ANALYTIC AND ALGEBRAIC INTEGRABILITY OF QUASI-SMOOTH DERIVED FOLIATIONS

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Abstract. We study integrability results for derived foliations in the holomorphic context. We prove a global integrability theorem by flat groupoids, as well as global algebraic integrability in the presence of a compact leaf with finite holonomy groups. These results are generalization to the derived (and thus singular) setting of well know results and constructions: integration of holomorphic foliations by smooth groupoids and global stability in the holomorphic situation (see [Per01]).

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Introduction

In [TV23] we have introduced the notion of derived foliation in the algebraic setting, and proved a Riemann-Hilbert correspondence for quasi-smooth and rigid (also called *transversally smooth and rigid* in this paper) derived foliations. This correspondence used analytic techniques at several places, as well as integrability results (formal and analytic).

The purpose of the present paper is to investigate further the integrability results of transversally smooth and rigid derived foliations, particularly at the analytic and algebraic level, including global aspects of integrability. For this, we reconsider the notion of derived holomorphic foliations on general derived analytic stacks, by introducing a holomorphic version of graded mixed algebras as used in [TV23]. We study several integrability results for (transversally smooth and rigid) derived foliations: formal and local integrability (Theorem 4.1), global analytic integrability by means of a holonomy groupoid (Theorem 6.2), and finally global algebraic integrability under the condition that at least one leaf is compact and has finite holonomy. On the way, we also prove that any differential ideal on a complex manifold, whose singularities lies in codimension 3 or higher, admits a unique derived enhancement; in particular our integrability results apply to such cases.

As a final comment, we remark that the content of this note is part of a book in preparation on derived foliations, and will appear with more details in its chapters.

1. Quick reminder on derived analytic geometry

We recall that we can define a set valued symmetric operad *hol*, of holomorphic rings as follows (see [Pir15, Pri20, Por19] and [CR13, Nui12] for the $\mathcal{C}^\infty$-version). The set $hol(n)$ is defined to be the set of holomorphic functions of $\mathbb{C}^n$, and the operadic structure is defined by substitution in a natural manner ($n = 0$ is included here, the operad *hol* is unital). By definition, a holomorphic ring is an (unital) algebra over the operad *hol*. In more concrete terms, a holomorphic ring consists of a set R, endowed with extra operations: for any holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ we are given applications

$$\gamma_f : R^n \to R,$$

satisfying some natural properties with respect to substitutions. Intuitively, $\gamma_f(a_1, \dots, a_n)$ stands for "$f(a_1, \dots, a_n)$", the "evaluation of f at the elements a_i". When f is restricted to just polynomials maps, the operations γ_f determines a commutative $\mathbb{C}$-algebra structure on R (the sum and multiplication being induced by the holomorphic maps $x + y$ and xy). The typical example of a holomorphic ring is the set $R = \mathcal{O}_X(X)$ of (globally defined) holomorphic functions on a given complex analytic space X, where the operations γ_f are actually defined by $\gamma_f(a_1, \dots, a_n) = f(a_1, \dots, a_n)$, for $a_i \in \mathcal{O}_X(X)$.

Morphisms of holomorphic rings are maps commuting with all the operations γ_f. They form a category denoted by $\mathbf{CR}^h$, which comes equipped with with a forgetful functor $\mathbf{CR}^h \to \mathbb{C}-\mathbf{CR}$, from holomorphic rings to commutative $\mathbb{C}$-algebras, given by restricting the f's to polynomial maps only. We will often use the same notations for a holomorphic ring R and its underlying $\mathbb{C}$-algebra, but in situations where the difference is important the latter will be denoted by $N(R)$. Note that the forgetful functor $\mathbf{CR}^h \to \mathbb{C} - \mathbf{CR}$ possesses a left adjoint, denoted by $R \mapsto R^h$, sending a $\mathbb{C}$-algebra to its "holomorphication". The construction $R \mapsto R^h$ sends the polynomials algebra $\mathbb{C}[x_1, \dots, x_n]$ to the ring $hol(n)$ of holomorphic functions on $\mathbb{C}^n$ (with its natural holomorphic structure), and commutes with colimits. For any $\mathbb{C}$-algebra R the holomoprhic ring R^h can thus be described by choosing a presentation of R. The forgetful functor $\mathbf{CR}^h \to \mathbf{CR}$ commutes with filtered colimits (and, more generally, with sifted colimits), but does not commute with push-outs in general. In order to avoid confusions we will use $\otimes^h$ to denote push-outs in $\mathbf{CR}^h$.

