GENERALIZED LOCAL JOCABIANS AND COMMUTATIVE GROUP STACKS

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Abstract. In [CS01, Page 109] Grothendieck sketches the construction of a complex $J_*(X)$ or commutative pro-algebraic groups, associated to a smooth variety X, and for which each $J_i(X)$ is a product of local factors called the local generalized jacobians. The purpose of this note is to recast this construction in the setting of higher algebraic group stacks for the fppf topology. For this, we introduce a notion of algebraic homology associated to a scheme which is a universal object computing fppf cohomology with coefficients in group schemes. We endow this algebraic homology with a filtration by dimension of supports, and prove that, when X is smooth, $J_*(X)$ appears as the E_1-page of the corresponding spectral sequence. In a final part we partially extends our constructions and results over arbitrary bases.

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Introduction

The main purpose of this note is to introduce a notion of *algebraic homology*, associated to schemes over a perfect field, which is a universal object controlling flat cohomology with coefficients in certain commutative group schemes. The idea of algebraic homology is certainly

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not new, and was already sketched in a slight different setting in a letter of Grothendieck to Serre (see letter from 8.9.1960 in [CS01, page 109]), as well as in [Gro] (cote 134-2 page 60) and in [Mil76] for instance. However, in the present work we realize algebraic homology as a universal (pro)object in the ∞-category of commutative groups in algebraic higher stacks, whereas in the above references it is constructed as a complex of commutative pro-algebraic groups. The main result of this work is a comparison statement for smooth schemes, and states that our notion of algebraic homology recovers Grothendieck's orginal construction. This comparison is not a formal statement as it relates two objects living in two non-equivalent categories, and requires to use an extra piece of data in the form of a filtration by dimension of supports. As filtered objects, Grothendieck's original construction is shown to appear as the 0-th truncation of algebraic homology, for the Beilinson's t-structure on filtered objects (see for instance [Pav23]). This is a fancy manner to states that Grothendieck's complex of pro-algebraic groups appears as the E_1-page of the spectral sequence associated to the filtration by dimension of supports on algebraic homology. We also show that both objects can be used in order to compute fppf cohomology with coefficients in commutative group schemes (with some affiness and unipotency conditions for higher cohomology groups). This last statement is true by definition for algebraic homology but requires a proof in the case of Grothendieck's construction which is done by gluing together the so-called *local generalized jacobians* appearing in our title.

We work over a perfect base field k, and we work in the ∞-topos of hypercomplete fppf stacks $\mathbf{St}_k$. In a first part of this work we introduce the notions of *commutative algebraic group stacks over* k (see definition 1.2), which are the commutative group objects inside the ∞-category of Artin stacks locally of finite type over k and whose diagonal are quasi-compact (in a strong sense). These are higher stacks analogues of *locally algebraic groups over* k, that is separated commutative group schemes locally of finite type. The commutative algebraic group stacks can also be characterized as connective complexes of sheaves of abelian groups E of the big fppf site, whose cohomology sheaves $H^i(E)$ are all representable by locally algebraic groups, and are moreover quasi-projective if $i < 0$. Among the commutative algebraic group stacks we isolate the subclass of *semi-unipotent* objects. These are characterized by the further conditions that $H^{-1}(E)$ is affine and that $H^i(E)$ is affine and unipotent when $i < -1$ (see definition 2.1). This kind of conditions already appears under the name *very presentable* in the context of non-abelian Hodge theory (see [Sim96]) and in the context of schematic homotopy types (see [Toe06]). The algebraic homology of a scheme X over k (or any stack over k) can then be defined as follows (see definition 3.1).

Definition 0.1. *For a scheme X over k, its algebraic homology $\mathbb{H}^{su}(X)$ is the pro-object in semi-unipotent commutative algebraic group stacks which pro-represents the ∞-functor $E \mapsto \mathbf{Map}(X, E)$.*

The existence of the pro-object $\mathbb{H}^{su}(X)$ in the definition above follows formally from the left exactness of $E \mapsto \mathbf{Map}(X, E)$. However, its existence already carries interesting cohomological information. For instance, the universal morphism $X \to \mathbb{H}^{su}(X)$ can be seen to be a generalization of the Albanese morphism of X which includes higher cohomological data in the picture.

In a way, $\mathbb{H}^{su}(X)$ could also be considered as a derived version of the classical Albanese variety of X (see end of §2).

In order to express $\mathbb{H}^{su}(X)$ in more explicit terms we introduce the standard filtration by dimension of supports

$$0 \to F_{dimX}\mathbb{H}^{su}(X) \cdots \to \dots F_i\mathbb{H}^{su}(X) \to F_{i-1}\mathbb{H}^{su}(X) \cdots \to F_0\mathbb{H}^{su}(X) = \mathbb{H}^{su}(X)$$

where $F_i\mathbb{H}^{su}(X)$ is the part of the algebraic homology which is supported in codimension i in X. The d-th graded piece of this filtration is given by the product of all local algebraic homologies over all the points of codimension d in X (see proposition 3.2)

$$Gr_d\mathbb{H}^{su}(X) \simeq \prod_{x \in X^{(dimX-d)}} \mathbb{H}^{su}_x(X).$$

When X is furthermore smooth over k, purity implies that each graded piece $Gr_d\mathbb{H}^{su}(X)$ is in fact $(n-d)$-connective (see corollary 3.5). In another word, the filtered object $F_*\mathbb{H}^{su}(X)$ is 0-connective with respect to Beilinson's t-structure on filtered objects. As being connective, we get a canonical projection of filtered objects

$$F_*\mathbb{H}^{su}(X) \longrightarrow H^0_B(F_*\mathbb{H}^{su}(X))$$

where H^0_B refers to the 0-th cohomology object for the Beilinson t-structure. This object is a genuine complex made of commutative pro-algebraic groups, and as such can be compared to Grothendieck's original construction $J_*(X)$. In the complex $J_*(X)$, the degree d part is the product of local generalized jacobian $J_x(X)$ over all points x of codimension d in X which are defined to pro-represent local cohomology in degree d (see definition 3.7).

Theorem 0.2. *We have a natural isomorphism of complexes of pro-algebraic groups over k*

$$H^0_B(F_*\mathbb{H}^{su}(X)) \simeq J_*(X)^1.$$

The theorem below, shows that $J_*(X)$ appears as the 0-th truncation of $F_*\mathbb{H}^{su}(X)$. The natural projection $F_*\mathbb{H}^{su}(X) \to J_*(X)$ is not an equivalence in general, and it is thus surprising that both objects compute the same cohomologies, as explained in the next result.

Theorem 0.3. *Let $i \geq 0$ and G a commutative algebraic group G, affine if $i > 0$ and unipotent if $i > 1$. Then, the canonical projection $F_*\mathbb{H}^{su}(X) \to J_*(X)$ induces isomorphisms*

$$Ext^i_{naive}(J_*(X), G) \simeq H^i_{fppf}(X, G).$$

In the theorem above the Ext^i_{naive} refers to the Yoneda ext-groups computed inside the derived category of complexes of commutative pro-algebraic groups. These ext-groups differ from the ext-groups computed as fppf sheaves, and thus are not given by ext-groups computed inside our category of commutative group stacks. We note that the theorem above is already stated in [CS01, page 109] where the fppf topology is replaced with the Zariski topology.

[1]The statement is slightly wrong here as there is a small discrepancy in degree 0, which is a consequence of some choices in the definitions. See corollary 3.8 for more details.

All the results mentioned above are true over a perfect field k, and we do not know a way to extend them over more general base schemes, at least in an interesting manner. However, when restricted to unipotent coefficients, we propose a generalization of algebraic homology over any base ring, based on our notion of affine stacks (see [Toe06]) and their abelian counter-parts which we call *affine homology types*. We prove that any (small) stack X, in particular any scheme, maps to a universal affine homology types called its affine homology and denoted by $\mathbb{H}^u(X)$ (see proposition 4.6 and definition 4.8). Over a field, $\mathbb{H}^u(X)$ is the unipotent completion of algebraic homology, and its advantage is that it makes sense over any base. Affine homology comes equiped with a filtration by dimension of supports whose associated graded are given by local affine homologies. Moreover, we prove that $\mathbb{H}^u(X)$ is completely characterized by the complex $\mathbb{H}(X, \mathbb{G}_a)$ considered as a dg-algebra over $\mathbb{R}End(\mathbb{G}_a)$, which we consider as a form of Dieudonné theory for affine homology types.

To finish this introduction we would like to mention several related works. As we already mentioned the notion of algebraic homology already appears in [CS01, page 109]), and and [Mil76]. The results of the present work does not pretend to be very original and the main point of this note was rather to recast these works in the more modern language of higher stacks. The only exception is possibly our last section on affine homology, which we think can be of independent interest.

In another direction, the results of this work have some limitations. First of all the fact that we work over a base field is restrictive. Over more general bases, the 1-truncation of algebraic homology essentially appears in the work of S. Brochard in [Bro21]. Extending the results of [Bro21] seems however difficult because of the presence of complicated higher ext groups between commutative group schemes which prevent to a perfect duality by mapping to $\mathbb{G}_m$. Also, the approach in [Bro21] is based on duality, which is an aspect we havent touched at all in this note. Grothendieck already mentions duality in [Gro, 134-2], and in degree 0 this duality of local generalized jacobians have been studied in [CC90]. Also, our comparison requires X to be smooth. In his letter Grothendieck suggests an extension of the definition of $J_*(X)$ to the singular setting as well, but we could not reconstruct from it anything meaningful at the moment. An extra limitations concern the unipotency conditions for higher homotopy groups we have in the definition of algebraic homology. It is certainly a very interesting question to drop this condition, for instance only assuming affiness, but that some of our results will not remain correct in this more general setting (typically proposition 3.9).

Finally, there are also relations with algebraic homotopy types, as studies for instance in [Toe06, KPT09, BV15, MR23]. Our notion of affine homology is the ablelian analogue of these, and the two should be related by means of the Hurewitz morphism. As a final comment, affine stacks are duals to cosimplicial commutative algebras, also called $LSym$-algebras. These are Koszul dual to partition Lie algebras, thanks to [BM23], and therefore our notion of affine homology should be closely related to abelian partition Lie algebras.

1. Commutative algebraic group stacks

We work over a perfect base field k. We denote by $\mathbf{Aff}_k$ the category of affine k-schemes, which will be endowed with the fppf topology. The ∞-category of (hypercomplete) stacks on $\mathbf{Aff}_k$ is denoted by $\mathbf{St}_k$. Similarly, $\mathbf{dAff}_k$ denotes the ∞-category of derived affine k-schemes.

1.1. **Reminders on abelian group schemes.** We denote by $\mathbf{AbSch}_k$, the category of commutative group schemes over k which are separated and of finite type over k (or equivalently quasi-projective over k, see for instance [sta] lemma 39.8.2, tag 0BF6). The Yoneda embedding provides a fully faithful functor

$$\mathbf{AbSch}_k \hookrightarrow \mathbf{Ab}_k,$$

from $\mathbf{AbSch}_k$ to the category $\mathbf{Ab}_k$ of sheaves of abelian groups on $\mathbf{Aff}_k$ endowed with the $fppf$-topology. It is well known that this full embedding commutes with the formations of kernels and cokernels, and that its essential image of this full embedding is also stable by extensions. We do not know a precise reference for this, so we include it as a lemma.

