}$.

Assume now that A is a connective cdga wich is moreover almost of finite presentation. We consider the ideal $I \subset H^0(A)$ of the induced augmentation $H^0(A) \to k$, and pick $(f_1, \ldots, f_p)$ a set of generators for I. For each integer $n > 0$ we define an object $A \to A_n$ of $A/\mathbf{dgart}_k^*$ as follows. Start with the Koszul cdga $K(A, f_*^n)$, which is the free A-linear cdga generated by elements $h_1, \ldots h_p$ in degree -1 with $dh_i = f_i^n$. We let $A_n := \tau_{\leq n} K(A, f_*^n)$ the n-truncation of $K(A, f_*^n)$. For $m > n$, we have a canonical morphism $p_{m,n} : A_m \to A_n$ which sends h_i to $f_i^{m-n}.h_i$ making the A_n into a projective system in $A/\mathbf{cdga}$. Moreover, $H^0(A_n) \simeq H^0(A)/(f_1^n, \ldots, f_p^n)$ is a local Artinian algebra, and each $H^i(A_n)$ is of finite type over $H^0(A_n)$. Indeed, we can see this by induction on the number of elements f_i, using that $K(A, f_*) \simeq K(K(A, f_{*<k}), f_k)$, and the long exact sequence

$$H^{i-1}(A) \longrightarrow H^i(A) \xrightarrow{f} H^i(A) \longrightarrow H^i(K(A, f)) \longrightarrow H^{i+1}(A) \xrightarrow{f} \ldots$$

for the Koszul algebra $K(A, f)$ of a single element $f \in H^0(A)$. Finally, A_n are truncated by construction. The family of objects A_n thus form a pro-object $"\lim_n" A_n$ in $A/\mathbf{dgart}_k^*$. There is an obvious morphism of pro-object $\underline{\hat{A}} \to "\lim_n" A_n$, as the diagram given by the canonical morphisms $A \to A_n$.

Propositon A.3.0.2. *With the notations above, the natural morphism $\underline{\hat{A}} \to "\lim_n" A_n$ is an equivalence of pro-objects in $A/\mathbf{dgart}_k^*$.*

Proof. Let $\mathcal{N}$ be the category of natural numbers with its natural order. The family of objects A_n defines an ∞-functor $\phi : \mathcal{N}^{op} \to A/\mathbf{dgart}_k^*$. The Proposition will follow from the fact that ϕ is cofinal. As a first observation, for any object $A \to A'$ in $A/\mathbf{dgart}_k^*$, the image of all f_i in $H^0(A')$ are nilpotent. Therefore, we can chose n such that for each i, the image of f_i^n is homotopic to zero in A'. By the universal property of the A-cdga $A_n = K(A, f_*^n)$, there exists a factorization $A \to A_n \to A''$.

Let $A' \in A/\mathbf{dgart}_k^*$, and let ϕ/A' be the slice ∞-category of ϕ. It is enough to show that ϕ/A' is cofiltered. For this, let $B_* : J \to \phi/A'$ be an ∞-functor $j \mapsto B_j$ with J a finite ∞-category. We can for the limit along J and get this way an dg-Artinian local algebra $A_J = \lim_{j \in J} B_j$ and a factorization $A \to A_J \to A'$. By our previous observation, there exists a factorization $A \to A_p \to A_J$.

For any $j \in J$, B_j is of the form A_{n_j} for some integer n_j. The morphism $A_p \to A_{n_j}$ is possibly not of the standard form, but it can be made so by increasing p thanks to the following lemma.

Lemma A.3.0.3. *For any integer $p \geq n$, and any morphism of A-cdga's $u : A_p \to A_n$, there exists $q \geq p$ such that the composition*

$$A_q \xrightarrow{p_{q,p}} A_p \xrightarrow{u} A_n$$

is homotopic to the standard morphism $p_{q,n}$.

Proof of the lemma. This follows easily from the universal property of Koszul algebras. Indeed, for any A-cdga B, the space of A-cdga morphisms $Map_{A-\mathbf{cdga}}(A_p, B)$ is $\Omega_{0,f^p}B$, the space of homotopies between 0 and the image of f^p in B. It is therefore either empty or non-canonically equivalent to the loop space $\Omega_0 B$, and a choice of a homotopy $0 \sim f^p$ induces an equivalence $\Omega_{0,f^p}B \simeq \Omega_0 B$. As a result, for any $p \geq n$, the set of homotopy classes of A-cdga morphisms $\pi_0(Map_{A-\mathbf{cdga}}(A_p, A_n))$ is isomorphic to $H^{-1}(A_n)$, and moreover this isomorphism is canonical as the universal homotopy $0 \sim f^n$ in A_n induces a homotopy $0 \sim f^p$ by multiplication by f^{p-n}.

Under the bijection $\pi_0(Map_{A-\mathbf{cdga}}(A_p, A_n)) \simeq H^{-1}(A_n)$, the canonical morphism $p_{p,n}$ corresponds to 0 in $H^{-1}(A_n)$. Therefore, the lemma simply follows by observing that for $q >> p$ (and precisely $q \geq 2p$), the morphism $p_{q,p}$ induces the zero map $H^{-1}(A_q) \to H^{-1}(A_p)$. □

Coming back to the proof of the Proposition. The lemma tells us that if p is chosen big enough, then all the morphisms $A_p \to B_j$ are the standard morphisms. This means that the

A_p defines indeed an object in ϕ/A' that admits a morphism to the diagram B_*. □

An important consequence of Proposition A.3.0.2 is the description of the pro-$\underline{\hat{A}}$-modules $\wedge^p\mathbb{L}_{\underline{\hat{A}}} := \text{"}\lim_{A\to A'}\text{"} \wedge^p \mathbb{L}_{A'}$ of p-forms.

COROLLARY A.3.0.4. *The natural morphism*

$$\text{"}\lim_{A\to A'}\text{"}\mathbb{L}_A \otimes_A A' \longrightarrow \mathbb{L}_{\underline{\hat{A}}}$$

is an equivalence of pro-$\underline{\hat{A}}$-modules.

Proof of Corollary A.3.0.4. For all $A \to A'$ we have an exact triangle of pro-$\underline{\hat{A}}$-modules

$$\mathbb{L}_A \otimes_A A' \to \mathbb{L}_{A'} \to \mathbb{L}_{A'/A}.$$

Therefore, in order to prove the statement we need to prove that the pro-object $\text{"}\lim_{A\to A'}\text{"}\mathbb{L}_{A'/A}$ is zero. To see this, it is enough by Proposition A.3.0.2 to see that $\text{"}\lim_n\text{"}\mathbb{L}_{A_n/A}$ is the zero pro-$\text{"}\lim_n A_n\text{"}$-module. For each n, $\mathbb{L}_{A_n/A} \simeq A_n[1]^p$, a basis being given by the de Rham differential $dR(h_i)$ (relative to A) of h_i, of the canonical elements h_i of degree -1 in A_n. Moreover, for $p \geq 2n$, the canonical morphism $A_p \to A_n$ sends $dR(h_i)$ to $f^{p-n}.dR(h_i)$, and thus is homotopic to the zero map because f^{p-n} is homotopic to zero for $p \geq 2n$. This implies that the pro-$\text{"}\lim_n A_n\text{"}$-module $\text{"}\lim_{A\to A'}\text{"}\mathbb{L}_{A'/A}$ is zero as wanted. □

By taking p-th wedge powers we find the following consequence.

COROLLARY A.3.0.5. *For any $p \geq 0$, the natural morphism*

$$\text{"}\lim_{A\to A'}\text{"}(\wedge^p_A\mathbb{L}_A) \otimes_A A' \longrightarrow \wedge^p_{\underline{\hat{A}}}\mathbb{L}_{\underline{\hat{A}}} \simeq \text{"}\lim_{A\to A'}\text{"} \wedge^p_{A'} \mathbb{L}_{A'}$$

is an equivalence of pro-$\underline{\hat{A}}$-modules.

Another important consequence is the following flatness result, stating the cohomology groups of the formal completion are the formal completion of the cohomology groups.

COROLLARY A.3.0.6. *Let $x : A \to k$ be an augmented connective cdga wich is almost of finite presentation and $\hat{A} = \lim_{A\to A'} A'$ be the realization of its formal completion at x. Then, we have*

$$H^0(\hat{A}) \simeq \widehat{H^0(A)} \qquad H^i(\hat{A}) \simeq \widehat{H^i(A)} \simeq H^i(A) \otimes_{H^0(A)} \widehat{H^0(A)}.$$

In particular, the morphism $A \to \hat{A}$ is a flat morphism of connected cdga's, which is moreover faithfully flat locally at x.

Proof of Corollary A.3.0.6. As A is almost of finite presentation over k, we can find a strict quasi-free model as a cdga of the form

$$\ldots \longrightarrow A^{-i} \longrightarrow A^{-i+1} \longrightarrow \ldots \longrightarrow A^0$$

where $A^0 \simeq k[x_1, \ldots, x_d]$ as a k-algebra, and each A^i is a free A^0-module of finite rank. We can moreover assume that the augmentation $A \to k$ precomposed with $A^0 \to A$ is given by the augmentation ideal $I = (x_1, \ldots, x_q)$. Therefore, the f_i are here represented by the x_i.