Any holomorphic ring R possesses a module of (holomorphic) *differential forms* Ω^1_R, which is an R-module with the following universal property: R-modules maps $\Omega^1_R \to M$ are in one-to-one correspondence with *holomorphic derivations* $\delta : R \to M$. Recall here that a derivation $\delta : R \to M$ is *holomorphic* if for all $f \in hol(n)$ we have the chain rule

$$\delta(\gamma_f(a_1, \ldots, a_n)) = \sum_i \frac{\partial f}{\partial z_i}(a_1, \ldots, a_n).\delta(a_i).$$

It is important here to make a difference between a holomorphic ring R and its underlying $\mathbb{C}$-algebra $N(R)$. Indeed, Ω^1_R is different from $\Omega^1_{N(R)/\mathbb{C}}$, the usual module of Kahler differential forms in $N(R)$. There is obvious morphism $\Omega^1_{N(R)/\mathbb{C}} \to \Omega^1_R$ coming from the fact that the universal holomorphic derivation $R \to \Omega^1_R$ is a derivation, but this is not an isomorphism in general. Also note that if we start with a commutative $\mathbb{C}$-algebra R, then there exists a natural isomorphism of R^h-modules (which follows from the fact that the forgetful functor $\mathbf{CR}^h \to \mathbb{C} - \mathbf{CR}$ commutes with trivial square zero extensions)

$$R^h \otimes_R \Omega^1_{R/\mathbb{C}} \simeq \Omega^1_{R^h}.$$

When R is a holomorphic ring Ω^1_R will always mean the R-module of holomorphic differentials, and we will use the notation $\Omega^1_{N(R)}$ for the (non-holomorphic) Kahler differentials.

The next definition is essentially similar to the notions used in [Pir15, Pri20]. Note however that in the non-connective case, our notions of holomorphic cdga differs slightly from the one in loc. cit., as the holomorphic structure will exist on the cohomological degree 0 subalgebra (and not merely on the subalgebra of 0-cocyles). Note also the extra condition on the differential which does not appear in the above mentioned reference. However, for connective cdga's it coincides with the notion of [Pir15, Pri20]. We will use holomorphic non-connective cdga in order consider de Rham theory in the holomorphic context.

Definition 1.1. (1) *Let $A \in \mathbf{cdga}_{\mathbb{C}}$ be a $\mathbb{C}$-linear cdga. A* holomorphic structure *on A consists of a holomorphic ring structure on the $\mathbb{C}$-algebra A^0 of elements of cohomological degree 0, compatible with its $\mathbb{C}$-algebra structure, such that the cohomological differential $d : A^0 \to A^1$ is a holomorphic derivation. A* holomorphic cdga *is a $\mathbb{C}$-linear cdga endowed with a holomorphic structure.*

(2) *A* morphism *$A \to B$, between two holomorphic cdga's, is a morphism in $\mathbf{cdga}_{\mathbb{C}}$ for which the induced morphism $A^0 \to B^0$ is a morphism of holomorphic rings.*

The holomorphic cdga's form a category $cdga^h$, endowed with a forgetful functor $cdga^h \to cdga_{\mathbb{C}}$, to the category of $\mathbb{C}$-linear cdga's. This functor is the right adjoint of an adjunction, whose left adjoint is denoted by $A \mapsto A^h$. Explicitly, for $A \in cdga_{\mathbb{C}}$, A^h is defined as follows. We start by the graded $\mathbb{C}$-algebra $(A^0)^h \otimes_{A^0} A$, by extending A^0 to its holomorphication. The cohomological differential $d : A^0 \to A^1$ extends uniquely into a holomorphic derivation

$d^h : (A^0)^h \to (A^0)^h \otimes_{A^0} A^1$. We then define the cohomological differential d^h on the whole graded algebra $(A^0)^h \otimes_{A^0} A$ by the formula

$$d^h(a \otimes x) = d^h(a).x + a.d(x),$$

for any homogenuous $x \in A$ and $a \in (A^0)^h$. This defines a structure of cdga on A^h. Moreover, $(A^h)^0 = (A^0)^h$ comes equipped with its natural holomorphic ring structure making A^h into a holomorphic cdga in the sense of the above definition.

The adjunction $cdga^h \leftrightarrows cdga_{\mathbb{C}}$ can be used in order to lift the model structure on $cdga_{\mathbb{C}}$ to a model structure on $cdga^h$, for which fibrations and equivalences are defined via the forgetful functor.

Propositon 1.2. *The above notions of equivalences and fibrations make $cdga^h$ into a model category for which the forgetful functor $cdga^h \to cdga$ is a right Quillen functor.*

Proof. We rely on the standard techniques to lift model structures (see for instance [BM03, §2.5, 2.6]). We use the standard set of generating cofibrations and trivial cofibrations for *cdga*. The forgetful functor commutes with filtered colimits, and thus the only non-trivial statement is to check that a push-out along the image by $(-)^h$ of a generating trivial cofibration is an equivalence.