Lemma 1.1. *The full sub-category of $\mathbf{Ab}_k$ formed by sheaves which are representable by commutative separated group schemes of finite type over k, is stable by taking kernels, cokernels and extensions.*

Proof. The stability by kernels is obvious as the Yoneda embedding is left exact. For the stability by cokernels, the quotient fppf-sheaf G/H, for a monomorphism of commutative algebraic groups $H \hookrightarrow G$, is always representable by an Artin stack whose diagonal is a monomorphism, that is by an algebraic space. This algebraic space is moreover quasi-separated (because H is quasi-compact) and of finite type. We can thus use Artin's theorem stating that a group object in the category of quasi-separated algebraic spaces over a field is representable by a scheme (see [Art69, Lem. 4.2]). Finally, if a sheaf of abelian groups is an extension of algebraic groups, then it is, by descent, representable by a quasi-separated algebraic space, and we can conclude using again Artin's result. □

The Barsotti-Chevalley decomposition theorem (see [Mil13]) states that any object $G \in \mathbf{AbSch}_k$ fits in an exact sequence

$$0 \to G_0 \to G \to A \to 0,$$

where G_0 is affine and A is an abelian variety. Moreover, by [DG70, II-5 Cor. 2.3] G_0 itself sits in an exact sequence $0 \to K \to G_0 \to H \to 0$, where K is smooth and affine, and H is finite infinitesimal (i.e. $K_{red} = 0$). We also recall from [DG70, IV-2 Porp. 2.5] that $G \in \mathbf{AbSch}_k$ is unipotent if it admits a finite filtration by sub-group schemes

$$\cdots \subset G_{i+1} \subset G_i \cdots \subset G_1 \subset G_0 = G$$

such that each G_i/G_{i+1} can be realized as a sub-group scheme the additive group $\mathbb{G}_a$. We can moreover arrange things so that each G_i/G_{i+1} is a twisted form of $\mathbb{Z}/p$ or of α_p. Unipotent group schemes are automatically affine, and form a full sub-category of $\mathbf{AbSch}_k$ stable by taking kernels, cokernels and extensions.

1.2. **Commutative algebraic group stacks.** We denote by $\mathbf{AbSt}_k$ the ∞-category obtained from $\mathbf{St}_k$ by $\mathbb{Z}$-linearization. It can be described explicitly as the ∞-category of ∞-functors

$$E : \mathbf{Aff}_k^{op} \to \mathbf{dg}_{\mathbb{Z}}^c,$$

from affine k-schemes to connective complexes of abelian groups, satisfying the usual $fppf$ descent condition. Any object $E \in \mathbf{Ab}_k$ possesses an underlying derived stack $|E| \in \mathbf{St}_k$ obtained by the Dold-Kan construction, explicitly defined by

$$|E|(S) := \mathbf{Map}_{\mathbf{dg}_{\mathbb{Z}}}(\mathbb{Z}, E(S))$$

for any $S \in \mathbf{Aff}_k$. This defines a forgetful ∞-functor $\mathbf{Ab}_k \to \mathbf{dSt}_k$, which admits a left adjoint sending F to $\mathbb{Z} \otimes F$. Recall that for $E \in \mathbf{AbSt}_k$ we denote by $\pi_i(E)$ the cohomology sheaves $H^{-i}(E)$, which are also the homotopy sheaves $\pi_i(|E|)$ of the underlying stack $|E|$ pointed at 0.

Definition 1.2. *A* commutative algebraic group stack *is an object $E \in \mathbf{AbSt}_k$ such that underlying stack $|E| \in \mathbf{St}_k$ is an Artin n-stack (for some n), locally of finite presentation and strongly quasi-separated over k. They form, by definition, a full sub-∞-category*

$$\mathbf{AbSt}_k^{alg} \subset \mathbf{AbSt}_k.$$

Remind that *strongly quasi-separated* means that the diagonal map $E \to E \times E$ is strongly quasi-compact in the sense of [TV08]. This means that all the higher diagonal maps $X \to X^{S^n}$ are quasi-compact for $n \geq 0$. Equivalently, the inertia stack $IE \to E$ is required to be strongly quasi-compact over E. As explained in [Toe05, Cor. 2.9] an Artin stack E locally of finite presentation which is strongly quasi-separated is such that for all field value point $s \in E(K)$, all its homotopy sheaves $\pi_i(E, s)$ are quasi-compact and quasi-separated algebraic spaces over K, and thus are representable by group schemes thanks to [Art69, Lem. 4.2]. As we are working over a field, the converse also holds.

Propositon 1.3. *An object $E \in \mathbf{AbSt}_k$ is a commutative algebraic group stack if and only if each homotopy sheaf $\pi_i(E) \in Ab_k$ is representable by a group scheme locally of finite type over k, which is moreover quasi-projective for $i > 0$ and vanish for all but a finite number of indices i.*

Proof. Assume first that E is a commutative algebraic group stack. In the definition 1.2. We have to prove that all the homotopy sheaves $\pi_i(E)$ are representable. We consider the inertia stack IE of E. We already saw that $\pi_i(E)$ are group schemes for $i > 0$ by means of [Toe05, Cor. 2.9], it remains to treat the case $i = 0$. As E is an abelian group object we have splitting $IE \simeq E \times \Omega_0 E$, where $\Omega_0 E$ is the pointed loop stack at 0. As k is a field, we thus have that $IE \to E$ is a fppf cover which is strongly quasi-compact. We can thus use [Toe05, Prop. 2.4], which remains valid only assuming the diagonal to be strongly quasi-compact, and deduce that E is a gerbe. In particular, $\pi_0(E)$ is representable by an algebraic space locally of finite type over k. This is a group object in algebraic space, and is moreover quasi-separated by assumptions on E, and by [Art69, Lem. 4.2] is representable by a group scheme locally of finite type.

Conversely, assume that E satisfies the conditions of the proposition. We will prove by induction on i that the i-th homotopical truncation $\tau_{\leq i}E$ is an Artin i-stack locally of finite presentation and strongly quasi-separated over k. For $i = 0$ this is obvious, as $\pi_0(E)$ is a scheme locally of finite type by assumption. Consider $E_{\leq i} \to E_{\leq i-1}$, which fits into a cartesian square

$$\begin{array}{ccc} E_{\leq i} & \longrightarrow & E_{\leq i-1} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & K(\pi_i(E), i+1). \end{array}$$

By assumption, $\pi_i(E)$ is a group scheme of finite type over k for $i > 0$, and thus $K(\pi_i(E), i+1)$ is an Artin $(i+1)$-stack locally of finite presentation and strongly quasi-separated k. By induction, this implies that $E_{\leq i}$ is an Artin i-stack locally of finite presentation and strongly quasi-separated. □

By using the long exact sequence in homotopy, we see that the proposition 1.3 easily implies the following corollary.

Corollary 1.4. *The full sub-∞-category $\mathbf{AbSt}_k^{alg} \subset \mathbf{AbSt}_k$ is a is stable by finite limits, retracts and the homotopical truncation ∞-functors $\tau_{\leq i}$.*

Remark 1.5. We warn the reader that if E is a commutative algebraic group stack then BE is in general not a commutative algebraic group stack anymore. It is so only if $\pi_0(E)$ is moreover assumed to be of finite type.

We will also use occasionally the following bigger category.

Definition 1.6. *A* commutative locally algebraic group stack *is an object $E \in \mathbf{AbSt}_k$ for all i the sheaf $\pi_i(E)$ is representable by an algebraic space locally of finite type. The ∞-category of commutative locally algebraic group stacks is denoted by $\mathbf{AbSt}_k^{lalg} \subset \mathbf{AbSt}_k$.*

As opposed to algebraic group stacks, the locally algebraic stacks are stable by suspension: if $E \in \mathbf{AbSt}_k^{lalg}$, then BE, computed in $\mathbf{AbSt}_k$, remains in $\mathbf{AbSt}_k^{lalg}$. More generally, $\mathbf{AbSt}_k^{lalg}$ is stable by finite limits and colimits, as this can be easily seen using the long exact sequence in homotopy.

2. Algebraic homology

In this section we define algebraic homology. We will work with certain coefficients and define algebraic homology accordingly. We start by the introduction of some extra conditions on commutative algebraic groups stacks.

Definition 2.1. *Let E be a commutative algebraic group stack over k.*

1. *We say that E is* semi-affine *if $\pi_i(E)$ is affine for $i>0$.*
2. *We say that E is* semi-unipotent *if E is semi-affine and $\pi_i(E)$ is unipotent for all $i>1$.*
3. *We say that E is* unipotent *if E is semi-unipotent and $\pi_0(E)$ is affine and unipotent.*

We denote by $\mathbf{AbSt}_k^{su} \subset \mathbf{AbSt}_k$ the full sub-∞-category consisting of semi-unipotent commutative algebraic group stacks over k.

In the sequel below we will focus on semi-unipotent objects, and will study unipotent coefficients in a more general setting in our last section on affine homology. Note that semi-unipotent commutative algebraic groups stacks are stable by finite limits and retracts taken in $\mathbf{AbSt}_k$, by considering the long exact sequence in homotopy.

We let X be any stack over k. We consider the ∞-functor

$$\mathbf{AbSt}_k^{su} \longrightarrow \mathsf{Top}$$

defined by $E \mapsto \mathbf{Map}(X,E)$. This ∞-functor will be denoted by $E \mapsto \mathbb{H}^*(X,E)$. By definition it obviously commutes with finite limits, and thus is pro-representable by an object

$$\mathbb{H}^{su}(X) \in Pro(\mathbf{AbSt}_k^{su}).$$

Definition 2.2. *With the notation above, the object $\mathbb{H}^{su}(X)$ is called the* (semi-unipotent) algebraic homology of X (relative to k)*.*

We will often drop the adjective semi-unipotent in the definition above and simply refer to $\mathbb{H}^{su}(X)$ as the algebraic homology of X. Note also that definition 2.2 makes sense for any stack X over k, possibly non-representable by an Artin stack. However, we will mainly be interested in the case where X is a scheme of finite type over k. Note also that algebraic homology is a relative notion and depends on the base field k. It should therefore be noted by $\mathbb{H}^{su}(X/k)$ when mentioning the base field is important.

Remark 2.3. The notion above of algebraic homology is very close to that of flat homology of [Mil76], but is different. A first difference is that these two objects live in two different, non-equivalent, ∞-categories. Flat homology lives in the ∞-category of complexes of commutative pro-algebraic groups. This has a canonical ∞-functor to $\mathbf{AbSt}_k$, by considering the stack pro-represented by a complex of pro-alegebraic groups, but this ∞-functor is known not to be fully faithful. This is due to the existence of two different, non-isomorphic, definitions of ext-groups of commutative algebraic groups: Yoneda ext-groups inside algebraic groups and ext-groups of the corresponding fppf-sheaves. It is known that these two notions are different (see for instance [Bre69]).

A second difference is the extra conditions of unipotency we use in definition algebraic homology which is not made in [Mil76]. These kind of extra conditions already appears in a different context as very presentable coefficients for the purpose of non-abelian Hodge theory, see [Sim96]. Definition 2.2 remains valid without the semi-unipotent condition, however the comparison with Grothendieck's original construction can only be made in the semi-unipotent context. The general study of algebraic homology outside of the semi-unipotent context is thus not considered in the present work (but is certainly interesting).

By definition the algebraic homology possesses the following basic properties.

Functoriality. The construction $X \mapsto \mathbb{H}^{su}(X)$ is functorial in X, and defines an ∞-functor from Artin k-stack to pro-object in derived commutative algebraic groups. Moreover, the identity of $\mathbb{H}^{su}(X)$ defines a canonical class $\alpha_X \in \mathbb{H}^0(X, \mathbb{H}^{su}(X))$, and thus a canonical morphism of pro-stacks

$$\alpha_X : X \to \mathbb{H}^{su}(X),$$

which is functorial in X as well.