By definition of the Koszul algebra $A_n = K(A, f_*^n)$ is thus given explicitly by

$$K(A, f_*^n) \simeq A \otimes_{A^0} K(A^0, x_*^n).$$

However, the x_i^n form a regular sequence and thus $K(A^0, x_*^n)$ is naturally quasi-isomorphic to the k-algebra $k[x_1, \ldots, x_q]/(x_1^n, \ldots, x_q^n)$. As A is cofibrant as an A^0-module, we thus have

$$K(A, f_*^n) := A \otimes_{A^0} K(A^0, x_*^n) \simeq A \otimes_{A^0} (A^0/(x_1^n, \ldots, x_q^n)).$$

Therefore, by the Proposition A.3.0.2, the realization $\hat{A}$ is simply given by

$$\hat{A} \simeq A \otimes_{A^0} \hat{A}^0,$$

where $\hat{A}^0$ is the formal completion of A^0 at the ideal $(x_1, \ldots, x_0)$. As formal completions of noetherian rings are flat, we deduce the Corollary. □

A.4. $\mathcal{O}_X$-modules, quasi-coherent and perfect modules

The ∞-topos $\mathbf{dSt}_k$, of derived stacks over k, admits a tautological sheaf of cdga's, namely the left Kan extension of the tautological ∞-functor sending $\mathbf{Spec}\, A$ to A. It is denoted by $\mathcal{O}_k$, and its restriction to $\mathbf{dSt}_k/X$ will be denoted by $\mathcal{O}_X$. Associated to this sheaf of cdga's we have ∞-categories of sheaves of modules, simply denoted by $\mathcal{O}_k - \mathbf{dg}$ and $\mathcal{O}_X - \mathbf{dg}$. In concrete terms, for $X \in \mathbf{dSt}_k$, and object E in $\mathcal{O}_X - \mathbf{dg}$ consists of the following data:

- for all $u : \mathbf{Spec}\, A \to X$ an A-dg-module $E(u)$,
- for all $\mathbf{Spec}\, A \xrightarrow{f} \mathbf{Spec}\, B \xrightarrow{u} X$ a morphism of B-dg-modules $\phi_f : E(u \circ f) \to E(u)$,

together with the usual coherence conditions. In different terms, the ∞-category $\mathcal{O}_X - \mathbf{dg}$ can also be described as a lax limit

$$\mathcal{O}_X - \mathbf{dg} \simeq lim^{lax}_{\mathbf{Spec}\, A \to X} A - \mathbf{dg}.$$

An object $E \in \mathcal{O}_X - \mathbf{dg}$ will be called *quasi-coherent* if all the transition morphisms $\phi_f : E(u \circ f) \to E(u)$ induces equivalences $A \otimes_B E(u \circ f) \simeq E(u)$. The full sub-$\infty$-category of quasi-coherent $\mathcal{O}_X$-dg-modules (also called quasi-coherent complexes) will be denoted by

$$j : \mathbf{QCoh}(X) \subset \mathcal{O}_X - \mathbf{dg}.$$

This is a full embedding of presentable ∞-catgeories, and obviously the inclusion ∞-functor commutes with arbitrary colimits. Therefore, the inclusion j admits a right adjoint

$$(-)^{qcoh} : \mathcal{O}_X - \mathbf{dg} \longrightarrow \mathbf{QCoh}(X),$$

called the *quasi-coherator* construction.

We remind also two more full sub-∞-categories $\mathbf{Perf}(X) \subset \mathbf{APerf}(X) \subset \mathbf{QCoh}(X)$. As a start let A be a connective cdga over k. We recall that an A-dg-module E is perfect if it is a compact object of $A - \mathbf{dg}$. It is equivalent to as for E being dualizable with respect to the natural symmetric monoidal structure on $A - \mathbf{dg}$, and also equivalent to the fact that E is a retract of a finite cell A-dg-module (see [**TV07**, Prop. 2.2]). Similarly, an A-dg-module E is *almost perfect* if it can be approximated by perfect objects: for all n, there exists a perfect A-dg-module E_n and a morphism $E_n \to E$ such that the induced morphism $H^i(E_n) \to H^i(E)$ is an isomorphism for all $i > n$. As A is connective, perfect A-dg-modules are bounded above, and thus so are almost perfect A-dg-modules. Moreover, if A is almost of finite presentation (i.e. $H^0(A)$ is of finite type over k and $H^i(A)$ is a finite type $H^0(A)$-module for all i) then $E \in A - \mathbf{dg}$ is almost perfect if and only if it satisfies the following two conditions:

- E is bounded above,
- $H^i(E)$ is a $H^0(A)$-module of finite type for all i.

To finish this section, remind that $\mathcal{O}_X - \mathbf{dg}$ comes equipped with a canonical symmetric monoidal structure $\otimes_{\mathcal{O}_X}$, determined for any $u : \mathbf{Spec}\, A \to X$ by the rule $(E \otimes_{\mathcal{O}_X} F)(u) := E(u) \otimes_A F(u)$ plus sheafification. This symmetric monoidal structure preserves the full sub-∞-categories $\mathbf{Perf}(X) \subset \mathbf{APerf}(X) \subset \mathbf{QCoh}(X) \subset \mathcal{O}_X - \mathbf{dg}$, and thus induces symmetric monoidal structures on all of them. Note also that $\mathbf{Perf}(X)$ consists of all dualizable objects in $\mathbf{QCoh}(X)$, and thus any perfect complex is also dualizable as an $\mathcal{O}_X$-dg-module.

We finish with an important statements concerning quasi-coherent sheaves on formal derived completions of derived affine schemes. Let $X = \mathbf{Spec}\, A$ be a derived affine scheme almost of finite presentation over a field k (so in particular $H^0(A)$ is noetherain and each $Hi(A)$ is a $H^0(A)$-module of finite type). Let $x \in X(k)$ be a global point and $\underline{\hat{A}}_x$ be the formal completion of A at the point x. It is a pro-object in $\mathbf{cdga}_k$ whose limit is denoted by $\hat{A}_x$.

Propositon A.4.0.1. *The natural ∞-functor*

$$\phi : \mathbf{APerf}(\hat{A}_x) \to \mathbf{Perf}(\underline{\hat{A}}_x) = \lim_{A \to A'} \mathbf{APerf}(A'),$$

sending E to the pro-module "$\lim_{A\to A'}$"$(A' \otimes_{\hat{A}_x} E)$, is an equivalence of ∞-categories.

Proof. From proposition A.3.0.2 $\underline{\hat{A}}_x$ can be represented as pro-object of the form

$$\hat{A}_x \longrightarrow \ldots \longrightarrow A_n \longrightarrow A_{n-1} \longrightarrow \ldots \longrightarrow A_0 \longrightarrow k$$

where each A_i is a connective local artinian cdga over k. Moreover each morphism $H^0(A_n) \to H^0(A_{n-1})$ is a surjective map of artinian local k-algebras. The proposition then becomes of statement about the ∞-functor

$$\phi : \mathbf{APerf}(\hat{A}_x) \to \lim_n \mathbf{APerf}(A_n).$$

The ∞-functor ϕ is the restriction of a left adjoint defined on the level of quasi-coherent complexes

$$\phi : \mathbf{QCoh}(\hat{A}_x) \leftrightarrows \lim_n \mathbf{QCoh}(A_n) : \psi.$$

The left adjoint has already been described, it sends E to " $\lim_n$ " $A_n \otimes_{\hat{A}_x} E$. The right adjoint sends a pro-object " $\lim_n E_n$" to its realization $\lim_n E_n \in \mathbf{QCoh}(\lim_n A_n)$.

A first observation is that the unit of the adjunction $\hat{A}_x \to \psi\phi(\hat{A}_x)$ is obviously an equivalence, and thus this is also the case for any $E \in \mathbf{Perf}(\hat{A}_x)$ (as $\hat{A}_x$ generates $\mathbf{Perf}(\hat{A}_x)$ by finite colimits, shifts and retracts).

Lemma A.4.0.2. *Suppose that $E \in \mathbf{QCoh}(\hat{A}_x)$ is k-connective ($H^i(E) \simeq 0$ for $i > -k$) and $H^k(E)$ is of finite type as a $H^0(\hat{A}_x)$-module. Then, $\psi\phi(E)$ remains k-connective and the adjunction morphism $E \to \psi\phi(E)$ induces an isomorphism*

$$H^k(E) \simeq H^k(\psi\phi(E)).$$

Proof of the lemma. We use the Milnor short exact sequences

$$0 \longrightarrow \lim^1_n H^{j-1}(E_n) \to \longrightarrow H^j(\lim_n E_n) \longrightarrow \lim^0_n H^j(E_n) \longrightarrow 0$$

for any $j \in \mathbb{Z}$ and any projective system of complexes

$$\dots \longrightarrow E_n \longrightarrow E_{n-1} \longrightarrow \dots \longrightarrow E_1 \longrightarrow E_0.$$

Here the $\lim^i_n$ denote the i-th right derived functor associated to the $\lim_n$ functor, and $\lim_n E_n$ is understood in the ∞-category $\mathbf{dg}_k$ of complexes of k-modules. To see that $\psi\phi(E)$ remains k-connective we thus have to show that $\lim^1_n H^k(E_n) = 0$. But this follows from the fact that the system $n \mapsto H^k(E_n)$ is of the form $n \mapsto H^0(A_n) \otimes_{H^0(\hat{A}_x))} H^k(E)$ and thus all maps $H^k(E_n) \to H^k(E_{n-1})$ are surjective. Once $\psi\phi(E)$ is known to be k-connective, the unit of the adjunction, on the k-th cohomology group can be written as

$$H^k(E) \to \lim_n H^0(A_n) \otimes_{H^0(\hat{A}_x))} H^k(E).$$

This is an isomorphism because $H^0(\hat{A}_x)) \simeq \lim_n H^0(A_n)$ and $H^k(E)$ is a module of finite type. $\square$

Let $E \in \mathbf{APerf}(\hat{A}_x)$. By definition, for any j, we can find $E_0 \in \mathbf{Perf}(\hat{A}_x)$ and a morphism $E_0 \to E$ whose cone E' is j-connective. We have a commutative diagram with exact rows

$$\begin{tikzcd} E_0 \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & E' \arrow[d] \\ \psi\phi(E_0) \arrow[r] & \psi\phi(E) \arrow[r] & \psi\phi(E'). \end{tikzcd}$$

The left vertical morphism is an equivalence because E_0 is perfect. Moreover, by the lemma A.4.0.2 both E' and $\phi(E')$ are j-connective. In particular, considering the long exact sequences on cohomology groups we see that the vertical morphism of the middle must induce isomorphisms H^k for all $k > -j$. As j is arbitrary this shows the unit of the adjunction $E\psi\phi(E)$ is an equivalence, and thus that the ∞-functor ϕ of the proposition is fully faithful.

To finish the proof we have to show that the right adjoint $\psi : \lim_n \mathbf{QCoh}(A_n) \leftrightarrows \mathbf{QCoh}(\hat{A}_x)$ sends objects in $\lim_n \mathbf{APerf}(A_n)$ to almost perfect complexes and that the induced ∞-functor

$$\psi : \lim_n \mathbf{APerf}(A_n) \to \mathbf{APerf}(\hat{A}_x)$$

is furthermore conservative. As we have seen during the proof of the lemma A.4.0.2, the Milnor short exact sequences implies that ψ preserves the j-connective objects. Any object $E = \{E_n\} \in \lim_n \mathbf{APerf}(A_n)$ is j-connective for some j (uniform in n). Indeed, if $E_n \in \mathbf{APerf}(A_n)$, then it is l-connective for some integer l. Moreover, A_n is an artinian local connective cdga, and thus the Yoneda lemma implies that if $E_n \otimes_{A_n} k$ is j-connective for some j then E_n was already j-connective as well. In particular, if E_0 is j-connective then all E_n are j-connective as well (for the same j), as

$$E_n \otimes_{A_n} k \simeq (E_n \otimes_{A_n} A_0) \otimes_{A_0} k \simeq E_0 \otimes_{A_0} k.$$

Therefore, $\psi(E)$ is also j-connective.