For this, we remind that the generating set J of trivial cofibrations in *cdga* is the set of morphisms $\mathbb{C} \to \mathbb{C}[x, y]$, where $deg(x) = deg(y) + 1$ and the differential is given by $d(x) = y$. When none of x or y sit in degree 0, $\mathbb{C}[x, y]^h$ is $\mathbb{C}[x, y]$ and its part of cohomological degree 0 is $\mathbb{C}$ with its canonical holomorphic structure. There are thus two cases to investigate: $deg(x) = 0$ and $deg(x) = -1$.

When $deg(x) = 0$, the holomorphic cdga $\mathbb{C}[x, y]^h$ can be written as $hol(1)[y]$, which is a holomorphic cdga zero outside of degree 0 and 1, and of the form

$$hol(1) \xrightarrow{d} hol(1)$$

in degrees 0 and 1. The holomorphic ring $hol(1)$ is here the ring of entire functions on $\mathbb{C}$, and the differential d is here simply the derivative $f \mapsto f'$ of entire functions. As a result, if A is any holomorphic cdga, the underlying complex of the push-out of holomorphic cdga's $A \otimes^h_{\mathbb{C}} \mathbb{C}[x, y]^h$ is the cocone of the morphism of complexes

$$A \otimes^h hol(1) \xrightarrow{d} A \otimes^h hol(1),$$

where d is again the derivative acting on the factor $hol(1)$. We have to prove that this cone is quasi-isomorphic to the complex A, sitting inside $A \otimes^h hol(1)$ via the unit $\mathbb{C} \to hol(1)$. In fact we will show that there is a short exact sequence of complexes

$$0 \longrightarrow A \longrightarrow A \otimes^h hol(1) \xrightarrow{d} A \otimes^h hol(1) \longrightarrow 0.$$

This means that for any i, the sequence of vector spaces

$$0 \longrightarrow A^i \longrightarrow (A \otimes^h hol(1))^i \xrightarrow{d} (A \otimes^h hol(1))^i \longrightarrow 0$$

is exact. For $i \neq 0$, $(A \otimes^h hol(1))^i \simeq A^i \otimes hol(1)$ is the usual tensor product, and thus the sequence is exact because it is obtained from $\mathbb{C} \to hol(1) \to hol(1)$ by tensoring with A^i. For $i = 0$, we can write A^0 as filtered colimit of holomorphic rings of finite presentation, and thus restrict to the case where $A^0 = \mathcal{O}(X)$ is the ring of holomorphic functions on a closed analytic subspace $X \subset \mathbb{C}^n$, defined by a finite number of global equations. The above sequence is then isomorphic to the exact sequence

$$0 \longrightarrow \mathcal{O}(X) \longrightarrow \mathcal{O}(X \times \mathbb{C}) \xrightarrow{\partial_t} \mathcal{O}(X \times \mathbb{C}) \longrightarrow 0,$$

where ∂_t is the partial derivative along the standard coordinates t on $\mathbb{C}$.

When $deg(x) = -1$ the proof follows the same lines and is left to the reader. □

The ∞-category obtained from $cdga^h$ by inverting the equivalences (i.e. the morphisms inducing quasi-isomorphisms on the underlying cdga's) will be denoted by $\mathbf{cdga}^h$. The Quillen adjunction of Proposition 1.2 produces an adjunction of ∞-categories

$$\mathbf{cdga}_{\mathbb{C}} \leftrightarrows \mathbf{cdga}^h,$$

whose left adjoint will be denoted by $A \mapsto A^h$. The right adjoint, if necessary, will be denoted by $A \mapsto N(A)$, but most of the time it will not be written explicitely (so we'll simply write again A for the underlying cdga).

A holomorphic cdga A possesses a *module of holomorphic differential forms* Ω^1_A. This is an A-dg-module such that morphisms of dg-modules $\Omega^1_A \to M$ are in one-to-one correspondence with holomorphic derivations $A \to M$. In order to define more precisely holomorphic derivations, we introduce the trivial square zero extension $A \oplus M$. This is a holomorphic cdga whose underlying cdga is the trivial square zero extension of $N(A)$ by M, while the holomorphic structure on $A^0 \oplus M^0$ is defined by formula

$$\gamma_f(a_1 + m_1, \ldots, a_n + m_n) = \gamma_f(a_1, \ldots, a_n) + \sum_i \frac{\partial f}{\partial z_i}(a_1, \ldots, a_n).m_i.$$

The holomorphic derivations from A to M are then *defined* as sections of the natural projection $A \oplus M \to A$.