Similarly, the canonical element $\alpha_X^u \in \mathbb{H}^0(X, \mathbb{H}^u(X))$, corresponding to the identity of $\mathbb{H}^u(X)$, induces an natural morphism $\mathbb{H}^{su}(X) \to \mathbb{H}^u(X)$ which is functorial in X. This morphism must be understood as *the universal pro-unipotent quotient.*

Homology of the point. We write $\mathbb{H}^{su}(k)$ for $\mathbb{H}^{su}(\mathbf{Spec}\, k)$. It is easy to see that $\mathbb{H}^{su}(k) \simeq \mathbb{Z}$ is the constant fppf sheaf $\mathbb{Z}$.

Homology groups. The pro-object $\mathbb{H}^{su}(X)$ possesses pro-homology groups $\mathbb{H}_i^{su}(X) := \pi_i(\mathbb{H}^{su}(X))$, which are commutative pro-algebraic groups over k. By definition of semi-unipotent these pro-algebraic groups are linear (or affine) for $i > 0$ and unipotent for $i > 1$. These are covariantly associated to X and are called the *algebraic homology groups.*

In general, the projection $X \to \mathbf{Spec}\, k$ provides a canonical morphism $\mathbb{H}^{su}(X) \to \mathbb{H}^{su}(\mathbf{Spec}\, k) \simeq \mathbb{Z}$. We thus have a canonical fibration sequence of commutative algebraic group stacks

$$\mathbb{H}^{su}(X)^{(0)} \to \mathbb{H}^{su}(X) \to \mathbb{H}^{su}(k) \simeq \mathbb{Z},$$

where $\mathbb{H}^{su}(X)^{(0)}$ is by definition the reduced algebraic homology of X. In general the fibration sequence above does not split (see below), and a choice of a rational point $x \in X(k)$ provides a canonical splitting $\mathbb{H}^{su}(X) \simeq \mathbb{Z} \oplus \mathbb{H}^{su}(X)^{(0)}$ and corresponding splitting on homology groups.

Relation to the Albanese variety. The pro-algebraic group $\mathbb{H}_0^{su}(X)$ is directly related to the Albanese variety of X. Indeed, for any commutative algebraic group G, we have a canonical bijection

$$H^0(X, G) = Hom(X, G) \simeq \mathbb{H}^0(X, G) \simeq Hom(\mathbb{H}_0^{su}(X), G).$$

Fixing a point $x \in X(k)$, we get a decomposition $\mathbb{H}_0^{su}(X) \simeq \mathbb{H}_0^{su}(X)^{(0)} \oplus \mathbb{Z},$. When X has no k-point, the projection $X \to \mathbf{Spec}\, k$ provides a (a priory non-split) short exact sequence

$$0 \to \mathbb{H}_0^{su}(X)^{(0)} \to \mathbb{H}_0^{su}(X) \to \mathbb{Z} \to 0.$$

We can consider the universal morphism $X \to \mathbb{H}^{su}(X)$, and check that the composite morphism $X \to \mathbb{H}_0^{su}(X) \to \mathbb{Z}$ is the constant function 1 on X. It thus factors through a canonical morphism

$$alb_X : X \to \mathbb{H}_0^{su}(X)^{(1)}$$

where $\mathbb{H}_0^{su}(X)^{(1)}$ is the fiber at 1 of the short exact sequence above, considered naturally as a torsor over the group $\mathbb{H}_0^{su}(X)^{(0)}$. This morphism is a generalization of the usual Albanese morphism to the Albanese variety. When X is smooth and proper, $\mathbb{H}_0^{su}(X)^{(1)}$ can be identified with the usual Albanese variety, and alb_X with the usual the Albanese map.

In general, the universal morphism $X \to \mathbb{H}^{su}(X)$, composed with the projection $\mathbb{H}^{su}(X) \to \mathbb{Z}$ is the constant map $X \to \mathbb{Z}$ equals to 1. Therefore, we have a canonical morphism $X \to \mathbb{H}^{su}(X)^{(1)}$ where $\mathbb{H}^{su}(X)^{(1)}$ is the fiber at 1 of $\mathbb{H}^{su}(X) \to \mathbb{Z}$. This morphism is a higher cohomological generalization of the Albanese morphism as being the universal map towards a torsor over an object in $Pro(\mathbf{AbSt}_k^{su})$.

Relation to the Albanese stack. When X is an Artin stack of finite type over X, the first Postnikov truncation $\mathbb{H}_{\leq 1}^{su}(X)$ is an abelian group object in algebraic pro-Artin 1-stacks. This object is closely related $D(Pic(X))$, the dual of the Picard stack of X (relative to k) studied in [Bro21]. Indeed, $Pic(X) := \underline{\mathbf{Map}}(X, B\mathbb{G}_m)$ is the internal Hom stack from X to $B\mathbb{G}_m$, and we thus have a canonical evaluation morphism $ev : X \times Pic(X) \to B\mathbb{G}_m$ which defines a canonical morphism

$$X \to \underline{\mathbf{Map}}(Pic(X), B\mathbb{G}_m) = D(Pic(X)).$$

When X is proper and the dual $D(Pic(X))$ is known to be a representable group stack, it is semi-unipotent. Indeed, we know that $\pi_1(D(Pic(X)))$ is an affine group scheme over k, and thus pro-algebraic and affine. Moreover, $Pic(X)$ is a commutative algebraic group stack and sits in an exact sequence $0 \to B\mathbb{G}_m \to Pic(X) \to \pi_0(Pic(X)) \to 0$. Therefore, by duality we get a fibration sequence of commutative group stacks

$$0 \to D(\pi_0(Pic(X))) \to D(Pic(X)) \to \mathbb{Z} \to 0.$$

The choice of a global point $x \in X(k)$ defines a splitting of this sequence, and thus the evaluation map $X \to D(Pic(X))$ can be composed to a morphism $X \to D(H^0(Pic(X))$. By the universal property of algebraic homology we get a canonical morphism of pro-objects in commutative algebraic 1-stacks

$$\mathbb{H}_{\leq 1}^{su}(X)^{(0)} \to D(H^0(Pic(X))).$$

In absence of global points we get a morphism $\mathbb{H}_{\leq 1}^{su}(X)^{(1)} \to D(H^0(Pic(X)))$ where $\mathbb{H}_{\leq 1}^{su}(X)^{(1)}$ is the fiber at 1 of the canonical fibration sequence

$$\mathbb{H}_{\leq 1}^{su}(X)^{(0)} \to \mathbb{H}_{\leq 1}^{su}(X) \to \mathbb{H}_{\leq 1}^{su}(\mathbf{Spec}\, k)$$

as described above. This map is an isomorphism when X satisfies some extra conditions expressed in [Bro21], as both objects $\mathbb{H}_{\leq 1}^{su}(X)^{(1)}$ and $D(H^0(Pic(X)))$ then satisfy the same universal property. This provides a direct relation between algebraic homology and the Albanese morphism constructed in [Bro21].

3. Grothendieck's generalized local jacobians

In this section we provide $\mathbb{H}^{su}(X)$ with a natural filtration, induced by the standard filtration by codimension on cohomology. This filtration will provide a description of $\mathbb{H}^{su}(X)$ in local terms. When working with smooth schemes over k, the local factors will directly related to Grothendieck's orginial construction of local generalized jacobians from his letter to Serre (letter from 9.8.1960, [CS01] page 109).

In this section X is a scheme of finite type over k. We denote by $X^{(d)}$ the subset of X consisting of all points of dimension d in X. Similarly, $X^{(\geq d)}$ will be the subset of points of dimension at least equal to d.

3.1. **The filtration by dimension.** We let $d \geq 0$ be an integer, and consider $Op_d(X)$ the poset of opens $U \subset X$ containing $X^{(\geq d)}$. An open U belongs to $Op_d(X)$ if and only if the closed complement $U - Z$ has dimension at most $d-1$. The only element in $Op_0(X)$ is X itself, whereas $\emptyset \in Op_{dimX+1}(X)$, and obviously $Op_d(X) \subset Op_{d+1}(X)$. We define $F_d\mathbb{H}^{su}(X)$ to be

$$F_d\mathbb{H}^{su}(X) := \lim_{U \in Op_d(X)} \mathbb{H}^{su}(U)$$

where the limit is taken over the poset $Op_d(X)$ and inside the ∞-category $Pro(\mathbf{AbSt}_k^{su})$ of pro-objects in semi-unipotent commutative algebraic group stacks. Note that $F_0\mathbb{H}^{su}(X) \simeq \mathbb{H}^{su}(X)$, and as $\emptyset$ is initial in $Op_{dimX+1}(X)$ we have $F_{dimX+1}\mathbb{H}^{su}(X) \simeq 0$.

The object $F_d\mathbb{H}^{su}(X)$ comes equipped with a canonical morphism $F_{d+1}\mathbb{H}^{su}(X) \to F_d\mathbb{H}^{su}(X)$, induced from the inclusion of posets $Op_d(X) \to Op_{d+1}(X)$, and we thus obtain this way a filtration

$$0 = F_{dimX+1}\mathbb{H}^{su}(X) = \mathbb{H}^{su}(X) \to \cdots \to F_d\mathbb{H}^{su}(X) \to F_{d-1}\mathbb{H}^{su}(X) \to \cdots \to F_0\mathbb{H}^{su}(X) = \mathbb{H}^{su}(X).$$

Definition 3.1. *The above filtration is called the* filtration by dimension *on* $\mathbb{H}^{su}(X)$.

The d-th graded piece $Gr_d\mathbb{H}^{su}(X)$ are defined as the cofiber of $F_{d+1}\mathbb{H}^{su}(X) \to F_d\mathbb{H}^{su}(X)$, computed in the stable ∞-category $Pro(\mathbf{AbSt}_k)$ (see comment 3.3) which is dual to local cohomology in dimension d on X. More precisely, we have the following universal characterization of $Gr_d\mathbb{H}^{su}(X)$.