We now proceed as before by approximation by perfect complexes as before. For any $E = \{E_n\} \in \mathbf{APerf}(\hat{A}_x)$, which is j-connective for some j, we can find a free A_0-module $D_0 = A_0^r$ and a morphism

$$u_0 : D_0[j] \to E_0$$

whose cones E'_0 is $(j+1)$-connective. As D_0 is free we can lift u_0 to a morphism $u : D[j] \to E$ whose cone E' is $(j+1)$-connective, and D is here of the form $\phi(\hat{A}_x^r)$ (i.e. free of rank r). Continuing the resolution on E', we construct inductively an object $F_r \in \mathbf{Perf}(\hat{A}_x)$ and a morphism $\phi(F_r) \to E$ whose cone E'_r is now $(j+r)$-connective. For a fixed r, we have a commutative diagram with exact rows

$$\begin{tikzcd}
F_r \arrow[r] & E \arrow[r] & E'_r \\
\phi\psi(F_r) \arrow[r] \arrow[u] & \phi\psi(E) \arrow[r] \arrow[u] & \phi\psi(E'_r). \arrow[u]
\end{tikzcd}$$

As E'_r and $\phi\psi(E'_r)$ are both $(j+r)$-connective, we see that the vertical morphism in the middle induces bijection on H^l for all $l \geq -r-j+1$. As r is arbitray, we deduce that $\phi\psi(E) \to E$ is an equivalence.

Finally, the same exact sequence $F_r \to E \to E'_r$ show that the cohomology groups of $\psi(E)$ are of finite type as $H^0(\hat{A}_x)$-modules, and thus that $\psi(E)$ is almost perfect as required. $\square$

A.5. Cotangent complexes of derived Artin stacks

For a derived Artin stack X over k, we have its big or fake cotangent complex $\mathbb{L}^{big}_{X/k}$. This is a non-necessarily quasi-coherent $\mathcal{O}_X$-dg-module. As an ∞-functor, it sends $u : \mathbf{Spec}\, A \to X$ to the A-dg-module $\mathbb{L}_{A/k}$. There is a canonical pull-back morphism $u^*\mathbb{L}_{X/k} \to \mathbb{L}_{A/k}$, which defines a morphism of $\mathcal{O}_X$-dg-modules

$$\mathbb{L}_{X/k} \to \mathbb{L}^{big}_{X/k}$$

where $\mathbb{L}_{X/k}$ is the global cotangent complex of X relative to k as constructed in [**TV08**, §1.4]. As $\mathbb{L}_{X/k}$ is itself quasi-coherent, we get a canonical morphism of quasi-coherent complexes $\mathbb{L}_{X/k} \to (\mathbb{L}^{big}_{X/k})^{qcoh}$. Considering exterior product, we also get this way natural morphisms for all $i \geq 0$

$$\wedge^i_{\mathcal{O}_X}\mathbb{L}_{X/k} \to (\wedge^i_{\mathcal{O}_X}\mathbb{L}^{big}_{X/k})^{qcoh}.$$

THEOREM A.5.0.1. *Let X be a derived Artin stack locally of finite presentation over k. For all $i \geq 0$ the natural morphism*

$$\wedge^i_{\mathcal{O}_X}\mathbb{L}_{X/k} \to (\wedge^i_{\mathcal{O}_X}\mathbb{L}^{big}_{X/k})^{qcoh}$$

is an equivalence of quasi-coherent complexes on X.

Proof. The proof is essentially the same as in [**PTVV13**, Prop. 1.14], together with the additional result from [**Mon21**]. Note that the statement of the theorem implies [**PTVV13**, Prop. 1.14] by taking global sections.

The proof goes by induction on the geometricity of X. When X is affine, the statement follow formally from the shape of the adjunction

$$i : \mathbf{QCoh}(X) \leftrightarrows \mathcal{O}_X - \mathbf{dg} : (-)^{qcoh}.$$

Indeed, the ∞-functor $E \mapsto E^{qcoh}$ is explicitely given by sending E to the A-dg-module of global sections $\Gamma(X, E) \in A - \mathbf{dg} \simeq \mathbf{QCoh}(X)$. From this, it is obvious that the canonical morphisms $\wedge^i_{\mathcal{O}_X}\mathbb{L}_{X/k} \to (\wedge^i_{\mathcal{O}_X}\mathbb{L}^{big}_{X/k})^{qcoh}$ are indeed equivalence. In the same manner, the result remains true for disjoint union of derived affine schemes.

We assume now that the statement has been proved for $(n-1)$-geometric derived Artin stack. We let $f : U \to X$ be a smooth atlas with U a disjoint union of affine derived schemes. Let $U_* \to X$ be the nerve of f, so each U_i is a $(n-1)$-geometric derived Artin stack. For a derived stack Y we denote by $TY[1]$ its shifted total tangent stack. It is formally defined by $\mathbf{Map}(\mathbf{Spec}\, k[\epsilon], Y)$, where ϵ sits in cohomological degree -1. The derived stack $TY[1]$ sits over Y, and as such is of the form $\mathbb{V}(\mathbb{L}_{Y/k}[-1])$, the linear stack associated to the perfect complex $\mathbb{L}_{Y/k}[-1]$. This is a $\mathbb{G}_m$-equivariant derived stack over Y, and according to [**Mon21**], the projection $\pi : [TY[1]/\mathbb{G}_m] \longrightarrow Y \times B\mathbb{G}_m$ is such that the quasi-coherent push-forward $\pi_(\mathcal{O})$ is given by

$$\pi_*(\mathcal{O}) \simeq \bigoplus_p Sym^p_{\mathcal{O}_Y}(\mathbb{L}_{Y/k}[-1]),$$

where the right hand side is a graded quasi-coherent complex on Y defined by the natural grading of the Sym. In other words, the weight i piece of $\pi_*(\mathcal{O})$ is $\wedge^i \mathbb{L}_{Y/k}[-i]$.

It is very easy to check, for instance using the infinitesimal lifting properties, that $TU_0[1] \to TX[1]$ remains a smooth epimorphism. In particular, $TU_*[1]$, which is equivalent to the nerve of $TU_0[1] \to TX[1]$, induces an equivalence of derived $\mathbb{G}_m$-equivariant stacks over X

$$TX[1] \simeq \mathrm{Colim}_p TU_p[1].$$

If we denote by $p_i : TU_p[1] \to X$ the natural projection, and similarly $p : TX[1] \to X$, we thus have an equivalence of $\mathbb{G}_m$-equivariant quasi-coherent complexes $p_*(\mathcal{O}) \simeq \mathrm{Lim}_p (p_p)_*(\mathcal{O})$. Applying the main result of [**Mon21**] and considering the weight p pieces, we get a canonical equivalence of quasi-coherent complexes

$$\wedge^i \mathbb{L}_{X/k} \simeq \mathrm{Lim}_p (f_p)_*(\wedge^i \mathbb{L}_{U_p/k})$$

where $f_i : U_i \to X$ is the natural projection.

Because the natural morphism $\mathrm{Colim}_p U_p \to X$ is an equivalence, we get, by descent, an equivalence of $\mathcal{O}_X$-dg-modules $\wedge^i \mathbb{L}^{big}_{X/k} \simeq \mathrm{Lim}_p (f_p)_*(f_p^*)(\wedge^i \mathbb{L}^{big}_{X/k})$. But we have a tautological identification $(f_p^*)(\wedge^i \mathbb{L}^{big}_{X/k}) \simeq \wedge^i \mathbb{L}^{big}_{U_p/k}$, and thus we get $\wedge^i \mathbb{L}^{big}_{X/k} \simeq \mathrm{Lim}_p (f_p)_*(\wedge^i \mathbb{L}^{big}_{U_p/k})$. As the ∞-functor $(-)^{qcoh}$ is right adjoint it commutes with limits. Also, as pull-backs of $\mathcal{O}$-dg-modules preserves quasi-coherent complexes, $(-)^{qcoh}$ also commutes with direct images. Finally, as each U_p is $(n-1)$-geometric, we know that the theorem holds for the U_p's. We thus get a natural equivalence

$$(\wedge^i \mathbb{L}^{big}_{X/k})^{qcoh} \simeq \mathrm{Lim}_p (f_p)_*(\wedge^i \mathbb{L}_{U_p/k}).$$

But we have already seen that the right hand side is naturally equivalent to $\wedge^i \mathbb{L}_{X/k}$, and thus we have

$$(\wedge^i \mathbb{L}^{big}_{X/k})^{qcoh} \simeq \wedge^i \mathbb{L}_{X/k}.$$

Unfolding the various equivalences we see that this equivalence is indeed induced by the canonical morphism $\wedge^i \mathbb{L}_{X/k} \to \wedge^i \mathbb{L}^{big}_{X/k}$, which proves the theorem. □

A.6. Cotangent complexes of formal completion

As in the previous section let X be a derived Artin stack of finite presentation over k. We assume here that k is a field and that we have given ourselves a global point $x : \mathbf{Spec}\, k \to X$ (over more general rings k, we can always reduce to this special case by base change). As explained in §2.3.2 we have the formal completion $(X, x)^\wedge$. This formal completion can be described as a ringed site as follows. We let X^{art} be the ∞-category $\mathbf{dgart}_k^*/X$, of derived local Artinian affine schemes over X (preserving the base point). The ∞-category is a ringed site, for the induced etale topology and for the canonical stack of simplicial rings sending $\mathbf{Spec}\, A \to X$

to A. We let $\mathcal{O}_{X^{art}} - \mathbf{dg}$ be the ∞-category of all $\mathcal{O}_{X^{art}}$-dg-modules. We have a full sub-∞-category $\mathbf{QCoh}(X^{art}) \subset \mathcal{O}_{X^{art}} - \mathbf{dg}$ consisting of objects E such that for all morphism in X^{art}

$$\begin{tikzcd} \mathbf{Spec}\, A \arrow[rr] \arrow[dr] & & \mathbf{Spec}\, B \arrow[dl] \\ & X & \end{tikzcd}$$

such that the induced morphism $E(B) \otimes_B A \longrightarrow E(A)$ is a quasi-isomorphism. Note that by definition $\mathbf{QCoh}(X^{art})$ identifies canonically with $\mathbf{QCoh}((X,x)^{\wedge})$, the ∞-category of quasi-coherent complexes on the formal completion $(X,x)^{\wedge}$ of X at x.