As in the algebraic case, the construction $A \mapsto \Omega^1_A$ can be left derived in order to produce a (holomorphic) *cotangent complex* $\mathbb{L}_A$ for any $A \in \mathbf{cdga}^h$. This cotangent complex satisfies the usual properties of stability by base change and functoriality inside $\mathbf{cdga}^h$. For $A \in \mathbf{cdga}_{\mathbb{C}}$ we have a natural equivalence

$$A^h \otimes_A \mathbb{L}_A \simeq \mathbb{L}_{A^h}.$$

Any object $A \in \mathbf{cdga}^h$ possesses a *connective cover* $\tau_{\leq 0}A$. This is the right adjoint of the inclusion ∞-functor $\mathbf{cdga}^h_c \hookrightarrow \mathbf{cdga}^h$, where $\mathbf{cdga}^h_c$ stands for the full sub-∞-category spanned by holomorphic cdga's whose underlying cdga are connective. The connective cover construction $A \mapsto \tau_{\leq 0}A$ is moreover compatible with the forgetful ∞-functor to $\mathbf{cdga}$. Indeed, we simply have to notice that if A is a holomorphic cdga, the kernel $Ker(d: A^0 \to A^{-1}) \subset A^0$ is a sub-holomorphic ring of A^0 (because, by definition, d is a holomorphic derivation).

A connective holomorphic cdga $A \in \mathbf{cdga}_c^h$ is called a *finite cell object* if there is a finite sequence of morphisms in $\mathbf{cdga}_c^h$

$$0 = A_0 \longrightarrow A_1 \longrightarrow \dots \longrightarrow A_{n-1} \longrightarrow A_n = A,$$

where each A_{i+1} is obtained from A_i by choosing some elements $a_i \in A_i^{d_i}$ of degree n_i, and freely adding a finite number of variables y_i of degrees $n_i - 1$ with $d(y_i) = a_i$. By weakening this definition, we say that $A \in \mathbf{cdga}_c^h$ is an *almost finite cell object* if there is a countable sequence of morphisms

$$0 = A_0 \longrightarrow A_1 \longrightarrow \dots \longrightarrow A_{n-1} \longrightarrow A_n \longrightarrow \dots,$$

with $A = co\lim_i A_i$ and where each $A_i \to A_{i+1}$ is as above but with the condition that the sequence of integers n_i tends to ∞. In other words, an almost finite cell object is a cell object with a finite number of cell in each dimension.

Definition 1.3. (1) *A connective holomorphic cdga A is* of finite presentation *if it is a retract, in the ∞-category $\mathbf{cdga}_c^h$ of a finite cell object.*
(2) *A connective holomorphic cdga A is* almost of finite presentation *if it is a retract of an almost finite cell object.*

Using [TV07, Prop. 2.2], it can be shown that finitely presented holomorphic cdga's are the compact objects in $\mathbf{cdga}_c^h$. Similarly, almost finitely presented holomorphic cdga's are the object A for which $Map(A, -)$ commutes with filtered colimits of uniformly truncated objects: for any filtered system $\{A_i\}$, with $H^j(A_i) = 0$ for all $j \leq n$ (for some fixed integer n), we have $colim_i Map(A, A_i) \simeq Map(A, colim_i A_i)$.

As in the algebraic context, cotangent complexes can be used in order to characterize (almost) finitely presented objects as follows. A holomorphic cdga A is (almost) finitely presented if and only if the holomorphic ring $H^0(A)$ is finitely presented (in the category of holomorphic rings $\mathbf{CR}^h$) and $\mathbb{L}_A$ is a (almost) perfect A-modules. Note here that $H^0(A)$ is the zero-th cohomology group of A, which comes equipped with a natural structure of a holomorphic ring such that $Ker(d) \to H^0(A)$ is a quotient of holomorphic rings.

Definition 1.4. *The* ∞-category of affine derived analytic spaces *is the full sub-∞-category of the opposite ∞-category of $\mathbf{cdga}_c^h$ formed by holomorphic cdga's almost of finite presentation. It is denoted by $\mathbf{dAff}^h$. The object in $\mathbf{dAff}^h$ corresponding to an object $A \in \mathbf{cdga}_c^h$ will be denoted symbolically by $\mathbf{Spec}^h\, A$.*

By definition/construction, we note that for a affine derived analytic space $X = \mathbf{Spec}^h\, A$, the holomorphic ring $H^0(A)$ can be canonically identified with the ring of holomorphic functions on a closed analytic subspace $\tau_0(X)$ in $\mathbb{C}^n$. The analytic space $\tau_0(X)$ is defined as follows. We can write A as a connective cell object with finitely many cells in each dimension, and for such a cell object, A^0 is a free holomorphic ring on n generators (for some n). Thus, $H^0(A)$ becomes a quotient of the holomorphic ring $hol(n) = \mathcal{O}(\mathbb{C}^n)$ by a finitely number of relations. These relations are given by elements $f_i \in hol(n)$ and therefore define a complex analytic subspace $\tau_0(X)$ inside $\mathbb{C}^n$. The holomorphic ring $H^0(A)$ is then naturally isomorphic to the holomorphic ring $\mathcal{O}_X(X)$ of holomorphic functions on $\tau_0(X)$. The complex space $\tau_0(X)$ is called the *truncation* of X. In the same manner, the cohomology groups $H^i(A)$ are finitely generated $H^0(A)$-modules, and thus corresponds to globally generated coherent sheaves on $\tau_0(X)$ (see [Pri20, Prop. 1.24, Lem. 1.25]).