Propositon 3.2. *With the notation above, and for any semi-unipotent commutative algebraic group stack G, there exists a canonical equivalence, functorial in G*

$$\mathbf{Map}(Gr_d\mathbb{H}^{su}(X), G) \simeq \bigoplus_{x \in X^{(d)}} \mathbb{H}^*_x(X, G),$$

where the sum runs over the set $X^{(d)}$ consisting of all points of X of dimension d, and where $\mathbb{H}^*_x(X,G)$ is local cohomology defined as the fiber of $\mathbb{H}^*(X_x, G) \to \mathbb{H}^*(X_x - \{x\}, G)$ with $X_x = \mathbf{Spec}\, \mathcal{O}_{X,x}$. Equivalently, we have a canonical identification

$$Gr_d\mathbb{H}^{su}(X) \simeq \prod_{x \in X^{(d)}} \mathbb{H}^{su}_{*,x}(X_x)$$

where $\mathbb{H}^{su}_{*,x}(X_x)$ *is local homology at* x*, defined as the cofiber of the natural morphism* $\mathbb{H}^{su}_*(X_x - \{x\}) \to \mathbb{H}^{su}_*(X_x)$*.*

Proof. For any inclusion of open $V \subset U$, with $U \in Op_d(X)$ and $V \in Op_{d+1}(X)$, the closed complement $U - V$ only has a finite number of points of dimension d. Indeed, if $(U-V)^{(d)}$ is infinite, then $U - V$ must be of dimension at least $d+1$ and thus must contain at least one point of dimension $d+1$, which contradicts $V \in Op_{d+1}(X)$. Note also that for $x \in (U-V)^{(d)}$, we have $X_x \cap V = X_x - \{x\}$, as x is the only point of X_x of dimension d in X. We thus get this way a commutative square of schemes

$$\begin{array}{ccc} \coprod_{x\in(U-V)^{(d)}} X_x - \{x\} & \longrightarrow & V \\ \downarrow & & \downarrow \\ \coprod_{x\in(U-V)^{(d)}} X_x & \longrightarrow & U, \end{array}$$

and thus an induced commutative square on the vertical cofibers on algebraic homologies

$$\prod_{x\in(U-V)^{(d)}} \mathbb{H}^{su}_{*,x}(X_x) \longrightarrow \mathbb{H}^{su}(U)/\mathbb{H}^{su}(V).$$

By passing to the limit over the pairs $V \subset U$ with $U \in Op_d(X)$ and $V \in Op_{d+1}(X)$ we find the required morphism

$$\phi : \prod_{x\in X^{(d)}} \mathbb{H}^{su}_{*,x}(X_x) \longrightarrow \lim_{V\subset U} \mathbb{H}^{su}(U)/\mathbb{H}^{su}(V) = Gr_d\mathbb{H}^{su}(X).$$

It remains to prove that this morphism ϕ is an equivalence. For this, by functoriality in X, we localize the construction on the small Zariski site of X. Both constructions

$$U \mapsto \prod_{x\in U^{(d)}} \mathbb{H}^{su}_{*,x}(U_x) \qquad U \mapsto Gr_d\mathbb{H}^{su}(U)$$

are costacks, with values in $Pro(\mathbf{AbSt}_k)$, on X_{Zar}. It is therefore enough to check that the above morphism induces an equivalence on the fibers at a given point $x \in X$. This fiber is clearly 0 if x is not of codimension d. If x is of codimension d the fiber of the right hand side is equivalent to

$$\lim_{x\in U\in Op_d(X)} \mathbb{H}^{su}(U)/\mathbb{H}^{su}(U - \overline{\{x\}}).$$

The fiber at x of the left hand side is clearly $\mathbb{H}^{su}_{*,x}(X_x)$. We are thus reduced to show that for any semi-unipotent commutative algebraic group stack $G \in \mathbf{AbSt}^{su}_k$, the canonical morphisms

$$co\lim_{x\in U} \mathbb{H}^*(U,G) \to \mathbb{H}^*(X_x,G) \qquad co\lim_{x\in U} \mathbb{H}^*(U - Z_x,G) \to \mathbb{H}^*(X_x - \{x\},G)$$

are equivalences. But this is true because G is an Artin stack of finite presentation, and thus its functor of points sends filtered colimits of commutative rings to filtered colimits. □

Remark 3.3. The semi-unipotent commutative algebraic group stacks are not stable by cofibers. So in general $Gr_d\mathbb{H}^{su}(X)$ are no longer (pro)semi-unipotent, and are merely belongs to $Pro(\mathbf{AbSt}_k^{lalg})$, the ∞-category of pro-objects in commutative locally algebraic groups stacks. It is however in general not too far from being semi-unipotent, and the problem arise mainly in low homotopical degrees. The pro-sheaves $\pi_i(Gr_d\mathbb{H}^{su}(X))$ are always pro objects in commutative groups algebraic spaces for all i. When $i > 0$, these pro-algebraic spaces are moreover pro-group schemes, and for $i > 2$ these pro-group schemes are always pro-unipotent. The group $\pi_1(Gr_d\mathbb{H}^{su}(X))$ might have some abelian and infinite discrete parts, and $\pi_2(Gr_d\mathbb{H}^{su}(X))$ might have non-unipotent affine parts. However, under regularity assumptions on X most of these graded pieces are in fact semi-unipotent as shown in the result below.

As the object $Gr_d\mathbb{H}^{su}(X)$ are pro-objects in $\mathbf{AbSt}_k^{lalg}$, their homotopy sheaves are pro-sheaves of abelian groups. In the comment above, as well as the results below we will simply say that $Gr_d(\mathbb{H}^{su}(X))$ is *connected and semi-unipotent*, to signify that the pro-sheaf $\pi_0(Gr_d\mathbb{H}^{su}(X))$ vanishes, that $\pi_1 Gr_d(\mathbb{H}^{su}(X))$ is representable by a pro-affine algebraic group, and that $\pi_2(Gr_d\mathbb{H}^{su}(X))$ is representable by a pro-unipotent algebraic group.

Propositon 3.4. *Assume that X smooth and geometrically connected of dimension n. Then, for all $d \neq n-1$ the (pro) commutative algebraic group stack $Gr_d\mathbb{H}^{su}(X)$ is connected and (pro) semi-unipotent. Moreover, $\pi_1(Gr_{n-1}\mathbb{H}^{su}(X))$ is a (pro) affine algebraic group.*

Proof. Wen $d = n$ there is nothing to prove, as $Gr_n\mathbb{H}^{su}(X) = F_n\mathbb{H}^{su}(X)$ is semi-unipotent by construction. We thus fix $d < n$, and for simplicity, we will denote by π_i the pro-algebraic group $\pi_i(Gr_d\mathbb{H}^{su}(X))$.

Being defined as a cofiber of a morphism between semi-unipotent commutative algebraic group stacks, it is easy to see from the long exact sequence in homotopy that π_i is unipotent for $i > 2$, and affine for $i = 2$. It remains to prove furthermore that π_0 vanishes, π_2 is unipotent when $d \neq n-1$, and that π_1 is always affine.

We first check that $Gr_d\mathbb{H}^{su}(X)$ is connected. Indeed, for $x \in X^{(d)}$ and for any commutative group scheme G locally of finite type over k the restriction map $G(X_x - \{x\}) \to G(X_x)$ is injective (because $X_x - \{x\}$ is Zariski dense in X_x as $d < n$). This shows that the morphism on algebraic homology induced by the inclusion of schemes $X_x - \{x\} \hookrightarrow X_x$ induces an epimorphism $\mathbb{H}_0^{su}(X_x - \{x\}) \twoheadrightarrow \mathbb{H}_0^{su}(X_x)$. As a result $\pi_0 \simeq 0$, proposition 3.2 implies that $Gr_d\mathbb{H}^{su}(X)$ is indeed connected.

We can argue similarly in order to prove that π_1 is always pro affine. Indeed, we consider again the morphism $u : \mathbb{H}_0^{su}(X_x - \{x\}) \twoheadrightarrow \mathbb{H}_0^{su}(X_x)$ of pro algebraic groups over k. For any étale commutative group scheme G (possibly infinite), we have $G(X_x - \{x\}) \simeq G(X_x)$ because X is geometrically connected. This shows that the morphism u induces an isomorphism on the groups of connected components. Similarly, for any abelian variety A, as X is regular, Weil extension theorem implies that $A(X_x) \simeq A(X_x - \{x\})$. This in turn implies that u induces also an isomorphism on the maximal quotients which are abelian varieties. By the Barasotti-Chevalley decomposition theorem we see that this implies that the kernel of u must be contained in the maximal pro affine sub-group of $\mathbb{H}_0^{su}(X_x - \{x\})$ and thus must itself be pro affine. As a result, the long exact sequence in homotopy implies that π_1 is indeed a pro affine algebraic group.

Finally, we now assume that $d < n-1$. We already know that $Gr_d\mathbb{H}^{su}(X)$ is connected. The assumption of the proposition insures that we have moreover $\pi_1 = 0$. Indeed, as X is regular, its local H^1 with coefficients in affine algebraic groups vanish. Moreover, for any commutative affine algebraic group G we have $Hom(\pi_1, G) \simeq H^1_x(X, G) = 0$. As π_1 is already known to be affine, we deduce that $\pi_1 = 0$. To finish the proof, let $u : \pi_2 \to \mathbb{G}_m$ be a multiplicative character. As $\pi_0 = \pi_1 = 0$ the morphism u can be considered as a morphism of commutative algebraic groups stacks

$$u : Gr_d\mathbb{H}^{su}(X) \to \mathbb{G}_m[2],$$

and thus as an element in $H^2_x(X, \mathbb{G}_m)$, which vanishes by assumptions. The group π_2 thus does not possesses any non-trivial multiplicative character and is therefore unipotent as required. □

The proposition 3.2 is important when combined with purity.

Corollary 3.5. *Let X be a connected and smooth k-scheme of dimension n. Then, for all integer d the object $Gr_d\mathbb{H}^{su}(X)$ is $(n-d)$-connective: its pro-homotopy groups $\pi_i(Gr_d\mathbb{H}^{su}(X))$ vanish for $i < n-d$.*

Proof of corollary. When $d = n$ there is nothing to prove. For $d = n-1$ this is already in our proposition 3.2. Let us assume that $d < n-1$, and therefore $Gr_d\mathbb{H}^{su}(X)$ is known to be semi-unipotent. To prove the corollary, we thus have to prove that any $G \in \mathbf{AbSt}_k^{su}$ which is $(n-d-1)$-truncated, then $\mathbf{Map}_{\mathbf{AbSt}_k}(Gr_d\mathbb{H}^{su}(X), G) \simeq *$. By proposition 3.2 this is equivalent to say that $\mathbb{H}^*_x(X, G) \simeq 0$ for all point $x \in X$ of dimension d. We then proceed by Postnikov decomposition on G, and we thus reduce the corollary to the following three statements.

(1) For $n-d > 2$, $x \in X^{(d)}$ and $2 \leq i < n-d$, and all commutative unipotent algebraic group G the local cohomology at x with coefficients in G vanishes: $H^i_x(X, G) = 0$.
(2) For $n-d > 1$, and all $x \in X^{(d)}$, and all commutative linear algebraic group G we have $H^1_x(X, G) = 0$.
(3) For $n-d > 0$, and all $x \in X^{(d)}$, and all commutative algebraic group G we have $H^0_x(X, G) = 0$.

The condition (3) is obvious, as by density the natural morphism $G(X - \{x\}) \to G(X)$ is injective. Condition (2) is true by Hartogs: any G-bundle on $X - \{x\}$ extends to a G-bundle on the whole X. It can also be checked easily by embedding G in a smooth commutative affine algebraic group and reducing the condition to the two cases $G = \mathbb{G}_a$ and $G = \mathbb{G}_m$. Finally, condition (1) can be proved the same manner. By embedding G in a smooth unipotent commutative algebraic group G' we can assume that G is itself smooth. When G is smooth it is a successive extensions of additive groups $\mathbb{G}_a$ and thus we are reduced to the case $G = \mathbb{G}_a$. In this last case the condition follows from the usual purity property for coherent sheaves on regular schemes. □

Remark 3.6. Even when X is smooth over k, we do not know if $Gr_{n-1}\mathbb{H}^{su}(X)$ is semi-unipotent. We have proven that it is connected with affine π_1, but the fact that its π_2 is pro-unipotent does not seem clear to us. Answering this question would require some knowledge of the first two homotopy groups π_1 and π_2, as well as their Postnikov extension class $Ext^2(\pi_1, \pi_2)$. The author does not claim to fully understand the situation.

3.2. **Grothendieck's construction.** We will now compare the general notion of alegbraic homology with the construction sketched by Grothendieck in [CS01, Page 109] (which also appears in a letter to L. Breen, see [Gro, cote 134-2] page 60). As we are working with the fppf topology, we will have to modify slightly the presentation that was originally done by Grothendieck in the context of the Zariski topology. For this, and all along this section, we will assume that X is a smooth scheme, connected and of dimension n over k. Let $x \in X^{(d)}$ be a point of dimension d on X.