By Freyd representability the natural inlcusion $j : \mathbf{QCoh}(X^{art}) \subset \mathcal{O}_{X^{art}} - \mathbf{dg}$ possesses a right adjoint denoted $E \mapsto E^{qcoh}$. We also have a canonical restriction ∞-functor

$$\mathcal{O}_X - \mathbf{dg} \longrightarrow \mathcal{O}_{X^{art}} - \mathbf{dg},$$

and thus the cotangent complex of X (relative to k) provides a canonical object $\mathbb{L}_{X^{art}/k} \in \mathcal{O}_{X^{art}} - \mathbf{dg}$ called the *formal cotangent complex of X at x*. It is obviously a quasi-coherent complex over X^{art}. In the same manner, we have the big or fake cotangent complex $\mathbb{L}^{big}_{X^{art}/k} \in \mathcal{O}_{X^{art}} - \mathbf{dg}$ given by $\mathbb{L}_{X^{art}/k}(\mathbf{Spec}\, A) := \mathbb{L}_{A/k}$.

The following theorem is the formal version of our previous result theorem A.5.0.1.

THEOREM A.6.0.1. *With the notations and conditions as above, for all $i \geq 0$ the canonical morphism*

$$\wedge^i_{\mathcal{O}_{X^{art}}} \mathbb{L}_{X^{art}/k} \longrightarrow (\wedge^i_{\mathcal{O}_{X^{art}}} \mathbb{L}^{big}_{X^{art}/k})^{qcoh}.$$

Proof. We proceed as in theorem by induction on the geometricity of X. We let $f_* : U_* \to X$ be the nerve of a smooth atlas for X with U_0 a disjoint union of derived affine schemes. Moreover, by galois descent we can replace k by a algebraic separable extension k'/k, and thus assume that x lifts to a k-point u in U_*.

As we have already seen during the proof of A.5.0.1 we have equivalences in $\mathcal{O}_X - \mathbf{dg}$ and in $\mathbf{QCoh}(X)$

$$\wedge^i \mathbb{L}^{big}_{X/k} \simeq \mathrm{Lim}_p (f_p)_* (\wedge^i \mathbb{L}^{big}_{U_p/k})$$
$$\wedge^i \mathbb{L}_{X/k} \simeq \mathrm{Lim}_p (f_p)_* \wedge^i \mathbb{L}_{U_p/k}.$$

As the restriction ∞-functor $\mathcal{O}_X - \mathbf{dg} \to \mathcal{O}_{X^{art}} - \mathbf{dg}$ commutes with limits, we see that we have a commutative square in $\mathcal{O}_{X^{art}} - \mathbf{dg}$

$$\begin{tikzcd} \wedge^i_{\mathcal{O}_{X^{art}}} \mathbb{L}_{X^{art}/k} \arrow[r] \arrow[d] & \mathrm{Lim}_p (\hat{f}_p)_* \wedge^i \mathbb{L}_{U_p^{art}/k} \arrow[d] \\ (\wedge^i_{\mathcal{O}_{X^{art}}} \mathbb{L}^{big}_{X^{art}/k})^{qcoh} \arrow[r] & \mathrm{Lim}_p (\hat{f}_p)_* (\wedge^i \mathbb{L}^{big}_{U_p^{art}/k})^{qcoh}, \end{tikzcd}$$

where $\hat{f}_* : U_*^{art} \to X^{art}$ is the induced atlas on formal completions. Using induction on the geometricity of X we see that we can reduce the theorem to the case X is disjoint union of affines. Restricting to the component containing x we can further assume that X is an affine derived scheme $X = \mathbf{Spec}\, A$, where A is of finite presentation over k.

We now fix $i \geq 0$, and consider the exact triangle of $\mathcal{O}_{X^{art}}$-dg-modules $\wedge^i \mathbb{L}_{X^{art}} \to \wedge^i \mathbb{L}_{X^{art}}^{big} \to E$. By construction, the $\mathcal{O}_{X^{art}}$-dg-module E sends $\mathbf{Spec}\, A' \to X$ to the cone of $\wedge^i \mathbb{L}_A \otimes_A A' \to \wedge^i \mathbb{L}_{A'}$. For any quasi-coherent complex $E_0 \in \mathbf{QCoh}(X^{art})$ we have a natural equivalence

$$\mathbf{Map}_{\mathcal{O}_{X^{art}}-\mathbf{dg}}(E_0, E) \simeq \mathrm{Lim}_{A \to A'} \mathbf{Map}_{A'-\mathbf{dg}}(E_0(A'), E(A')).$$

In other words, this mapping space if also the mapping space of pro-$\underline{\hat{A}}$-modules between "$\mathrm{Lim}_{A\to A'}$"$E_0(A')$ and "$\mathrm{Lim}_{A\to A'}$"$E(A')$. By Corollary A.3.0.5 we know that E is equivalent to zero as a pro-object, and thus that $\mathbf{Map}_{\mathcal{O}_{X^{art}}-\mathbf{dg}}(E_0, E) \simeq 0$ for any quasi-coherent complex E_0, or equivalently that $E^{qcoh} \simeq 0$. This shows as required that $\wedge^i \mathbb{L}_{X^{art}} \simeq (\wedge^i \mathbb{L}_{X^{art}}^{big})^{qcoh}$. □

A.7. Deformation to the normal bundle

In this section we denote by $\mathcal{A}$ the quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$. As explained in [**Mou21**], derived stacks over $\mathcal{A}$ should be considered as filtered derived stacks. We remind in particular (see §1.3.2), that there exists a natural symmetric monoidal equivalence of ∞-categories

$$\mathbf{QCoh}(\mathcal{A}) \simeq \mathbf{dg}_k^{fil},$$

between quasi-coherent complexes over $\mathring{A}$ and filtered complexes of k-modules. This equivalence is induced by a natural (decreasing) filtered object $\mathcal{O}(-*)$ in $\mathbf{QCoh}(\mathcal{A})$, namely the families of $\mathcal{O}(-i)$ for $i \in \mathbb{Z}$. Here $\mathcal{O}(-i)$ stands for the line bundle on $\mathcal{A}$ which is trivial on $\mathbb{A}^1$ together with its canonical $\mathbb{G}_m$-action of weight $-i$. The families of $\mathcal{O}(-i)$ is endowed with canonical morphisms $\mathcal{O}(-i) \to \mathcal{O}(-i+1)$ induced by multiplication by $X \in \Gamma(\mathring{A}, \mathcal{O}(1))$. The ∞-functor $\mathbb{R}\underline{Hom}(\mathcal{O}(*), -) : \mathbf{QCoh}(\mathring{A}) \longrightarrow \mathbf{dg}_k^{fil}$ is the equivalence mentionned above.

We consider the inclusion of the origin $\mathbf{Spec}\, k \to \mathbb{A}^1$, and the induced morphism on quotient stacks

$$j : B\mathbb{G}_m \hookrightarrow [\mathbb{A}^1/\mathbb{G}_m] = \mathcal{A}.$$

As an object over $\mathcal{A}$, $B\mathbb{G}_m$ will be denoted by $\mathcal{Q} \in \mathbf{dSt}_k/\mathcal{A}$. This is a relative derived affine scheme, whose fiber at any non-zero point is empty, and whose fiber at 0 is $\mathbf{Spec}\, k[\epsilon_{-1}]$, where ϵ_{-1} is of degree -1 and $\epsilon_{-1}^2 = 0$. The induced $\mathbb{G}_m$-action acts by weight 1 on ϵ_{-1}.

We now let $X \to Y$ be any morphism of derived stacks. we consider $Y \times \mathcal{Q}$ and $X \times \mathcal{A}$ as objects over $Y \times \mathcal{A}$, and we form the relative derived mapping stack

$$Def(X/Y) := \underline{\mathbf{Map}}_{/Y\times\mathcal{A}}(Y \times \mathcal{Q}, X \times \mathcal{A}) \in \mathbf{dSt}_k/Y \times \mathcal{A}.$$

Definition A.7.0.1. *The* deformation to the normal bundle of a morphism $f : X \to Y$ *is the derived stack defined above*

$$Def(X/Y) = Def(f) := \underline{\mathbf{Map}}_{/Y\times\mathcal{A}}(Y \times \mathcal{Q}, X \times \mathcal{A}) \in \mathbf{dSt}_k/Y \times \mathcal{A}.$$

The deformation to the normal bundle $Def(X/Y)$ has the following two important properties. Its fiber over the open $Y = [\mathbb{G}_m/\mathbb{G}_m] \times Y \hookrightarrow \mathcal{A} \times Y$ is $\mathbf{Map}_{/Y}(\emptyset, X) \simeq Y$, whereas its fiber at zero $Y \times B\mathbb{G}_m \hookrightarrow Y \times \mathcal{A}$ is $[\mathbf{Map}_{/Y}(Y \times \mathbf{Spec}\, k[\epsilon_{-1}], X)/\mathbb{G}_m]$. Note that $[\mathbf{Map}_{/Y}(Y \times \mathbf{Spec}\, k[\epsilon_{-1}], X)$ is identified with the shifted relative tangent bundle $T(X/Y)[1]$, where $\mathbb{G}_m$ acts by weight -1 dilatation along the fibers of $T(X/Y)[1] \to X$. When X and Y are derived Artin stacks locally of finite presentation, $T(X/Y)[1]$ is also the linear stack $\mathbb{V}(\mathbb{L}_{X/Y}[-1])$, associated to the perfect complex $\mathbb{L}_{X/Y}[-1]$ on X, endowed with its natural $\mathbb{G}_m$-action. Therefore, $Def(X/Y)$ provides indeed an interpolation, by means of filtration, between X and $T(X/Y)[1]$, which is at the origin of the relative Hodge filtration.