The natural analytic topology on $\mathbb{C}$ induces a Grothendieck topology on the ∞-category $\mathbf{dAff}^h$, as follows. If $i : \mathbf{Spec}^h A \to \mathbf{Spec}^h B$ is a morphism of affine derived analytic spaces, we say that i is an *open immersion* if the induced morphism $i_0 : \mathbf{Spec}^h H^0(A) \to \mathbf{Spec}^h H^0(B)$ is an open immersion of analytic spaces, and if furthermore the natural morphisms $H^i(A) \otimes_{H^0(A)} H^0(B) \to H^i(B)$ are isomorphisms (i.e. the coherent sheaves $H^i(A)$ on $\mathbf{Spec}^h H^0(A)$ restrict to $H^i(B)$ along the open immersion i_0). An *open covering* of affine derived analytic spaces is then a family of open immersion $\{U_i \to X\}_i$ such that $\coprod_i \tau_0(U_i) \to \tau_0(X)$ is a surjective map of complex spaces.

We can also define smooth and étale morphisms, simply by using the holomorphic cotangent complexes as usual. We say that $\mathbf{Spec}^h A \to \mathbf{Spec}^h B$ is smooth (resp. étale) if the relative cotangent complex $\mathbb{L}_{B/A}$ is a projective B-module of finite rank (resp. $\mathbb{L}_{B/A} \simeq 0$). .

We are now in a situation in which we have a Grothendieck ∞-site $\mathbf{dAff}^h$ together with a notion of smooth morphisms. We can thus define Artin stacks by using the usual formal procedure (see [TV08, 1.4.3]). In particular, we have the ∞-topos of stacks on this ∞-site, which will be denoted by $\mathbf{dSt}^h$, as well as full sub-∞-category of Artin stacks. Note that by convention here the site $\mathbf{dAff}^h$ consists only of almost finitely presented objects, and thus Artin stacks in these settings will be automatically locally almost finitely presented. Therefore, for any Artin derived analytic stack X, its truncation $\tau_0 X$ is an (underived) Artin analytic stack locally of finite presentation, and the homotopy groups $H^i(\mathcal{O}_X)$ define *coherent* sheaves on $\tau_0(X)$.

2. De Rham algebras in the holomorphic context

We introduce a category of *holomorphic* $\mathbb{C}$*-linear graded mixed cdga's*. Its objects are graded mixed cdga's B together with a holomorphic structure on the cdga $B^{(0)}$, and such that the mixed structure $\epsilon : B^{(0)} \to B^{(1)}[-1]$ is a holomorphic derivation on the holomorphic cdga $B^{(0)}$. We can endow this category with a model category structure simply by defining equivalences and fibrations on the underlying complexes $B^{(i)}$ (by forgetting the holomorphic structures).

Propositon 2.1. *The category of holomorphic graded mixed cdgas, with the above notions of fibrations and equivalences, define a model category structure such that the forgetful functor to graded mixed cdga's is right Quillen.*

Proof. This is similar to the proof of Proposition 1.2, and in fact easier as here the forgetful functor commutes with push-outs. We leave the proof to the reader. □

The ∞-category obtained from the model category of holomorphic graded mixed cdga's will be denoted by $(\epsilon - \mathbf{cdga}^{gr})^h$, and will be called the ∞-category of *holomorphic graded mixed cdga's*. For any holomorphic cdga A its de Rham algebra $\mathbf{DR}(A) = Sym_A(\mathbb{L}_A[1])$, endowed with the mixed structure induced by the de Rham differential, obviously is an object in $(\epsilon - \mathbf{cdga}^{gr})^h$. More precisely, the ∞-functor $(\epsilon - \mathbf{cdga}^{gr})^h \to \mathbf{cdga}^h$, sending B to $B^{(0)}$ has a left adjoint given by $B \to \mathbf{DR}(B)$. This adjunction of ∞-categories is moreover induced by a Quillen adjunction on the level of model categories. Of this adjunction we retain the following property

Propositon 2.2. *Let A be a holomorphic cdga and $\mathbf{DR}(A)$ its de Rham complex as an object in $(\epsilon - \mathbf{cdga}^{gr})^h$. For any $B \in (\epsilon - \mathbf{cdga}^{gr})^h$ the morphism induced on mapping spaces*

$$\mathbf{Map}_{(\epsilon - \mathbf{cdga}^{gr})^h}(\mathbf{DR}(A), B) \longrightarrow \mathbf{Map}_{\mathbf{cdga}^h}(A, B^{(0)})$$

is an equivalence.