Let us first consider the case where $n - d > 1$. We consider the functor $G \mapsto H_x^{n-d}(X_x, G)$, of local cohomology at x, defined on the category of abelian and unipotent algebraic groups over k. By purity, we have $H_x^{n-d-1}(X_x, G) \simeq 0$ and thus the above functor is left exact in G. It is therefore pro-representable by a commutative pro-unipotent abelian algebraic group $J_x(X)$. Similarly, when $n - d = 1$, we consider the functor $G \mapsto H_x^1(X_x, G)$, now defined on the category of abelian affine algebraic groups. It is again left exact and thus pro-representable by $J_x(X)$. Finally, for $n = d$, we consider functor $G \mapsto G(K)$, where $K = Frac(X)$ is the fraction field of X, but now defined on the whole category of all abelian group schemes locally of finite type over k. It is left exact and thus pro-representable by a pro object $J_x(X)$.

For an integer d we then set

$$J_m(X) := \prod_{x \in X^{(n-m)}} J_x(X)$$

which is a pro object in the category of commutative group schemes locally of finite type over k. It is moreover pro-affine if $m = 1$ and pro-unipotent if $m > 1$. We denote by $J_*(X)$ the graded object $\oplus_m J_m(X)$. It comes equipped with a homological differential $J_m(X) \to J_{m-1}(X)$ defined as follows. For a fixed $z \in X^{(n-m+1)}$, we define a morphism

$$d_z : \prod_{x \in X^{(n-m)}} J_x(X) \to J_z(X)$$

by defining a natural transformation on the functor they pro-represent. We start by the case where $m > 2$, so that these functors are both defined on the category of unipotent commutative algebraic groups. The right hand side pro-represents the functor $G \mapsto H_z^{m-1}(X_z, G)$, whereas the left hand side pro-represents the functor $G \mapsto \oplus_{x \in X^{(n-m)}} H_x^m(X_x, G)$. In order to define d_z we thus have to construct a morphism on local cohomologies

$$H_z^{m-1}(X_z, G) \longrightarrow \oplus_{x \in X^{(n-m)}} H_x^m(X_x, G),$$

functorial in G. This morphism is the usual differential in the Cousin complex (see [CTHK97, §1]). Any element $\alpha \in H_z^{m-1}(X_z, G)$ comes from an element $\alpha_U \in H_Z^{m-1}(U, G)$ for an open

$U \subset X$ containing z, and where Z is the closure of z in U. We denote by $x_1, \dots, x_q$ the points of X of dimension $n-m$ which are specializations of z but not belonging to U, and we let $T = X - U$. We consider the boundary map $H^{m-1}(U,G) \to H^{m-1}_T(X,G)$, together with the restriction morphism $H^m_T(X,G) \to H^m_x(X_x,G)$ along the inclusion of pairs $(X_x,\{x\}) \subset (X,T)$. We get this way a composed morphism

$$H^{m-1}_Z(U,G) \to H^{m-1}(U,G) \to H^m_T(X,G) \to \bigoplus_{1\leq i\leq q} H^m_{x_i}(X_{x_i},G).$$

The image of α_U by the morphism is by definition $d_z(\alpha)$. It is well known that this morphism is independant of the choice of α_U and squares to zero.

It remains to understand the cases in low degrees where $m=2$ or $m=1$, for which mild modifications are required. Notice first that the differential defined on local cohomologies $H^{m-1}_z(X_z,G) \longrightarrow \oplus_{x\in X^{(n-m)}} H^m_x(X_x,G)$ makes perfect sense for any sheaf of abelian groups G, the unipotence or affiness conditions on G, or even representability, not being necessary. Assume first that $m=2$, $z \in X^{(n-1)}$ is a point of codimension 1 and $x \in X^{(n-2)}$ is a specialization of z of codimension 2. The pro-algebraic group $J_z(X)$ is an affine commutative group scheme over k, whereas $J_x(X)$ is defined to corepresent the functor $G \mapsto H^2_x(X,G)$ on the category of unipotent group schemes G only. The important fact here is that $J_x(X)$ also corepresents the functor $G \mapsto H^2_x(X,G)$ defined on all affine group schemes G. Indeed, let us denote by $J'_x(X)$ the pro-algebraic group representing $G \mapsto H^2_x(X,G)$ for G running in the category of affine algebraic groups over k. The existence of $J'_x(X)$ is again guaranteed by the fact that the functor considered is left exact: as X is smooth, $H^1_x(X,G) \simeq 0$ for any $x \in X^{(n-2)}$ and any affine group scheme G. By the universal properties of $J_x(X)$ and $J'_x(X)$ we have a natural projection $J'_x(X) \to J_x(X)$, making $J_x(X)$ into the maximal unipotent quotient of $J'_x(X)$. We claim that this projection is an isomorphism. For this, we split $J'_x(X)$ as a direct sum of a unipotent and a diagonalisable part (see [DG70, IV-3 Thm. 1.1.]), and we see that we only need to prove that the diagonalisable part is trivial. This, in turn, is equivalent to the fact that there are no non-trivial maps $J'_x(X) \to \mathbb{G}_m$, or equivalently that $H^2_x(X,\mathbb{G}_m)=0$. This last assertion is true because X is smooth and x is of codimension 2, therefor its local Brauer group at x vanishes ($H^1(X_x,\mathbb{G}_m) \to H^1(X_x - \{x\})$ is surjective by Hartogs and $H^2(X_x,\mathbb{G}_m) \to H^2(X_x-\{x\})$ is injective, see [Gro68, Cor. 1.10]). We thus now that $J_x(X)$ corepresents the functor $G \mapsto H^2_x(X,G)$ on all affine group scheme G, and thus the differential $J_z(X) \to J_x(X)$ can be defined by duality using the above mentioned differential on local cohomologies $H^1_z(X_z,G) \longrightarrow H^2_x(X_x,G)$.

Finally, for $m=1$ the argument is similar to the case $m=2$. We let $z \in X^{(n)}$ be the generic point of X and denote by $x \in X^{(n-1)}$ a point of codimension 1. We introduce as before $J'_x(X)$, the pro-algebraic group corepresenting the functor $G \mapsto H^1_x(X,G)$ without any conditions on G now. We have to prove as before that the natural projection $J'_x(X) \to J_x(X)$ is an isomorphism. As this projection identifies $J_x(X)$ with the maximal affine quotient of $J'_x(X)$, it is enough to show that $J'_x(X)$ is itself affine. By the Barsotti-Chevalley decomposition theorem this is equivalent to prove that for any abelian variety A we have $Hom(J'_x(X),A) \simeq H^1_x(X,A) \simeq 0$.

But again, X being smooth implies that $H^1_x(X, A) \simeq 0$ for all abelian variety A. We can then extend the differential also to $J_1(X) \to J_0(X)$, and we are thus finished with the definition of the complex $J_*(X)$. We gather this in the definition below, which is originally due to Grothendieck.

Definition 3.7 (Grothendieck)**.** *Let X be a smooth and connected k-scheme of dimension n as above.*

(1) *The pro-algebraic group $J_d(X) = \prod_{x \in X^{(n-i)}} J_x(X)$ is called* Grothendieck's local generalized jacobian of X at x.
(2) *The complex of local jacobians, with the differentials define above, will be denoted by*

$$J_*(X) := \; J_n(X) \to J_{n-1}(X) \to \cdots \to J_1(X) \to J_0(X)\;.$$

As a first relation between the complex $J_*(X)$ and algebraic homology, we have the following direct observation, which is a direct consequence of corollary 3.5, proposition 3.4 and and the definitions of $J_*(X)$.

Corollary 3.8. *Let X be a smooth k-scheme of dimension n. Then, the* 0*-th truncation of the graded pieces of $\mathbb{H}^{su}(X)$ are the Grothendieck's local generalized jacobians*

$$\pi_0(Gr_{n-d}\mathbb{H}^{su}(X)) \simeq J_d(X),$$

as soon as $d < n$. The canonical morphism $Gr_n\mathbb{H}^{su}(X) \to J_0(X)$ is such that for any commutative group scheme of finite type G we have

$$Hom(J_0(X), G) \simeq Hom(Gr_n\mathbb{H}^{su}(X), G).$$

In the previous corollary there is a slight discrepancy in degree 0, as $\pi_0(Gr_n\mathbb{H}^{su}(X))$ is allowed to have an infinite discrete part, as opposed to $J_0(X)$. However, these two pro-objects, even though different, share the same morphisms towards any commutative group scheme of finite typ (so $J_0(X)$ is some sort of pro-completion of finite type of $Gr_n\mathbb{H}^{su}(X)$).

The above result can be made more precise. Indeed, the filtration by dimension on $\mathbb{H}^{su}(X)$ induces a spectral sequence

$$\pi_p(Gr_q\mathbb{H}^{su}(X)) \Rightarrow \pi_{p+q}(\mathbb{H}^{su}(X)),$$

and under the identification of the corollary the differential in the complex $J_*(X)$ corresponds to the differential on the E_1-term of the spectral sequence. Therefore, by the purity 3.5, Grothendieck's local generalized jacobian $J_d(X)$ appear as the lowest non-trivial homotopy group of $Gr_{(n-d)}\mathbb{H}^{su}(X)$. However, the graded pieces $Gr_{(n-d)}\mathbb{H}^{su}(X)$ are not concentrated in a single degree, and in general $Gr_{(n-d)}\mathbb{H}^{su}(X)$ and $J_d(X)$ differs (this is due to the use of the fppf topology here). We will now see however that the complex $J_*(X)$ is essentially enough to recover the cohomology of X with coefficients in a single commutative algebraic group, affine and unipotent when necessary, as stated in [CS01, Page 109].

The corollary 3.8 can also be phrased by stating that the complex of pro-algebraic groups $J_*(X)$ is the 0-truncation of the filtered object $\mathbb{H}^{su}(X)$ with respect to the Beilinson's t-structure on filtered objects. As a result, we do have a canonical projection of filtered objects

$$q : \mathbb{H}^{su}(X) \to J_*(X)$$

where the filtration on the right hand side is the naive filtration $J_{*\leq n-d}(X) \subset J_*(X)$. This projection is not an equivalence, as the graded pieces $Gr_d\mathbb{H}^{su}(X)$ are not purely concentrated in a single degree in general. However, the projection q induces an isomorphisms on certain ext groups showing that the complex $J_*(X)$ can be used in order to compute the individual cohomology groups $H^i_{fppf}(X, G)$ with coefficient in a commutative algebraic group G (affine if $i > 0$, and unipotent if $i > 1$). In order to explain this, we need to first introduce a comparison functor, between the naive derived category of commutative group schemes and our category of commutative derived group stacks.