Note that the projection $Y \times \mathcal{Q} \to Y \times \mathcal{A}$ induces on derived mapping stacks a morphism $j : X \times \mathcal{A} \to Def(f)$. Over the open point $[\mathbb{G}_m/\mathbb{G}_m]$ of $\mathcal{A}$ the morphism j recovers $f : X \to Y$, whereas over $B\mathbb{G}_m$ it is the zero section $X \to T(X/Y)[1]$.

We finish this section by the following comment. When $f : X = \mathbb{V}(E') \to Y = \mathbb{V}(E)$ is a morphism of derived linear stacks over $\mathbf{Spec}\, A$, induced by a morphism of bounded above A-dg-modules $u : E \to E'$, then $Def(f)$ is itself globally, as a derived stack over $\mathbf{Spec}\, A \times \mathcal{A}$, a derived linear stack. Indeed, this can be seen directly using that for any derived stack $Z \to \mathbf{Spec}\, A$ we have functorial equivalences

$$\mathbf{Map}_{\mathbf{dSt}_A}(Z, \mathbb{V}(E)) \simeq \mathbf{Map}_{\mathbf{dg}_A}(E, \mathcal{O}_Z(Z)).$$

In this case, the derived linear stack $Def(f)$, relative to $\mathbf{Spec}\, A \times \mathcal{A}$, can be described explicitly as $\mathbb{V}(u)$, where $u : E \to E'$ is now considered as a one step filtered quasi-coherent complex on $\mathbf{Spec}\, A$, and thus as a quasi-coherent complex over $\mathbf{Spec}\, A \times \mathcal{A}$.

A.8. Internal derived de Rham theory

We remind from [MRT22] the group stack (over k) $\mathcal{H}_0 := B\mathbb{G}_a \ltimes \mathbb{G}_m$, semi-direct product of the multiplicative group acting, by its standard weight 1 action, on $B\mathbb{G}_a$. As we have seen, the ∞-category $\mathbf{QCoh}(B\mathcal{H}_0)$ is canonically equivalent to the ∞-catgeory of graded mixed complexes

$$\mathbf{QCoh}(B\mathcal{H}_0) \simeq \epsilon - \mathbf{dg}_k^{gr}.$$

In the same manner, there is an equivalence of ∞-categories of commutative algebras

$$\mathcal{O}_{B\mathcal{H}_0} - \mathbf{cdga}_{qcoh} \simeq \epsilon - \mathbf{cdga}_k^{gr},$$

where $\mathcal{O}_{B\mathcal{H}_0} - \mathbf{cdga}_{qcoh}$ is the ∞-category of quasi-coherent cdga's over the stack $B\mathcal{H}_0$, defined by the usual left Kan extension formula

$$\mathcal{O}_{B\mathcal{H}_0} - \mathbf{cdga}_{qcoh} = \lim_{\mathbf{Spec}\, A \to B\mathcal{H}_0} \mathbf{cdga}_A.$$

For any derived stack X, we define the ∞-category of quasi-coherent graded mixed $\mathcal{O}_X$-cdga's by

$$\mathcal{O}_X - \epsilon - \mathbf{cdga}^{gr}_{qcoh} := \mathcal{O}_{X \times B\mathcal{H}_0} - \mathbf{cdga}_{qcoh}.$$

Note that, by definition, an object in $\mathcal{O}_X - \epsilon - \mathbf{cdga}^{gr}_{qcoh}$ is a sheaf of graded mixed cdga's over X which is moreover linear over $\mathcal{O}_X$ and quasi-coherent. That is, it is given by a family of graded mixed A-linear cdga D_A, for each $\mathbf{Spec}\, A \to X$, together with compatible by base change equivalences

$$D_A \otimes_A B \simeq D_B$$

for each $\mathbf{Spec}\, B \to \mathbf{Spec}\, A \to X$.

Remark A.8.0.1. One should not confuse $\mathcal{O}_X - \epsilon - \mathbf{cdga}^{gr}_{qcoh}$ with the notion of graded mixed cdga's over X used in the definition of derived foliations over the derived stack X: the latter objects are not quasi-coherent by any means. We hope that the wording "quasi-coherent graded mixed $\mathcal{O}_X$-cdga" will help avoiding the confusion.

Taking the weight 0 part defines an ∞-functor $\mathcal{O}_X - \epsilon - \mathbf{cdga}^{gr}_{qcoh} \longrightarrow \mathcal{O}_X - \mathbf{cdga}_{qcoh}$, where $\mathcal{O}_X - \mathbf{cdga}_{qcoh}$ is the ∞-category of sheaves of quasi-coherent $\mathcal{O}_X$-linear cdga's over X (defined by $\mathcal{O}_X - \mathbf{cdga}_{qcoh} = \lim_{\mathbf{Spec}\, A \to F} \mathbf{cdga}_A$). This forgetful ∞-functor possesses a left adjoint called the relative or internal de Rham algebra construction

$$\mathbf{DR}^{int}(-/\mathcal{O}_X) : \mathcal{O}_X - \mathbf{cdga}_{qcoh} \longrightarrow \mathcal{O}_X - \epsilon - \mathbf{cdga}^{gr}_{qcoh}.$$

Concretely, if $B \in \mathcal{O}_X - \mathbf{cdga}_{qcoh}$, then for any $\mathbf{Spec}\, A \to X$ we consider $\mathbf{DR}(B_A/A) \in \epsilon{-}\mathbf{cdga}^{gr}_A$. The base change equivalences for B induces base change equivalences for $\mathbf{DR}(B_A/A)$ making it into a quasi-coherent $\mathcal{O}_X$-linear graded mixed cdga.

We will now be interested in the specific case where $X = B\mathcal{H}_0$. We get this way an ∞-functor

$$\mathcal{O}_{B\mathcal{H}_0} - \mathbf{cdga}_{qcoh} \longrightarrow \mathcal{O}_{B\mathcal{H}_0} - \epsilon - \mathbf{cdga}^{gr}_{qcoh} \simeq \mathcal{O}_{B\mathcal{H}_0 \times B\mathcal{H}_0} \mathbf{cdga}_{qcoh}.$$

Note that, using the same argument as in the proof of the equivalence $\mathcal{O}_{B\mathcal{H}_0} - \mathbf{cdga}_{qcoh} \simeq \epsilon - \mathbf{cdga}^{gr}$, the ∞-category $\mathcal{O}_{B\mathcal{H}_0 \times B\mathcal{H}_0} - \mathbf{cdga}_{qcoh}$ is naturally equivalent to the ∞-category of *double graded mixed cdga*: that a cdga endowed with two multiplicative $\mathbb{Z}$-graduations and two compatible mixed structures ϵ_1 and ϵ_2 such that $\epsilon_1\epsilon_2 + \epsilon_2\epsilon_1 = 0$ (ϵ_1 is of weight bigrading $(1,0)$ and ϵ_2 of bigrading $(0,1)$, and both are of cohomological grading -1). The internal de Rham construction is then explicitly given by sending an graded mixed cdga A to its graded de Rham algebra $\mathbf{DR}^{gr}(A/k)$, which is endowed with a double graded mixed structure: ϵ_1 is induced by the original mixed structure on A, and ϵ_2 is the de Rham differential of $\mathbf{DR}^{gr}(A/k)$.

Double graded mixed cdga's have two different realizations constructions, depending if we realize the first of the second mixed structure. These corresponds to considering direct images along the two projections

$$B\mathcal{H}_0 \leftarrow B\mathcal{H}_0 \times B\mathcal{H}_0 \to B\mathcal{H}_0$$

which induced the two (non-Tate) realization ∞-functors

$$|-|_1\,, |-|_2 : \mathcal{O}_{B\mathcal{H}_0\times B\mathcal{H}_0} - \mathbf{cdga}_{qcoh} \longrightarrow \mathcal{O}_{B\mathcal{H}_0} - \mathbf{cdga}_{qcoh}.$$

Applied to $\mathbf{DR}^{int}(A/k)$, for A a graded mixed cdga, we get this way two possible de Rham complexes of A

$$\widehat{C}^{I}_{DR}(A/k) := |\mathbf{DR}(A/k)|_1 \qquad \widehat{C}^{II}_{DR}(A/k) := |\mathbf{DR}(A/k)|_2,$$

which both of them are graded mixed cdga's.

Definition A.8.0.2. *Let A be a graded mixed cdga (over k).*

(1) The (derived) de Rham complex of the first kind of A relative to k *is the graded mixed cdga given by*

$$\widehat{C}^{I}_{DR}(A/k) := |\mathbf{DR}^{int}(A/k)|_1.$$

(2) The (derived) de Rham complex of the second kind of A relative to k *is the graded mixed cdga given by*

$$\widehat{C}^{II}_{DR}(A/k) := |\mathbf{DR}^{int}(A/k)|_2.$$

(3) The absolute (derived) de Rham complex of A relative to k *is the graded mixed cdga given by*

$$\widehat{C}^{abs}_{DR}(A/k) := ||\mathbf{DR}^{int}(A/k)|_1|_2 \simeq ||\mathbf{DR}^{int}(A/k)|_2|_1.$$

Visually, the two versions of the de Rham complexes of definition above can be summarized as follows.

$$\widehat{C}^{I}_{DR}(A/k) = |A| \to |\mathbb{L}_{A/k}| \to |\wedge^2_A \mathbb{L}_{A/k}| \to \dots |\wedge^i_A \mathbb{L}_{A/k}| \to |\wedge^{i+1}_A \mathbb{L}_{A/k}| \to \dots$$

$$\widehat{C}^{II}_{DR}(A/k) = |A \to \mathbb{L}_{A/k} \to \wedge^2_A\mathbb{L}_{A/k} \to \cdots \wedge^i_A \mathbb{L}_{A/k} \to \wedge^{i+1}_A\mathbb{L}_{A/k} \to \dots|$$

Note also that the two graded mixed cdga's $\widehat{C}^{I}_{DR}(A/k)$ and $\widehat{C}^{II}_{DR}(A/k)$ are different, but their realizations are canonically equivalent as cdga's

$$|\widehat{C}^{I}_{DR}(A/k)| \simeq |\widehat{C}^{II}_{DR}(A/k)|.$$

In a more explicit form, $\widehat{C}^{I}_{DR}(A/k)$ is a graded mixed cdgas whose piece of weight i is $|\wedge^i \mathbb{L}_{A/k}|$, where $\mathbb{L}_{A/k}$ is the graded cotangent complexe of A, endowed with the induced mixed structure $\epsilon(dR(a)) = -dR(\epsilon(a))$, where ϵ is the original mixed structure on A. The mixed structure is then induced by the de Rham differential of A.