We will also use a second adjunction between holomorphic graded mixed cdga's and holomorphic cdga's. Specifically, we consider the ∞-functor $A \mapsto A(0)$, sending a holomorphic cdga A to the holomorphic graded mixed cdga $A(0)$ which is A purely in weight 0 with the trivial mixed structure. This defines an ∞-functor $\mathbf{cdga}^h \to (\epsilon - \mathbf{cdga}^{gr})^h$ which commutes with colimits and thus admits a right adjoint. This right adjoint is called the *holomorphic realization* and is denoted by

$$| - | : (\epsilon - \mathbf{cdga}^{gr})^h \to \mathbf{cdga}^h.$$

Propositon 2.3. *For any $B \in (\epsilon - \mathbf{cdga}^{gr})^h$, the underlying cdga $N(|B|)$ associated to $|B|$ is canonically equivalent to $|N(B)|$, the realization of the underlying graded mixed cdga $N(B)$ associated to B.*

Proof. This is clear by adjunction and from the fact that $A(0)^h$ obviously identifies with $A^h(0)$. For $A \in \mathbf{cdga}_{\mathbb{C}}$, we have a chain of natural equivalences

$$Map_{\mathbf{cdga}_{\mathbb{C}}}(A, N(|B|)) \simeq Map_{\mathbf{cdga}^h}(A^h, |B|) \simeq Map_{(\epsilon - \mathbf{cdga}^{gr})^h}(A(0)^h, B)$$
$$\simeq Map_{\epsilon - \mathbf{cdga}^{gr}}(A(0), N(B)) \simeq Map_{\mathbf{cdga}_{\mathbb{C}}}(A, |N(B)|).$$

□

Definition 2.4. *Let A be a (connective) holomorphic cdga and $\mathbf{DR}(A)$ its de Rham holomorphic graded mixed cdga. The* holomorphic derived de Rham complex of A *is defined by $\widehat{C}^*_{DR}(A) := |\mathbf{DR}(A)| \in \mathbf{cdga}^h$.*

Remark 2.5. In most cases we will only consider the underlying cdga of $\widehat{C}^*_{DR}(A)$ and will seldom use its natural holomorphic structure. However, this holomorphic structure will be useful for some specific arguments. Because of this we will still denote by $\widehat{C}^*_{DR}(A)$ the underlying cdga and simply call it the derived de Rham complex of A.

3. Analytic derived foliations

For any affine derived analytic space $\mathbf{Spec}^h A$ we remind that we have a holomorphic graded mixed cdga $\mathbf{DR}(A)$ whose underlying graded cdga is $Sym_A(\mathbb{L}_A[1])$, and for which the mixed structure is given by the de Rham differential. The construction $A \mapsto \mathbf{DR}(A)$ defines an ∞-functor

$$\mathbf{DR}(-) : (\mathbf{dAff}^h)^{op} \longrightarrow (\epsilon - \mathbf{cdga}^{gr})^h.$$

We then consider the ∞-functor

$$\mathcal{F}ol^{pr} : (\mathbf{dAff}^h)^{op} \longrightarrow \mathbf{Cat}_\infty,$$

sending $X = \mathbf{Spec}^h A \in \mathbf{dAff}^h$ to the ∞-category of holomorphic graded mixed $\mathbf{DR}(X)$-cdga's B such that the following two conditions are satisfied

(1) the natural morphism $A \to B^{(0)}$ is a quasi-isomorphism
(2) $B^{(1)}$ is an almost perfect complex of A-dg-modules
(3) the natural morphism of graded cdga's
$$Sym_A(B^{(1)}) \to B$$
is a graded quasi-isomorphism.