We know that the category $\mathbf{AbSch}_k$, of separated commutative group schemes of finite type over k is an abelian category. Its category of pro-objects $Pro(\mathbf{AbSch}_k)$ is known to be abelian and to possess enough projective objects (see [Mil70]). As such it possesses a bounded derived category $\mathcal{D}^b(Pro(\mathbf{AbSch}_k))$, obtained by localizing the category of bounded complexes in $Pro(\mathbf{AbSch}_k)$ with respect to the quasi-isomorphisms. Because the functor sending $G \in Pro(\mathbf{AbSch}_k)$ to the fppf-sheaf it represents is exact, there is a canonical ∞-functor

$$h : \mathcal{D}^b(Pro(\mathbf{AbSch}_k)) \longrightarrow \mathbf{AbSt}_k,$$

It is well known that this functor is not fully faithful in general, as this can be seen on higher ext-groups. The ext-groups defined in $\mathcal{D}^b(Pro(\mathbf{AbSch}_k))$ are the *naive extension groups* of commutative pro-algebraic groups, and can be described in terms of Yoneda extensions. On the other side, the extensions groups in $\mathbf{AbSt}_k$ are the ext-groups of sheaves. It is know that, when k is algebraically closed, Ext^i vanish when $i > 2$ and when computed in $\mathcal{D}^b(Pro(\mathbf{AbSch}_k))$ (see [Mil70]), but this is not the case in $\mathbf{AbSt}_k$ (see [Bre69]).

Using the projection q and the ∞-functor h, we get for any commutative group scheme G of finite type over k, and any integer i a natural morphism of groups

$$\phi^i_G : [J_*(X), G[i]]_{\mathcal{D}^b(Pro(\mathbf{AbSch}_k))} \longrightarrow [h(J_*(X)), h(G)[i]] \longrightarrow [\mathbb{H}^{su}(X), h(G)[i]].$$

If we assume moreover that G is affine if $i > 0$, and moreover unipotent if $i > 1$, the right hand side can be identified with $H^i_{fppf}(X, G)$ by definition of algebraic homology. We thus get under this restriction a natural morphism

$$\phi^i_G : [J_*(X), G[i]]_{\mathcal{D}^b(Pro(\mathbf{AbSch}_k))} \to H^i_{fppf}(X, G).$$

For simplicity we denote by $Ext^i_{naive}(J_*(X), G)$ the left hand side of the above morphism.

Propositon 3.9. *For all $i \geq 0$ and all commutative group scheme G which is affine if $i > 0$ and unipotent if $i > 1$, the morphism*

$$\phi^i_G : Ext^i_{naive}(J_*(X), G) \to H^i_{fppf}(X, G)$$

is an isomorphism.

Proof. We start by using that the projection $q : \mathbb{H}^{su}(X) \to J_*(X)$ is compatible with filtrations on both sides: the filtration by dimension on $\mathbb{H}^{su}(X)$ and the naive filtration on $J_*(X)$. These filtrations induce convergent spectral sequences

$$[Gr_{n-q}\mathbb{H}^{su}(X), G[p]] \Rightarrow H^{p+q}_{fppf}(X, G) \qquad Ext^p_{naive}(J_q(X), G) \Rightarrow Ext^{p+q}_{naive}(J_*(X), G).$$

The morphism ϕ_G of the proposition is then compatible with these spectral sequences, and we are thus reduced to show that, for fixed p and q, the induced morphism

$$Ext^p_{naive}(J_q(X), G) \to [Gr_{n-q}\mathbb{H}^{su}(X), G[p]]$$

is an isomorphism. By proposition 3.2, the right hand side can be identified with local cohomology $\coprod_{x\in X^{(n-q)}} H^p_x(X, G)$. We also note that by construction $J_q(X) = \prod_{x\in X^{(n-q)}} J_x(X)$, and thus we are reduced to show that for a given point $x \in X$, of codimension d, and all $i \geq 0$, the natural morphism

$$\psi^i_G : Ext^i_{naive}(J_x(X), G) \to H^{d+i}_x(X, G),$$

under the extra condition stated in the proposition: G is affine if $d + i > 0$ and unipotent if $d + i > 1$. Recall here that $J_x(X)$ corepresents the functor $G \mapsto H^d_x(X, G)$, where d is the codimension of x in X.

We first deal with the two special cases $i + d = 0$ and $i + d = 1$. When $i + d = 0$, and thus $i = d = 0$, we have $Ext^0(J_x(X), G) = Hom(J_x(X), G) \simeq H^d_x(X, G)$ and it is then obvious that ψ^0_G is bijective. Suppose now that $d + i = 1$, and thus G is assumed to be affine. If x is of codimension 1, we have $i = 0$, and again ψ^0_G is bijective by definition of $J_x(X)$. If x is of codimension 0 and $i = 1$, we have to prove that the natural morphism

$$Ext^1_{naive}(J_0(X), G) \to H^1_{fppf}(K(X), G),$$

where $K(X)$ is the fraction field of X. To show this we start by the following lemma treating the case of $G = \mathbb{G}_a$ and $G = \mathbb{G}_m$.

Lemma 3.10. *With the notation above we have*

$$\psi^1_G : Ext^1_{naive}(J_0(X), \mathbb{G}_a) \simeq Ext^1_{naive}(J_0(X), \mathbb{G}_m) \simeq 0.$$

Proof of the lemma. Suppose that we are given an extension of commutative pro-algebraic groups $0 \longrightarrow \mathbb{G}_a \longrightarrow H \longrightarrow J_0(X) \longrightarrow 0$. By the universal property of $J_0(X)$, construction a section $J_0(X) \to H$ is equivalent to give a point $s \in H(K(X))$, which projects to the universal element in $J_0(X)(K(X))$. But we have an exact sequence in $fppf$ cohomology of $K(X)$ $H(K(X)) \to J_0(X)(K(X)) \to H^1_{fppf}(K(X), \mathbb{G}_a) = 0$, showing that $H(K(X)) \to J_0(X)(K(X))$ is surjective. Therefore the short exact sequence splits showing that $Ext^1_{naive}(J_0(X), \mathbb{G}_a) = 0$. The proof for $\mathbb{G}_m$ is similar. □

For a general commutative affine group scheme G over k, we can decompose G as a direct sum $G^u \oplus G^m$, of a unipotent part and a multiplicative part. We can moreover use compositions series on each components G^u and G^m to reduce to the case where G is either $\mathbb{G}_a$ or $\mathbb{G}_m$, or a subgroup of $\mathbb{G}_a$ or $\mathbb{G}_m$ with short exact sequences

$$0 \to G \to \mathbb{G}_m \to \mathbb{G}_m \to 0 \qquad 0 \to G \to \mathbb{G}_a \to \mathbb{G}_a \to 0.$$

We then consider the action of ψ^1_G on the associated long exact sequences, and lemma 3.10 shows that indeed ψ^1_G is bijective for any G.

We now turn to the case where $i+d > 1$ and G is unipotent, which is proven in a very similar fashion than the case $i = 1$ and $d = 0$. To begin with, we prove that ψ^i_G is an isomorphism for $G = \mathbb{G}_a$. Indeed, when $i = 0$ this is obvious by the definition of $J_x(X)$. For $i > 0$ the domain and codomain of $\psi^i_{\mathbb{G}_a}$ vanishes. Indeed, $H^{d+i}_x(X, \mathbb{G}_a) \simeq H^{d+i}_x(X, \mathcal{O}) \simeq 0$ for $i > 0$ by local duality for coherent sheaves. On the other hand, $Ext^i_{naive}(J_x(X), \mathbb{G}_a) = 0$ by the following lemma.

Lemma 3.11. *With the notation above and for all $i > 0$*

$$\psi^i_{\mathbb{G}_a} : Ext^i_{naive}(J_x(X), \mathbb{G}_a) \simeq 0.$$

Proof of the lemma. Suppose that we are given an extension of commutative pro-algebraic groups $0 \longrightarrow \mathbb{G}_a \longrightarrow H \longrightarrow J_x(X) \longrightarrow 0$. By the universal property of $J_0(X)$, construction a section $J_0(X) \to H$ is equivalent to give an element $s \in H^d_x(X, H)$, which projects to the universal element in $H^d_x(X, J_x(X))$. But we have an exact sequence in $fppf$ cohomology of X local at x $H^d_x(X, H) \to H^d_x(X, J_x(X)) \to H^{d+1}_x(X, \mathbb{G}_a) = 0$, showing that $H^d_x(X, H) \to H^d_x(X, J_x(X))$ is surjective. Therefore the short exact sequence splits showing that $Ext^1_{naive}(J_x(X), \mathbb{G}_a) = 0$. Moreover, it is well known, and can be proven using Dieudonné theory, that $Ext^i_{naive}(A, \mathbb{G}_a) = 0$ for all $i > 1$ and all commutative unipotent pro-algebraic group A over k (see [DG70] proposition V-1 5.1 and 5.2). □

We thus know that $\psi^i_{\mathbb{G}_a}$ is an isomorphism for all i. It remains to treat the case of a general commutative unipotent group G. Any such group possesses a decomposition series whose layers G_α can be realized in a short exact sequence $0 \to G_\alpha \to \mathbb{G}_a \to \mathbb{G}_a$. The associated long exact sequences and the lemma easily implies that ψ^i_G is also an isomorphism for all i (note that the domains and codomains of ψ^i_G can be non-zero for $i = 1$). □

Remark 3.12.
1. The previous proposition provides a close relation between our algebraic homology $\mathbb{H}^{su}(X)$ and the original Grothendieck's construction $J_*(X)$ of. The former is a truncated version of the later, but both can be used in order to express the fppf cohomology of X with coefficients in a commutative group scheme.
2. The extra conditions on G, being affine for $i > 0$ and unipotent for $i > 1$, does not appear in Grothendieck's original construction. This is because the construction is made using the Zariski topology instead othe fppf topology, and higher Zariski cohomology with coefficients in group schemes vanishes outside of the unipotent context.

4. Partial generalization to more general schemes

In this section we partially extend the results of the previous sections to the more general case of schemes over arbitrary bases. For this, we restrict ourselves to the unipotent part of algebraic homology and use the formalism of commutative groups in affine stacks (in the sense of [Toe06]) as replacement for complexes of abelian sheaves with unipotent cohomolgy sheaves. On the way, we develop the general framework of *affine homology*, the natural homology theory associated to affine stacks.

We fix a base commutative ring k. We will work from now with the fpqc topology on affine k-schemes, so $\mathbf{St}_k$ will denote the ∞-category of hypercomplete fpqc-stacks on $\mathbf{Aff}_k$.

4.1. **Affine homology.** When $C = \mathbf{St}_k$ is the ∞-category of stacks over k, $\mathbf{Ab}C$ is naturally equivalent to $\mathbf{AbSt}_k$ (already used previously in this work, but in the setting of the fppf topology), and is identified with the ∞-category of connective complexes of sheaves of abelian groups on the site of affine k-schemes with the fppf topology.

Definition 4.1. *The ∞-category of* commutative affine group stacks over k *is defined to be the full sub-∞-category of $\mathbf{AbSt}_k$ consisting of all commutative group stacks E such that for all $n \geq 0$ the underlying stack $|E[n]| \in \mathbf{St}_k$ is an affine stack in the sense of [Toe06]. The ∞-category of all commutative affine group stacks will be denoted by $\mathbf{AbChAff}_k$.*

We warn the reader that the notation $\mathbf{AbChAff}_k$ can be misleading. Indeed, the ∞-category $\mathbf{ChAff}_k$ defines another ∞-category $Ab(\mathbf{ChAff}_k)$, of abelian group objects in $\mathbf{ChAff}_k$. Recall that this is formally defined as the full sub-∞-category of $Fun^{\infty}(T^{op}, \mathbf{ChAff}_k)$, consisting of all ∞-functors sending finite sums in T to finite products in C, and where T is the algebraic theory of abelian group (i.e. the opposite of the category of free abelian group of finite rank). The ∞-category $\mathbf{AbChAff}_k$ is smaller than $Ab(\mathbf{ChAff}_k)$. Indeed, any affine commutative group scheme G defines an object in $Ab(\mathbf{ChAff}_k)$. However, we will see below (see proposition 4.2) that $\mathbb{G}_m$ is not an object in $\mathbf{AbChAff}_k$. We thus have canonical fully faithful ∞-functors $\mathbf{AbChAff}_k \hookrightarrow Ab(\mathbf{ChAff}_k) \hookrightarrow \mathbf{AbSt}_k$, but these three ∞-categories are all distinct.