On the other hand, the graded piece of weight i of $\widehat{C}^{II}_{DR}(A/k)$ is given by the complex

$$\prod_{j\geq 0}(\wedge^j\mathbb{L}_{A/k})^{(i)}[-j],$$

endowed with the total differential $d + dR$. The mixed structure relating these graded pieces is then induced by the mixed structure ϵ of A. Note that for each i, we have a decomposition

$$(\wedge^j \mathbb{L}_{A/k})^{(i)} \simeq \bigoplus_{n_1k_1+\cdots+n_jk_j=i} \bigotimes_l \wedge^{k_l}(\mathbb{L}_{A/k}^{(n_l)})[-j],$$

so the weight i piece of $\widehat{C}^{*,II}_{DR}(A/k)$ can also be written

$$\prod_{j\geq 0}\Big(\bigoplus_{n_1k_1+\cdots+n_jk_j=i} \bigotimes_l \wedge^{k_l}(\mathbb{L}_{A/k}^{(n_l)})[-j]\Big).$$

As an example of applications of the notions of internal de Rham complex we prove the following well known lemma.

PROPOSITON A.8.0.3. *Let $A \to k$ be a local augmented k-linear Artinian dg-algebra, almost of finite presentation over k. Then, the canonical morphism induces a quasi-isomorphism of cdga's*

$$k \simeq \widehat{C}_{DR}(A/k).$$

Proof. Let $D := \mathbf{DR}(k/A) \in \epsilon - \mathbf{cdga}_k^{gr}$, and we consider the canonical morphism $A \to \mathbf{DR}(k/A)$, where A is considered as a graded mixed cdga of weight 0 with trivial mixed structure. This morphism induces a morphism on internal de Rham algebras

$$\phi : \mathbf{DR}(A/k) \simeq \mathbf{DR}^{int}(A/k) \to \mathbf{DR}^{int}(D/k).$$

This is a morphism of double graded mixed cdga's, which can be realized in the absolute sense to get a morphism of cdga's

$$\widehat{C}_{DR}(A/k) \to \widehat{C}^{abs}_{DR}(D/k).$$

This morphism can be described as first realizing along the first graded mixed structure and then realizing along the second one. The morphism $|\phi|_1$ is given by $\mathbf{DR}(A/k) \to \widehat{C}^{I}_{DR}(D/k)$, as the first graded mixed structure on $\mathbf{DR}(A/k)$ is trivial of weight 0. On the peice of weight i the morphism $|\phi|_1$ is the canonical morphism

$$\wedge^i \mathbb{L}_{A/k}[i] \longrightarrow |\wedge^i \mathbb{L}_{D/k}[i]|_1.$$

For $i = 0$, this morphism is the canonical morphism of cdga's $A \to \widehat{C}_{DR}(k/A)$, which is a quasi-isomorphism as shown in [**CPT$^+$17**, Lem. 2.2.4]. The morphism $|\phi|_1$ in weights $i > 0$ is the canonical morphism

$$\wedge^i \mathbb{L}_{A/k} \longrightarrow \widehat{C}_{DR}(k/A, \wedge^i \mathbb{L}_{A/k} \otimes_A k),$$

where the right hand side is the derived de Rham cohomology of k relative to A, with coefficients in $\mathbb{L}_{A/k} \otimes_A k$. As $\mathbb{L}_{A/k}$ is almost perfect A-dg-module, we also know that this morphism is a quasi-isomorphism (see [**CPT$^+$17**, Cor. 2.2.5] for the case of perfect modules which can be extended to almost perfect modules in a straightforward manner by postnikov approximation).

This shows that $|\phi|_1 : \mathbf{DR}(A/k) \to \widehat{C}^{I}_{DR}(D/k)$ is a quasi-isomorphism of graded mixed cdga's, and thus induces an equivalence after realizations $\widehat{C}_{DR}(A/k) \simeq \widehat{C}^{abs}_{DR}(D/k)$. Now,

$\widehat{C}^{abs}_{DR}(D/k)$ can also be computed as$||\mathbf{DR}^{int}(D/k)|_2|_1$. As a graded cdga, we have $D \simeq Sym_k(E[2])$, where $E := \mathbb{L}_{A/k} \otimes_A k$. Therefore, its de Rham cohomology is trivial, and we thus have $|\mathbf{DR}^{int}(D/k)|_2 \simeq k$. This shows that $\widehat{C}^{abs}_{DR}(D/k) \simeq k$ and finishes the proof of the Proposition. □

Remark A.8.0.4. Proposition A.8.0.3 is valid in a more general context, with a similar proof. For any derived affine k-scheme $X = \mathbf{Spec}\, A$ almost of finite presentation over k, the natural morphism

$$\widehat{C}_{DR}(A/k) \to \widehat{C}_{DR}(A_{red}/k)$$

is an equivalence.

Another useful statement is the following reconstruction result.

Propositon A.8.0.5. *Let B be a graded mixed cdga with $B^{(0)} = A$ connective and $Sym_A(B^{(1)}) \simeq B$ as graded cdgas. Then, there is a canonical equivalence of graded mixed cdga's*

$$B \simeq \widehat{C}^{I}_{DR}(A/B).$$

Proof. We first notice that the canonical morphism $B \to \mathbf{DR}^{int}(A/B)$ induces an equivalence of graded mixed cdga's $B \simeq \widehat{C}^{II}_{DR}(A/B)$. Indeed, this statement is independent of the mixed structure on B, as it is enough to show that $B \to \widehat{C}^{II}_{DR}(A/B)$ is an equivalence of graded cdga's. To see this it is enough to write down $B = Sym_A(E)$ for a perfect A-dg-module E, and write A as $Sym_A(E')$ where E' is the cone of the identity $E \to E$. as a graded cdga, the de Rham cohomology of the second kind $\widehat{C}^{II}_{DR}(A/B)$ can be modeled by the (graded) de Rham complex of $Sym_A(E')$ over B, which is explicitly given by $Sym_A(K[-1])$ where K is the cone of the natural projection $E' \to E[1]$. The A-dg-module K is canonically quasi-isomorphic to $E[1]$ and the corresponding morphism $B \to Sym_A(K[-1])$ sends E to $K[-1]$ by its canonical identification and thus is an equivalence.

To finish the proof of the Proposition it is thus enough to identify $\widehat{C}^{I}_{DR}(A/B)$ canonically with $\widehat{C}^{II}_{DR}(A/B)$. For this we use the following statement about double graded mixed complexes.

Lemma A.8.0.6. *There exists a fully faithful symmetric monoidal ∞-functor*

$$\mathbf{QCoh}(B\mathcal{H}_0) \hookrightarrow \mathbf{QCoh}(B\mathcal{H}_0 \times B\mathcal{H}_0)$$

from graded mixed complexes to double graded mixed complexes, whose essential image consists of all object E which, as a bigraded complex, is concentrated on the diagonal: $E^{(i,j)} \simeq 0$ for all $i \neq j$.

Proof of the lemma. The most easy proof consists of using the equivalence between graded mixed complexes and complete filtered complexes (see Proposition 1.3.3.3 and Corollary 1.3.4.4). Via this translation the lemma becomes obvious: the right adjoint d_* consists of sending a filtered complex to the corresponding "diagonal" bilfiltered complex. More explicitely, if $\{F^j\}_{j\in\mathbb{Z}}$ is a filtered complex, the corresponding bifiltered complex X is given by $X^{i,j} := F^{min(i,j)}$ with the natural transition morphisms. In terms of functors category, if we denote by $\mathbb{Z}$ the poset of integers, this full embedding is given by

$$d_! : Fun(\mathbb{Z}, \mathbf{dg}_k) \longrightarrow Fun(\mathbb{Z} \times \mathbb{Z}, \mathbf{dg}_k)$$

given by left Kan extension along the diagonal $d : \mathbb{Z} \hookrightarrow \mathbb{Z} \times \mathbb{Z}$. This is clearly a fully faithful construction and its essential image correspond precisely to complete bifiltered complexes E whose associated bigraded are concentrated on the diagonal. □

The full embedding of the previous lemma being symmetric monoidal ∞-functor and thus extends to a similar statement concerning commutative algebras object. This means that in particular, that any double graded mixed cdga D, which is concentrated on the diagonal as a bi-graded complex, is naturally symmetric with respect to the action of $\mathbb{Z}/2$ which exchanges the two gradings and the two mixed structures. In particular, for any such object D, we have a canonical equivalence $|D|_1 \simeq |D|_2$, and both are equivalent to $|E|$, where E is the graded mixed cdga obtained from the lemma A.8.0.6. This can therefore be applied to $D = \mathbf{DR}^{int}(A/B)$, which as a bi-graded cdga is equivalent to $Sym_A(E[2])$ and thus is indeed concentrated on the diagonal. □

A.9. Derived affine stacks

A.9.1. Cosimplicial-simplicial algebras. We let denote by sCR be the category of simplicial commutative rings. It is endowed with its usual *model category* structure for which equivalences and fibrations are defined on the underlying simplicial sets.

Similarly, we denote by $csCR$ the category of cosimplicial objects in sCR, or equivalently of cosimplicial-simplicial commutative rings. Its objects will be denoted by A_*^*, where the superscript will refer to the cosimplicial direction and subscript to the simplicial direction. The category $csCR$ can be endowed with a *tensored* and *cotensored* structure over the category $sSet$ of simplicial sets. We warn the reader here that there are several natural such structures and that the choice we make here is important for the sequel. For a simplicial set K and an object A of $csCR$, we define the cotensored structure by A^K by the formula

$$(A^K)_p^q := (A_p^q)^{K_q}$$

for $[p] \in \Delta^{op}$ and $[q] \in \Delta$. The tensored structure $K \otimes A$, for $A \in csCR$ and $K \in sSets$, is defined by the usual adjunction formula $Hom(K \otimes A, B) \simeq Hom(A, B^K)$, for arbitrary $B \in csCR$. Similarly, we have a canonical simplicial enrichment for which the simplicial Homs are denoted by $\underline{Hom}$ and are defined by

$$\underline{Hom}(A, B)_p := Hom_{csCR}(A, B^{\Delta^p}).$$

By considering the natural embedding $Mod_{\mathbb{Z}} \subset C(\mathbb{Z})$, from $\mathbb{Z}$-modules to cochain complexes, any object $A \in csCR$ can be considered, by forgetting the multiplicative structure, as a cosimplicial-simplicial object in $C(\mathbb{Z})$. We denote by $Tot^{\pi}(A)$ the complex equals to $\prod_{q-p=n} A_p^q$

in cohomological degree n and whose differential is given by the alternated sums of the faces and cofaces of the cosimplicial-simplicial diagram. Equivalently, A can be first turned into a bi-complex, by individual totalisations along the cosimplicial and the simplicial directions, and $Tot^{\pi}(A)$ can then be realized as the product-total complex of this bicomplex. Note also that $Tot^{\pi}(A)$ is a model for the homotopy limit-colimit of A

$$Tot^{\pi}(A) \simeq \mathrm{holim}_{[p]\in\Delta^{op}} \left(\mathrm{hocolim}_{[q]\in\Delta} A_p^q\right) \in C(\mathbb{Z}).$$

DEFINITION A.9.1.1. *A morphism $A \to B$ in $csCR$ is a* completed quasi-isomorphism *if the induced morphism $Tot^{\pi}(A) \to Tot^{\pi}(B)$ is a quasi-isomorphism of complexes.*

We warn the reader that even though the categories $csCR$ and $scCR$, of cosimplicial-simplicial commutative rings and of simplicial-cosimplicial rings, are equivalent, the notion of completed quasi-isomorphisms is not the same as the notion of weak equivalences used for instance in [**BCN21**].