As opposed to the algebraic situation, the ∞-functor $\mathcal{F}ol^{pr} : (\mathbf{dAff}^h)^{op} \longrightarrow \mathbf{Cat}_\infty$ is only a prestack i.e. itf does not satisfy descent. Indeed, if $X \subset \mathbb{C}^n$ is a closed analytic subspace, globally defined by a finite number of equations, and with holomorphic ring of functions A, then almost perfect A-modules correspond to almost perfect complexes on X than can be written as bounded complex of globally generated vector bundles on X. Of course, these are not all almost perfect complexes on X. The associated stack $\mathcal{F}ol$ to $\mathcal{F}ol^{pr}$ can however be computed easily. For this, we sheafify everything on X. We first consider the small site of affine opens $U \subset X$. The holomorphic ring A sheafifies to a sheaf of holomorphic cdga's $\mathcal{O}_X$. We next consider $\mathbf{DR}_X$ as the sheaf $U \mapsto \mathbf{DR}(U)$ of holomorphic graded mixed cdga's. The objects in $\mathcal{F}ol(X)$ can then be described as sheaves of holomorphic graded mixed $\mathbf{DR}_X$-cdga's B such that

(1) the natural morphism $\mathcal{O}_X \to B^{(0)}$ is a quasi-isomorphism of sheaves of cdga's on X
(2) $B^{(1)}$ is a almost perfect complex of $\mathcal{O}_X$-dg-modules
(3) the natural morphism of graded cdga's
$$Sym_{\mathcal{O}_X}(B^{(1)}) \to B$$
is a quasi-isomorphism of sheaves of graded complexes.

We thus get an ∞-functor $\mathcal{F}ol : (\mathbf{dAff}^h)^{op} \to \mathbf{Cat}_\infty$ satisfying the descent property, and thus can be left Kan extended to a colimit preserving ∞-functor

$$\mathcal{F}ol : (\mathbf{dSt}^h)^{op} \longrightarrow \mathbf{Cat}_\infty.$$

Definition 3.1. *For a derived analytic stack $X \in \mathbf{dSt}^h$, the* ∞-category of derived foliations on X *is* $\mathcal{F}ol(X)$. *The initial object of* $\mathcal{F}ol(X)$*, when it exists, will be denoted by* 0_X *(or just by* 0*), and called the* initial foliation *on* X*. For a morphism of derived analytic stacks* $f : X \to Y$ *the induced* ∞*-functor* $\mathcal{F}ol(Y) \to \mathcal{F}ol(X)$ *is called the* pull-back of derived foliations *and is denoted by* f^*.

Remark 3.2.
(1) By definition, derived foliations in the analytic settings are always almost perfect. This is only because we want to avoid technical complications related to the notion of quasi-coherent sheaves in the analytic setting, and moreover all the examples treated in this paper will be almost perfect.
(2) In the definition of derived foliation on an affine $\mathbf{Spec}\, A$, it would have been equivalent to consider $\mathbf{DR}_X$ as a sheaf of (non-holomoprhic) graded mixed cdga's, and define derived foliations simply as sheaves of graded mixed $\mathbf{DR}_X$-cdga's B with the same conditions (this was the point of view used in [TV23]). Indeed, the holomorphic structure on $B^{(0)}$ is here implicit by the first condition, as $\mathcal{O}_X$ is equipped with a canonical holomorphic structure. Moreover, $\epsilon : B^{(0)} \to B^{(1)}[-1]$ would then be compatible with $dR : \mathcal{O}_X \to \mathbb{L}_X$, and thus would carry a canonical holomorphic structure as well. However, it seems slightly easier to work with the definition we gave above because of the abvious universal property of proposition 2.2.

We finish this Section by mentioning how most of the general results and notions we have seen for derived foliations on derived stacks (see [TV23]) remain valid, possibly with mild modifications, in the current analytic case. As a start, if $\mathcal{F} \in \mathcal{F}ol(X)$ is a derived foliation on a derived analytic stack $X \in \mathbf{dSt}^h$, we can define its fake (or big) cotangent complex $\mathbb{L}_{\mathcal{F}}$, which is an $\mathcal{O}_X$-module, simply by sending $u : S := \mathbf{Spec}^h A \to X$ to the $\mathcal{O}_S$-module $\mathbb{L}_{u^*(\mathcal{F})}$. Here, $u^*(\mathcal{F}) \in \mathcal{F}ol(\mathbf{Spec}^h A)$ and can thus be represented by a sheaf of graded mixed $\mathbf{DR}_S$-algebras $\mathbf{DR}_{u^*(\mathcal{F})}$ satisfying the conditions for being a derived foliations. The $\mathcal{O}_S$-module $\mathbb{L}_{u^*(\mathcal{F})}$ is then by definition $\mathbf{DR}^{(1)}_{u^*(\mathcal{F})}[-1]$, the ($-1$-shifted) weight one piece of $\mathbf{DR}_{u^*(\mathcal{F})}$, which by assumption is an almost perfect $\mathcal{O}_S$-module. We get this way a family of $\mathcal{O}_S$-modules, functorial in $S \to X$, or in other words an $\mathcal{O}_X$-module denoted by $\mathbb{L}^{big}_{\mathcal{F}}$. It comes equipped with a canonical morphism $\mathbb{L}^{big}_X \to \mathbb{L}^{big}_{\mathcal{F}}$.