Note also that because affine stacks are stable by small limits in $\mathbf{St}_k$, $\mathbf{AbChAff}_k \subset \mathbf{AbSt}_k$ is clearly stable by small limits.

An important first observation is the following statement, which identifies the ∞-category $\mathbf{AbChAff}_k$ when k is a field. It is the commutative group version of the characterization of affine stacks by means of their homotopy sheaves which is proven in [Toe06].

Propositon 4.2. *If k is a field, then the natural inclusion*

$$\mathbf{AbChAff}_k \subset \mathbf{AbSt}_k$$

identifies $\mathbf{AbChAff}_k$ with the sub-∞-category consisting of all E such that for all $i \geq 0$ the sheaf $\pi_i(E)$ is representable by a commutative and unipotent affine group scheme over k.

Proof. We start by the following lemma.

Lemma 4.3. *Let E be an affine stack over k which is endowed with a group structure. Then, for all $i > 0$ the sheaves $\pi_i(E)$ are representable by affine unipotent group schemes.*

Proof of the lemma. This is already contained in the proof of [Toe06, Cor. 3.2.7] which we reproduce here for simplicity. Let $A = \mathcal{O}(E)$ be the commutative cosimplicial algebra corresponding to E, which is endowed with a H_∞-Hopf algebra structure in the sense of [Toe06, Def. 3.1.4], or in other words which is a cogroup in the ∞-category of commutative cosimplicial k-algebras. As a consequence, $H^*(A)$ is a graded commutative Hopf k-algebra and therefore each $H^0(A)$-module $H^i(A)$ is free (because it corresponds to an equivariant quasi-coherent sheaf on the group scheme $\mathbf{Spec}\, H^0(A)$), and in particular A is flat as a complex of $H^0(A)$-modules. Let $G = \mathbf{Spec}\, H^0(A)$ and $p : E \to G$ the natural projection induced by the canonical morphism $H^0(A) \to A$. The fiber of p is $\mathbf{Spec}\, A'$, where $A' = A \otimes_{H^0(A)} k$, which, by the flatness of A over $H^0(A)$, is reduced $H^0(A') \simeq k$. By [Toe06, Thm. 2.4.5] the corresponding affine stack $E' = \mathbf{Spec}\, A'$ is pointed and connected, and thus all the sheaves $\pi_i(E')$ are representable by unipotent group schemes. The long exact sequence in homotopy therefore implies the lemma, as $\pi_i(E) \simeq \pi_i(E')$ for all $i > 0$. □

The lemma 4.3 as the following important consequence: for any object $E \in \mathbf{AbChAff}_k$, the homotopy groups $\pi_i(E)$ are all representable by unipotent group schemes, for any $i \geq 0$. Indeed, by the lemma we already know that this is the case for $i > 0$. We now consider $E[1]$, and by definition we know that $E[1]$ is again a group object in affine stacks. In particular the lemma implies that $\pi_1(E[1]) \simeq \pi_0(E)$ is representable by a unipotent group scheme.

Conversely, let $E \in \mathbf{AbSt}_k$ be an object such that the sheaves $\pi_i(E)$ are representable by unipotent group schemes. We have to show that each stack $|E[n]| \in \mathbf{St}_k$ is affine. But, the classifying stack $B(|E[n]|)$ is affine because of [Toe06, Thm. 2.4.1], and thus so is $\Omega_* B(|E[n]|) \simeq |E[n]|$ as affine stacks are stable by limits. □

As a direct consequence of the proposition 4.2 we have the following corollary, which is the abelian analogue of the characterizations of affine stacks as being limits of stacks of the form $K(\mathbb{G}_a, n)$.

Corollary 4.4. *We continue to assume that k is a field. The sub-∞-category $\mathbf{AbChAff}_k \subset \mathbf{AbSt}_k$ is the smallest sub-∞-category which is stable by small limits and contains all the objects $\mathbb{G}_a[n]$ for all $n \geq 0$.*

Proof of the corollary. Let $C \subset \mathbf{AbSt}_k$ be the smallest sub-∞-category containing $\mathbb{G}_a[n]$ and stable by small limits. Clearly C is contained in $\mathbf{AbChAff}_k$, and we know that $\mathbf{AbChAff}_k$ is stable by limits and that it contains all the objects $\mathbb{G}_a[n]$. The converse is a direct consequence of Postnikov decomposition and the proposition. By [Toe20, Prop. B1], Postnikov towers always converge for the fpqc topology (see also [MR22] for a more general statement), and thus any $E \in \mathbf{AbChAff}_k$ can be written as $E \simeq \lim_n \tau_{\leq n} E$. Moreover, the proposition implies that each $\tau_n(E)$ belongs to $\mathbf{AbChAff}_k$. Finally, we proceed by induction on n to show that each $\tau_n(E)$ belongs to C, by means of the cartesian squares

$$\begin{array}{ccc} \tau_n(E) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \tau_{n-1}(E) & \longrightarrow & \pi_n(E)[n+1]. \end{array}$$

As $\pi_n(E)$ is a unipotent group scheme by proposition 4.2, it is itself contains in C, and thus so is $\pi_n(E)[n+1]$. □

We do not know if the corollary 4.2 remains valid over an arbitrary base ring k. We thus restrict our notion of commutative affine group stacks to a notion of *affine homology types*, by considering all the objects generated by limits of $\mathbb{G}_a[n]$.

Definition 4.5. *The* ∞-category of affine homology types over k *is the smallest sub-∞-category of* $\mathbf{AbSt}_k$ *which is stable by small limits and contains the objects* $\mathbb{G}_a[n]$ *for all* $n \geq 0$*. It is denoted by*

$$\widehat{\mathbf{AbChAff}}_k \subset \mathbf{AbSt}_k.$$

As a first observation, all the objects $\mathbb{G}_a[n]$ are commutative affine group stacks, and thus we clearly have $\widehat{\mathbf{AbChAff}}_k \subset \mathbf{AbChAff}_k$. However, outside of the case of a base field we do not know if this inclusion is strict. In any case, we will see below that $\widehat{\mathbf{AbChAff}}_k$ is better behaved than $\mathbf{AbChAff}_k$, thanks to the following proposition. We do not know if the similar statement holds true for $\mathbf{AbChAff}_k$.

Propositon 4.6. *The forgetful ∞-functor of ∞-categories* $\widehat{\mathbf{AbChAff}}_k \hookrightarrow \mathbf{ChAff}_k$*, sending* E *to* $|E|$*, possesses a left adjoint.*

Proof. We consider the inclusion of opposite ∞-categories. We know that $(\mathbf{ChAff}_k)^{op}$ is equivalent to the ∞-category associated with the combinatorial model category of commutative cosimplicial k-algebras by means of the **Spec** functor (see [Toe06]). Therefore, $(\mathbf{ChAff}_k)^{op}$ is a presentable ∞-category. Moreover, $(\widehat{\mathbf{AbChAff}}_k)^{op}$ is generated by colimits by the small set of objects $\mathbb{G}_a[n]$. Moreover, when considered as objects in $(\widehat{\mathbf{AbChAff}}_k)^{op}$, the objects $\mathbb{G}_a[n]$ are small, or more precisely $\mathbb{G}_a[n]$ are cosmall objects in $\mathbf{AbChAff}_k$ as this is implied by the following lemma.

Lemma 4.7. *Let* C *be a complete ∞-category and* $Ab(C)$ *be the ∞-category of abelian group objects in* C*. Let* $G \in Ab(C)$ *be an object satisfying the following two conditions.*

1. *The underlying object* G *is cocompact in* C*:* $Map_C(-, G)$ *sends filtered limits to filtered colimits.*
2. *The object* G *is truncated in* C*: there is an* n *such that* $Map(E, C)$ *is* n*-truncated for all* $E \in c$*.*

Then, G is cocompact as an object in $Ab(C)$.

Proof of the lemma. The mapping spaces between two abelian group objects H and G in C can be computed as a certain homotopy limit over Δ of the form

$$\mathbf{Map}_{Ab(C)}(H,G) \simeq \lim_{i\in\Delta} \mathbf{Map}_C(H^{n(i)}, G^{m(i)}),$$

where $i \mapsto n(i)$ and $i \mapsto m(i)$ are certain functions. The existence of such a limit decomposition can be seen for instance using MacLane resolutions of abelian group objects as presented in [Bre70, §2]. Because G is n-truncated this homotopy limit reduces to a finite homotopy limit, which in turn will commute with filtered colimit. The result follows easily as all objects $G^{m(i)}$ are cocompact in C. □

The lemma implies that $(\widehat{\mathbf{AbChAff}}_k)^{op}$ is a presentable ∞-category It is therefore a formal consequence of Freyd representability theorem (see [NRS20, Thm. 4.1.3]) that the forgetful ∞-functor $(\mathbf{AbChAff}_k)^{op} \to (\mathbf{ChAff}_k)^{op}$, which commutes with small colimits, admits a right adjoint. □

We warn the reader that the forgetful ∞-functor $\mathbf{AbSt}_k \to \mathbf{St}_k$ also possesses a left adjoint, given by $F \mapsto \mathbb{Z} \otimes F$, but that the square

$$\begin{array}{ccc}
\mathbf{ChAff}_k & \longrightarrow & \mathbf{St}_k \\
\downarrow & & \downarrow \\
\widehat{\mathbf{AbChAff}}_k & \longrightarrow & \mathbf{AbSt}_k
\end{array}$$

of course does not commute, even when k is a field. Indeed, the construction $\mathbf{ChAff}_k \to \widehat{\mathbf{AbChAff}}_k$ involves considering some free abelian group objects in $\mathbf{ChAff}_k$, and thus involves certain colimits, which are computed in $\mathbf{ChAff}_k$ (and the inclusion $\mathbf{ChAff}_k \subset \mathbf{AbSt}_k$ does not preserve colimits).

We also remind that there are set theory issues in the situation as the site of affine schemes Aff_k is not a small site. As a result, the inclusion $\mathbf{ChAff}_k \subset \mathbf{St}_k$ does not possess a left adjoint as objects in $\mathbf{St}_k$ might not be small. However, we can restrict ourselves to small stacks $\mathbf{St}_k^{sm} \subset \mathbf{St}_k$, which are all objects that can be written as small colimits of affine schemes. By definition affine stacks are small, as they are spectrum of small commutative cosimplicial k-algebras, so we have an inclusion $\mathbf{ChAff}_k \hookrightarrow \mathbf{St}_k^{sm}$. This last inclusion does have a left adjoint, called the *affinization* and denote by $F \mapsto (F \otimes k)^u$. It is explicitely given by considering $\mathcal{O}(F)$, the commutative cosimplicial k-algebra of functions on F, which is small as F is assumed to be small, and setting $(F \otimes k)^u \simeq \mathbf{Spec}\, \mathcal{O}(F)$.

As a result, and including our proposition 4.6, we get an ∞-functor

$$\mathbb{H}^u(-) : \mathbf{St}_k^{sm} \longrightarrow \widehat{\mathbf{AbChAff}}_k,$$

which is a left adjoint to the forgetful ∞-functor $|-| : \widehat{\mathbf{AbChAff}}_k \to \mathbf{St}_k^{sm}$.