THEOREM A.9.1.2. *The category $csCR$ is endowed with a model category structure for which the weak equivalences are the completed quasi-isomorphism, and the fibrations are the epimorphisms. This model category structure is moreover cofibrantly generated and is a simplicial model category for the above mentioned simplicial enrichment.*

Proof. As fibrations[5] and equivalences are defined via the functor Tot^{π}, the proof of the theorem consists of transferring the projective model structure on $C(\mathbb{Z})$ to $csCR$ via the functor Tot^{π}. For this we apply the path object argument (see for instance [**BM03**, §2.6]).

The functor Tot^{π} is a right adjoint whose left adjoint sends a complex $E \in C(\mathbb{Z})$ to the free commutative ring over the cosimplicial-simplicial module $\phi(E)$ given by

$$(p, q) \mapsto (C^*(\Delta^p) \otimes C_*(\Delta^q) \otimes E)^0 / Im\, d^{-1},$$

where $C^*(\Delta^p)$ and $C_*(\Delta^q)$ are the cohomology and homology complexes of the standard simplex Δ^p and Δ^q. This left adjoint will be denoted by $L\phi$, as being the composition of the free commutative ring functor $L : csMod \to csCR$, with the functor $\phi : C(\mathbb{Z}) \to csMod$ defined by the above formula. Clearly, the functor Tot^{π} preserves small objects (but does not commute with filtered colimits in general). In order to apply [**BM03**, §2.6] we thus simply have to prove that $csCR$ possesses a fibrant replacement functor and that all fibrant object possesses a path object. But all objects are fibrant by definition, and we are thus reduced to show the existence of path objects. For this we use the following lemma.

LEMMA A.9.1.3. *For any simplicial set K and any $A \in csCR$, there exists a canonical quasi-isomorphism*

$$Tot^{\pi}(A^K) \simeq Tot^{\pi}(A)^K,$$

where for $E \in C(\mathbb{Z})$ and $K \in sSet$ we define

$$E^K := \mathrm{holim}_{[p]\in\Delta} E^{K_p}.$$

[5]Note that $f : A \to B$ in $csCR$ is surjective iff $Tot^{\pi}(f)$ is.

Proof of the lemma. We remind that for any functor $F : \Delta\times\Delta \to \mathbf{dg}$ (or more generally any bi-cosimplicial object in a complete ∞-category), the homotopy limit of F can be computed, up to a quasi-isomorphism, either component-wise, or diagonally

$$\mathrm{holim}_{[p]\in\Delta}F(p,p) \simeq \mathrm{holim}_{[p]\in\Delta}\mathrm{holim}_{[q]\in\Delta}F(p,q).$$

We apply this to the functor sending $([p],[q])$ to the complex $N(A^q_*)^{K_p}$, where $N : sMod \to dg$ is the normalization construction. Computing the homotopy limit diagonally yields $Tot^\pi(A^K)$, whereas the double homotopy limit construction yields $Tot^\pi(A)^K$. $\square$

Lemma A.9.1.3 now implies the existence of path objects. Indeed, for $A \in csCR$ an explicit path object is given by $A \to A^{\Delta^1} \to A^{\partial\Delta^1} = A\times A$. The fact that the constant map morphism $A \to A^{\Delta^1}$ is a completed quasi-isomorphism follows from the lemma A.9.1.3. The fact that $A^{\Delta^1} \to A^{\partial\Delta^1} = A\times A$ is a fibration simply follows from the fact that it is a levelwise epimorphism as this can be checked easily using the explicit formula for A^K.

In order to finish the proof of the theorem, we are left to showing that the simplicial enrichement is compatible with the model category structure. In other words, we must show that for any cofibration of simplicial sets $i : K \hookrightarrow L$ and any fibration $f : A \to B$ in $csCR$, the induced morphism

$$f^i : A^L \longrightarrow B^L \times_{B^K} A^K$$

is a fibration, which is also an equivalence if i is a trivial cofibration or f is a trivial fibration. The fact that the morphism f^i is a fibration is clear because fibrations are epimorphisms and because of the explicit formula for the exponentiation by a simplicial set in $csCR$. Finally, the fact that f^i is also a weak equivalence when i or f is so simply follows from Lemma A.9.1.3. $\square$

Definition A.9.1.4. *The* ∞-category of cs-rings *is the ∞-category obtained from $csCR$ by inverting the completed quasi-isomorphisms. It is denoted by* $\mathbf{csCR}$ *(or by* $\mathbf{csCR}_{\mathbb{U}}$ *if one wants to specify cosimplicial-simplicial rings belonging to a given universe* $\mathbb{U}$*).*

Remark A.9.1.5. We note also that the model structure on $csCR$ can be equivalently be obtained by first constructing a model structure on $csMod_{\mathbb{Z}}$, the category of cosimplicial-simplicial abelian groups, where again the equivalences and fibrations are defined via the Tot^π functor. The model structure on $csCR$ can then be obtained by transferring along the forgetful functor $csCR \to csMod_{\mathbb{Z}}$, with left adjoint given by the free commutative cosimplicial-simplicial ring.

Remark A.9.1.6. It is possible to a construct an ∞-functor $Tot^\pi : \mathbf{csCR} \longrightarrow LSym - Alg$, from our ∞-category of cosimplicial-simplicial commutative rings to the ∞-category of *derived rings*, namely modules over the $LSym$-monad. Here we denote by $LSym$ the monad associated to the derived commutative operad as in [**BCN21**, Ex. 3.71]. This ∞-functor simply sends A to $\lim_q A^q_*$, where the limit is taken in $LSym - Alg$ and A^q_* is considered as a connective $LSym$-algebra and thus as an object in $LSym - Alg$. This ∞-functor is the right adjoint of an adjunction, and preserves free objects over perfect complexes. It is likely that Tot^{π} is an equivalence of ∞-categories when restricted to nice enough objects.

A.9.2. Spectrum of cosimplicial-simplicial rings. For a fixed commutative simplicial ring $k \in sCR$, we can work relatively over k and define the model category $k - csCR$ of cosimplicial-simplicial commutative k-algebras, and its associated ∞-category $k - \mathbf{csCR}$. As usual, we have a natural equivalence of ∞-categories

$$k - \mathbf{csCR} \simeq k/\mathbf{csCR},$$

where k is considered as an object in $\mathbf{csCR}$ which is constant is the cosimplicial direction.

We consider $dAff_k := (k - sCR)^{op}$, the model category of derived affine k-schemes, which is defined to be the opposite category of that of simplicial commutative k-algebras. Remind from [**TV08**] that it can be endowed with the fpqc model topology, and that we can consider the model category of (hyper-complete) stacks $dAff_k^{\sim,fpqc}$. There are here some set-theoretical issues, that can be solved, as usual, by fixing two universes $\mathbb{U} \in \mathbb{V}$. By definition $k - sCR$ refers here to $\mathbb{U}$-small simplicial k-algebras, and $dAff_k^{\sim,fpqc}$ is then the category of functors $k - sCR \to sSet_{\mathbb{V}}$ to $\mathbb{V}$-small simplicial sets.

We then consider a functor

$$\mathbf{Spec}^{\Delta} : k - csCR_{\mathbb{V}}^{op} \longrightarrow dAff_k^{\sim,fpqc},$$

from $\mathbb{V}$-small cosimplicial-simplicial commutative k-algebras to $dAff_k^{\sim,fpqc}$. It is defined by sending $A \in k - csCR_{\mathbb{V}}$ to the functor $\mathbf{Spec}^{\Delta} A : sCR \to sSet_{\mathbb{V}}$ defined by

$$\mathbf{Spec}^{\Delta} A : B \mapsto \underline{Hom}(A, B) = Hom_{csCR}(A, B^{\Delta^{\bullet}}),$$

where $\underline{Hom}$ are the simplicial Hom's of the simplicial enrichement described in the previous section, and B is considered as an object in $k - csCR_{\mathbb{V}}$ by viewing it as constant in the cosimplicial direction.

Propositon A.9.2.1. *The functor* $\mathbf{Spec}^{\Delta} : k - csCR_{\mathbb{V}}^{op} \longrightarrow dAff_k^{\sim,fpqc}$ *is right Quillen, and the induced functor on ∞-categories*

$$\mathbb{R}\mathbf{Spec}^{\Delta} : k - \mathbf{csCR}_{\mathbb{U}} \longrightarrow \mathbf{dSt}_k$$

is fully faithful. The essential image of $\mathbf{Spec}^{\Delta}$ *is the smallest full sub-∞-category of* $\mathbf{dSt}_k$ *containing the objects $K(\mathbb{G}_a, n)$ for various n, and which is stable by $\mathbb{U}$-small limits.*

Proof. This is proven in a very similar manner than [**Toe06**, Cor. 2.2.3]. The left adjoint to $\mathbf{Spec}^{\Delta}$, denoted by $\mathcal{O}$, sends a representable presheaf $h^B = Hom(B, -) : k - sCR \to sSet$ to $B \in k - csCR$, which is considered as constant in the cosimplicial direction. It also sends an object of the form $K \times h^B$, for a simplicial set $K \in sSet$, to $B^K \in k - csCR$. Finally, it is uniquely defined by these properties together with the requirement that it sends colimits in $dAff_k^{\sim,fpqc}$ to limits in $k - csCR$.