Definition 4.8. *For a small stack $X \in \mathbf{St}_k$, its* affine homology *is defined as $\mathbb{H}^u(X)$.*

Affine homology can be hard to compute. We give here the simplest example namely the affine homology of a constant stack with finite 1-connected fibers. Let K be a 1-connected finite simplicial set, which is considered as a constant stack $\underline{K} \in \mathbf{St}_k$. As such it is a colimit of the punctual stack $*$ taken over the category of simplices in K. Therefore, its affine homology is given as

$$\mathbb{H}^u(\underline{K}) \simeq co \lim_{\Delta(K)} \mathbb{H}^u(*),$$

and we are thus reduced to compute the affine homology of the point. For this, we recall the existence of the commutative group scheme $\mathbb{H}$ of [Toe20], which can be defined as the spectrum of the Hopf algebra of integral valued polynomials. Equivalently, $\mathbb{H}$ sits inside the group scheme of big Witt vectors $\mathbb{W}$ as the sub-functor of simultaneously fixed points by all Frobenius F_p. The group scheme $\mathbb{H}$ is the additive part of ring scheme structure, for which the multiplication is induced by the multiplication of Witt vectors.

We claim that $\mathbb{H}^u(*) \simeq \mathbb{H}$, considered as a commutative group for the additive structure. First of all, as shown in [Toe20] all the $K(\mathbb{H}, n)$ are abelian group objects in affine stacks, and thus $\mathbb{H}$ is indeed an object of $\mathbf{AbChAff}_k$. Using the presentation as fixed points of Frobenius on $\mathbb{W}$, it can be checked that $\mathbb{H}$ lies furthermore in $\widehat{\mathbf{AbChAff}}_k$ and thus is an affine homology type. In [Toe20] is also known that when $n > 0$ $K(\mathbb{H}, n)$ is the affinization of the space $K(\mathbb{Z}, n)$, and thus the natural morphism $K(\mathbb{Z}, n) \to K(\mathbb{H}, n)$, which is a morphism of abelian group objects induced by the canonical map $\mathbb{Z} \to \mathbb{H}$, makes $K(\mathbb{H}, n)$ into the affinization of $K(\mathbb{Z}, n)$. Indeed, for any abelian group object E in $\mathbf{ChAff}_k$, the mapping space $Map_{\mathbf{AbSt}_k}(K(\mathbb{Z}, n), E)$ can be computed by a certain homotopy end

$$Map_{\mathbf{AbSt}_k}(K(\mathbb{Z}, n), E) \simeq hoend\left((i, j) \mapsto Map_{\mathbf{St}_k}(K(\mathbb{Z}, n)^i, E^j)\right),$$

where i and j lives in the category of free abelian groups of finite rank. Similarly, we have

$$Map_{\mathbf{AbChAff}_k}(K(\mathbb{H}, n), E) \simeq hoend\left((i, j) \mapsto Map_{\mathbf{ChAff}_k}(K(\mathbb{H}, n)^i, E^j)\right).$$

The affinization of $K(\mathbb{Z}, n)^i \simeq K(\mathbb{Z}^i, n)$ being $K(\mathbb{H}^i, n)$ (see [Toe20]), we deduce easily that the natural morphism

$$Map_{\mathbf{AbChAff}_k}(K(\mathbb{H}, n), E) \longrightarrow Map_{\mathbf{AbSt}_k}(K(\mathbb{Z}, n), E) \simeq \Omega^n_*(E(k))$$

is an equivalence for any $E \in \mathbf{AbChAff}_k$. In other words, the reduced affine homology of S^n is $K(\mathbb{H}, n)$, for any $n > 0$. But we then have $Map_{\mathbf{AbSt}_k}(\mathbb{H}, E) \simeq Map_{\mathbf{AbSt}_k}(K(\mathbb{H}, 1), K(E, 1)) \simeq \Omega_*(K(E, 1)(k)) \simeq E(k)$, and thus $\mathbb{H}$ is indeed the affine homology of $*$ as wanted.

To put things in a more concise manner, we can write, with the notations and conditions above

$$\mathbb{H}^u(\underline{K}) \simeq K \otimes \mathbb{H},$$

where the tensored structure is computed in the ∞-category $\mathbf{AbChAff}_k$ of commutative affine group stacks over k.

For our purpose, two general properties of affine homology are of some importance, namely base change and filtration by dimension.

Base change. As a start, affine homology is base sensitive, and should rather be written as $\mathbb{H}^u(X/k)$ in order to mention the base ring. Let $k \to k'$ be a morphism of commutative rings, and $X \in \mathbf{St}_k^{sm}$ a small stack over k. We denote by $\otimes_k k'$ the base change from stacks over k to stacks over k', and similarly for affine stacks. The base change commutes with limits and thus induces a well defined base change $\mathbf{AbChAff}_k \to \mathbf{AbChAff}_{k'}$. As this sends $\mathbb{G}_a$ to $\mathbb{G}_a$, base change restricts to an ∞-functor

$$\otimes_k k' : \widehat{\mathbf{AbChAff}}_k \to \widehat{\mathbf{AbChAff}}_{k'}.$$

Therefore, by universal property we have a canonical morphism of affine homology types over k'

$$\mathbb{H}^u(X \otimes_k k'/k') \to \mathbb{H}^u(X/k) \otimes_k k'.$$

This morphism is not an equivalence in general, but we claim it is so when k' is finite and of local complete intersection over k. Indeed, by the very definition of affine homology this would follow formally from the fact that the direct image ∞-functor $R_{k'/k} : \mathbf{AbSt}_{k'} \to \mathbf{AbSt}_k$, right adjoint to the base change, preserves affine homology types. As these are generated by limits from the $\mathbb{G}_a[n]$, it is enough to show that $R_{k'/k}(\mathbb{G}_a[n])$ is an affine homology type over k. But $R_{k'/k}(\mathbb{G}_a[n])$ is nothing else that $|p_*(\mathbb{G}_a)[n]|$, where $p : \mathbf{Spec}\, k' \to \mathbf{Spec}\, k$. Moreover, by assumption on k'/k the stack $p_*(\mathbb{G}_a)[n]$ is a linear stack over k associated to the perfect complex $E := (k')^\vee[-n]$, which is the dual of $k'[n]$. More explicitly, $p_*(\mathbb{G}_a)[n]$ can be written as the spectrum of the cosimplicial commutative k-algebras $Sym_k(E)$, with E a perfect complex of k-modules. These linear stacks, as objects in $\mathbf{AbSt}_k$ can be decomposed by considering a projective resolution of E, and are easily seen to be expressible by limits of stacks of the form $V[n]$ where V is a vector bundle on $\mathbf{Spec}\, k$, which are retracts of $\mathbb{G}_a^r[n]$ and thus are affine homology types over k.

An important cases of base change is when k is regular, and $k \to k(p)$ is the quotient corresponding to a closed point $p \in \mathbf{Spec}\, k$. In this case base change tells us that $\mathbb{H}^u(X/k)$ specializes to $\mathbb{H}^u(X_p/k(p))$ when pull-backed to the point p.

Filtration by dimension. We let X be a k-schemes separated and of finite type over k. For simplicity we assume that k is noetherian and X is of finite dimension d. We can proceed as in definition 3.1 and construct a finite filtration on $\mathbb{H}^u(X/k)$

$$0 = F_{d+1}\mathbb{H}^u(X/k) \longrightarrow F_d\mathbb{H}^u(X/k) \longrightarrow \dots \longrightarrow F_1\mathbb{H}^u(X/k) \longrightarrow F_0\mathbb{H}^u(X/k) = \mathbb{H}^u(X/k).$$

The associated graded for this filtration, $Gr_d\mathbb{H}^u(X/k)$, defined considering cofibers in $\widehat{\mathbf{AbChAff}}_k$, are affine homology types over k which are related to local cohomology in dimension d as follows. We first introduce affine homology types of finite presentation, as being the objects $E \in \widehat{\mathbf{AbChAff}}_k$ being obtained by finite limits and retracts of objects of the form $\mathbb{G}_a[n]$. As we have already seen in lemma 4.7 the objects $\mathbb{G}_a[n]$ are cocompact in $\mathbf{AbChAff}_k$, and thus any affine homology type of finite presentation are cocompact objects amoung affine homology types.

Let then E be an affine homology type of finite presentation. We then have

$$\mathbf{Map}_{\mathbf{AbChAff}_k}(Gr_d\mathbb{H}^u(X), E)) \simeq \oplus_{x\in X^{(d)}}\mathbb{H}^*_x(X, E).$$

Because affine homology type of finite presentation generate the whole ∞-category $\widehat{\mathbf{AbChAff}}_k$ by limits, we deduce that we have we have a general formula in terms of local affine homology

$$Gr_d\mathbb{H}^u(X) \simeq \prod_{x\in X^{(d)}} \mathbb{H}^u_x(X).$$

4.2. **Affine homology as a dg-module.** We consider the object $\mathbb{G}_a \in \mathbf{AbSt}_k$ and the corresponding $\mathbb{Z}$-linear dg-algebra of endomorphisms

$$C_{\mathbb{G}_a} := \mathbb{R}End(\mathbb{G}_a).$$

We get this way the usual adjunction of ∞-categories

$$M : (\mathbf{AbSt}_k)^{op} \leftrightarrows C_{\mathbb{G}_a} - \mathbf{dg} : D,$$

where $M(E) := \mathbb{R}Hom(E, \mathbb{G}_a)$. As $\mathbb{G}_a[n]$ generates $\widehat{\mathbf{AbChAff}}_k$ by limits, and as $\mathbb{G}_a[n]$ are cocompact objects, it is formal to see that for any $E \in \widehat{\mathbf{AbChAff}}_k$ the adjunction morphism

$$E \longrightarrow D(M(E))$$

is an equivalence. As a result we get the following corollary.

Corollary 4.9. *The ∞-functor M restricts to a full embedding*

$$M : (\widehat{\mathbf{AbChAff}}_k)^{op} \longrightarrow (C_{\mathbb{G}_a} - \mathbf{dg})^{op}.$$

Its essential image consists of all $C_{\mathbb{G}_a}$-dg-modules which are colimits of $C_{\mathbb{G}_a}$.

The above corollary implies that for a small stack X its affine homology $\mathbb{H}^u(X/k)$ is characterized by $\mathbb{H}(X, \mathbb{G}_a)$, as a dg-module over $C_{\mathbb{G}_a}$. The dg-algebra $C_{\mathbb{G}_a}$ is however quite complicated in general. It simplifies when $\mathbb{G}_a$ is restricted to sites of perfect schemes (see [Bre81]). In particular, things tend to simplify when the base ring k is perfect (or at least perfectoid). Another simplification consists of using the group of big Witt vectors $\mathbb{W}$ instead of the additive group $\mathbb{G}_a$, as for instance $Ext^1(\mathbb{W}, \mathbb{W}) = 0$, and again higher ext vanish when computed over the site of perfect schemes. Moreover, $End(\mathbb{W})$ is the usual Dieudonné ring so dg-modules over $\mathbb{R}End(\mathbb{W})$ are closed related to usual Dieudonné modules. However, $\mathbb{W}$ is not a cocompact object in $\widehat{\mathbf{AbChAff}}_k$, and therefore the corresponding adjunction with dg-modules is not as well behaved and relates dg-modules over $\mathbb{R}End(\mathbb{W})$ and certain pro-objects inside $\mathbf{AbChAff}_k$.

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