To prove that $\mathbf{Spec}^{\Delta}$ is right Quillen we use that $dAff_k^{\sim,fpqc}$ is a left Bousfield localization of the levelwise projective model structure on the category of simplicial presheaves

$Fun(k - sCR_{\mathbb{U}}, sSet)$ obtained by inverting fpqc-local equivalences and equivalences in sCR (see [**TV08**]). Therefore, by general facts about Bousfield localizations, it is enough to show that $\mathbf{Spec}^{\Delta}$ is a right Quillen functor for the levelwise projective model structure, and moreover that for any cofibrant $A \in k - csCR$, $\mathbf{Spec}^{\Delta}(A)$ is a fibrant object in $dAff_k^{\sim,fpqc}$. The first of these statements easily follows from the fact that $k - csCR$ is a simplicial model category in which every object is fibrant, and thus that if $A \to A'$ is a (trivial) cofibration in $k - csCR$, for any $C \in k - sCR$ the morphism $\underline{Hom}(A', C) \to \underline{Hom}(A, C)$ is a (trivial) fibration of simplicial sets. For the second of these statements, let $A \in k - csCR$ be a cofibrant object. Any equivalence $B \to B'$ in $k - sCR$ obviously induces an equivalence in $k - csCR$ when considered both B and B' as constant in the cosimplicial direction, and thus $\underline{Hom}(A, B) \to \underline{Hom}(A, B')$ is an equivalence of simplicial sets. This shows that $\mathbf{Spec}^{\Delta}(A)$ is fibrant when the topology is the trivial topology. Finally, if $B = \mathrm{holim}_i B_i$ is a homotopy limit in $k - sCR$, it is also a homotopy limit in $k - csCR$, and thus the natural morphism $\underline{Hom}(A, B) \to \mathrm{holim}_i \underline{Hom}(A, B_i)$ is an equivalence. As the fpqc model topology is subcanonical, this implies that $\mathbf{Spec}^{\Delta}(A)$ is indeed a stack for the fpqc topology, and thus fibrant as an object in $dAff_k^{\sim,fpqc}$ (see [**TV08**]).

To finish the proof of the Proposition, let $A \in k - csCR_{\mathbb{U}}$ and $X = \mathbf{Spec}^{\Delta}(A)$. By definition of the simplicial structure on $k - csCR$, the functor X is the geometric realization of the simplicial object $q \mapsto h^{A_*^q}$, where $h^{A_*^q} : k - sCR \to sSet$ is corepresented by $A_*^q \in k - sCR$. In other words, we have $\mathbf{Spec}^{\Delta}(A) \simeq \mathrm{hocolim}_q h^{A_*^q}$. Therefore, the adjunction morphism

$$A \longrightarrow \mathcal{O}(\mathbf{Spec}^{\Delta}(A))$$

is the canonical morphism $A \to \mathrm{holim}_q A_*^q$ in $Ho(k - csCR_{\mathbb{U}})$. This canonical morphism is obviously an equivalence, showing the fully faithfulness property in the Proposition. It is then formal that the essential image of $\mathbb{R}\mathbf{Spec}^{\Delta}$ is stable by limits. It also contains the objects $K(\mathbb{G}_a, n)$, as these are the images of $L\phi(\mathbb{Z}[-n])$, where $L\phi : C(\mathbb{Z}) \to k - csCR$ is the left adjoint to Tot^{π}. Finally, as $k - csCR$ is cofibrantly generated, any object is equivalent to a I-cell object, where I is the image by $L\phi$ of the generating cofibrations in $C(\mathbb{Z})$. The images of I by $\mathbf{Spec}^{\Delta}$ are equivalent to morphisms of the form $* \to K(\mathbb{G}_a, n)$, so that any object is the essential image of $\mathbb{R}\mathbf{Spec}^{\Delta}$ lies in the smallest sub-∞-category containing the $K(\mathbb{G}_a, n)$ and stable by limits. □

Definition A.9.2.2. *The ∞-category of* (k-linear) derived affine stacks *is the essential image of* $\mathbb{R}\mathbf{Spec}^{\Delta} : k - \mathbf{csCR}_{\mathbb{U}}^{op} \hookrightarrow \mathbf{dSt}_k$. *It is denoted by* $\mathbf{dChAff}_k$.

The above notions and results also have *graded versions*, for which the details are left to the reader. We denote by $k - csCR^{gr}$ the category of graded cosimplicial-simplicial commutative k-algebras. We endow this category with the model category structure for which equivalences and fibrations are defined by the graded Tot functor to graded complexes $Tot^{\pi} : k - csCR^{gr} \to C(\mathbb{Z})^{gr}$, defined by taking the Tot^{π} of each graded component. The proof of the existence of this model category structure follows the same lines as in the non-graded case. The corresponding ∞-category will be denoted by $k - \mathbf{csCR}^{gr}$.

We consider the strict multiplicative group object $\mathbb{G}_m^{st}$ in the model category $dAff_k^{\sim,fpqc}$ of fpqc (hyper-complete) stacks (see Section A.9.2). $\mathbb{G}_m^{st}$ is defined as the functor sending $B \in k - sCR$ to the group B_0^* of invertible elements in the ring B_0 of 0-simplices in B. This is a group object in the model category $dAff_k^{\sim,fpqc}$, which is not a fibrant object (it does not preservers equivalences in B), but its image in $\mathbf{dSt}_k$ is the usual multiplicative group scheme $\mathbb{G}_m$. Indeed, $\mathbb{G}_m^{st} = h^{k[t,t^{-1}]}$ is corepresented by $k[t,t^{-1}]$, and, as shown in [**TV08**], a fibrant model for h^A is the derived scheme $\mathbf{Spec}\, A : B \mapsto Map(A,B)$. In particular, we have that the model category $\mathbb{G}_m^{st} - dAff_k^{\sim,fpqc}$, of $\mathbb{G}_m^{st}$-equivariant objects in $dAff_k^{\sim,fpqc}$, is a model for the ∞-category $\mathbf{dSt}_k^{\mathbb{G}_m}$, of $\mathbb{G}_m$-equivariant derived stacks, called *graded derived k-stacks.*

The graded version of the spec functor is the functor

$$\mathbf{Spec}^{\Delta,gr} : (k - csCR^{gr})^{op} \longrightarrow \mathbb{G}_m^{st} - dAff_k^{\sim,fpqc},$$

sending $A \in k - csCR^{gr}$ to $\underline{Hom}(A,-)$, endowed with the natural $\mathbb{G}_m^{st}$-action coming from the grading on A. More explicitly, for $B \in k - sCR$, the set of q-simplices of $\mathbf{Spec}^{\Delta,gr}(A)(B)$ is the set of morphisms $Hom(A^q, B)$, which is endowed with a B_0^*-action as follows. For $b \in B_0^*$ and $f : A^q \to B$, we define $bf : A^q \to B$ by the formula $(bf)(x) = b^n.f(x)$ for a homogeneous element x of degree n. As in the non-graded case, $\mathbf{Spec}^{\Delta,gr}$ is a right Quillen functor, and the induced ∞-functor

$$\mathbb{R}\mathbf{Spec}^{\Delta,gr} : (k - \mathbf{csCR}_{\mathbb{U}}^{gr})^{op} \longrightarrow \mathbf{dSt}_k^{\mathbb{G}_m}$$

is fully faithful. Its essential image is the smallest sub-∞-category containing all the objects $K(\mathbb{G}_a^{\chi}, n)$, for $n \geq 0$ and $\chi \in \mathbb{Z}$, where $\mathbb{G}_m$ acts on $\mathbb{G}_a$ with weight $\chi \in \mathbb{Z}$, and which is stable by $\mathbb{U}$-small limits. This is the graded analog of Proposition A.9.2.1.

DEFINITION A.9.2.3. *The ∞-category of* graded derived affine k-stacks *is the essential image of $\mathbb{R}\mathbf{Spec}^{\Delta,gr} : (k - \mathbf{csCR}_{\mathbb{U}}^{gr})^{op} \hookrightarrow \mathbf{dSt}_k^{\mathbb{G}_m}$. It is denoted by $\mathbf{dChAff}_k^{gr}$.*

We finish this section by recalling the notion of *linear derived stack*, already introduced and studied in [**Mon21**]. We first notice that the Tot functor $Tot^{\pi} : csMod_k \longrightarrow C(\mathbb{Z})$ admits a natural lax monoidal structure. Therefore, it gives rise to an ∞-functor

$$Tot^{\pi} : \mathbf{csMod}_k \longrightarrow \mathbf{dg}_k,$$

where $\mathbf{csMod}_k$ is the ∞-category of cosimplicial-simplicial k-modules. This ∞-functor turns out to be an equivalence of ∞-categories, and we denote by $\phi : \mathbf{dg}_k \to \mathbf{csMod}_k$ its inverse.

For any object $E \in \mathbf{dg}_k$, we consider $\phi(E) \in \mathbf{csMod}_k$, and define

$$Sym^{\Delta}(E) := L\phi(E) \in csCR^{gr}$$

the free cosimplicial-simplicial commutative ring generated by $\phi(E)$ (here, $L\phi$ is the left adjoint to the Tot^{π} functor, same notation as in the proof of Theorem A.9.1.2). Being a free commutative ring, $Sym^{\Delta}(E)$ comes naturally equipped with a graduation, and thus is considered as an object in the ∞-category $\mathbf{csCR}^{gr}$.

Definition A.9.2.4. *The* k-linear derived stack associated to a complex $E \in \mathbf{dg}_k$, *is the graded derived affine* k*-stack defined by*

$$\mathbb{V}(E) := \mathbb{R}\mathbf{Spec}^{\Delta,gr}\left(Sym^{\Delta}(E)\right) \in \mathbf{dSt}_k^{\mathbb{G}_m}.$$

Note that by construction the functor of points of $\mathbb{V}(E)$ sends $B \in k{-}sCR$ to $Map_{\mathbf{dg}_k}(E, N(B))$, where $N(B)$ is the normalisation of B as a simplicial k-module. The $\mathbb{G}_m$-action is then the natural action of invertible elements of B on B itself.

It is proven in [**Mon21**] that the ∞-functor sending E to $\mathbb{V}(E)$ is fully faithful when restricted to bounded above k-dg-modules E, and in particular for perfect k-dg-modules E.

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