A first observation is that the cone of this morphism is an almost perfect $\mathcal{O}_X$-module. Indeed, this is due to the fact that conormal complexes of derived foliations are stable by pull-backs. We denote this cone by $\mathcal{N}^*_{\mathcal{F}}$. Suppose now that X is a derived analytic Artin stack, and let us consider its cotangent complex $\mathbb{L}_X$, which comes equipped with a canonical morphism $\mathbb{L}^{big}_X \to \mathbb{L}_X$.

Definition 3.3. *Let X be an analytic derived Artin stack and $\mathcal{F} \in \mathcal{F}ol(X)$ be a derived foliation on X. The* cotangent complex of $\mathcal{F}$, $\mathbb{L}_{\mathcal{F}}$, *is defined by the cartesian square of $\mathcal{O}_X$-modules*

$$\begin{array}{ccc} \mathbb{L}_X^{big} & \longrightarrow & \mathbb{L}_{\mathcal{F}}^{big} \\ \downarrow & & \downarrow \\ \mathbb{L}_X & \longrightarrow & \mathbb{L}_{\mathcal{F}}. \end{array}$$

We note that the cone of $\mathbb{L}_X \to \mathbb{L}_{\mathcal{F}}$ is again $\mathcal{N}^*_{\mathcal{F}}$, that we have seen to be almost perfect. Therefore, $\mathbb{L}_{\mathcal{F}}$ is itself an almost perfect $\mathcal{O}_X$-module. Also, from the definition, it is easy to deduce the functorial nature of cotangent complexes. For a morphism $f : X \to Y$ of analytic derived Artin stacks, and $\mathcal{F} \in \mathcal{F}ol(Y)$, we have a cartesian square of almost perfect $\mathcal{O}_Y$-modules

$$\begin{array}{ccc} f^*(\mathbb{L}_Y) & \longrightarrow & f^*(\mathbb{L}_{\mathcal{F}}) \\ \downarrow & & \downarrow \\ \mathbb{L}_X & \longrightarrow & \mathbb{L}_{f^*(\mathcal{F})}. \end{array}$$

Cotangent complexes are useful in order to define smooth, quasi-smooth and rigid derived foliations, as done in [TV23]. We remind here the analog notion that is important for the present paper.

Definition 3.4. *A derived foliation $\mathcal{F} \in \mathcal{F}ol(X)$ on an analytic derived Artin stack X is called* transversally smooth and rigid *if the fiber of the morphism $\mathbb{L}_X \to \mathbb{L}_{\mathcal{F}}$ is a vector bundle on X. It is* smooth *if $\mathbb{L}_{\mathcal{F}}$ is a vector bundle on X.*

An important special case is when X is a *complex manifold*. In this case transversally smooth and rigid derived foliations have a two terms cotangent complex $V \to \Omega^1_X$, where V is a vector bundle. These are the derived foliations which are the closest to be genuine holomorphic foliations (i.e. those corresponding to the case where V is a subbundle of Ω^1_X).

We finish by defining the notion of *derived enhancement of a differential ideal* $K \subset \Omega^1_X$ on a complex manifold X. Remind that a coherent subsheaf $K \subset \Omega^1_X$ is called a *differential ideal* if the sheaf image of the de Rham differential $dR(K) \subset \Omega^2_X$ is contained in the image of $K \otimes \Omega^1_X \to \Omega^2_X$ (induces by the wedge product). Any derived analytic foliation $\mathcal{F}$ defines a differential ideal $K_{\mathcal{F}} \subset \Omega^1_X$ by considering the kernel of the morphism of coherent sheaves $\Omega^1_X \to H^0(\mathbb{L}_{\mathcal{F}})$.

Definition 3.5. *Let X be a complex manifold and $K \subset \Omega^1_X$ a differential ideal. A* derived enhancement of K *is the datum of $\mathcal{F} \in \mathcal{F}ol(X)$ with $K_{\mathcal{F}} = K$.*

4. Analytic integrability and existence of derived enhancements

The following local integrability result is specific to the analytic setting, and is a version of the classifical Frobenius theorem for transversally smooth and rigid derived foliation. It is a direct consequence of a result of Malgrange combined with the analytic version of [TV23, Cor. 1.5.4], that is proven the same manner as in the formal algebraic setting. However, we still display this result as a theorem to stress its importance.

Theorem 4.1. *Let X be a complex manifold (or more generally a smooth complex analytic space), and $\mathcal{F} \in \mathcal{F}ol(X)$ be a derived foliation on X. We suppose that the following two conditions are satisfied.*

1. *The derived foliation $\mathcal{F}$ is transversally smooth and rigid.*
2. *The derived foliation is smooth in codimension 2.*

Then, locally on X, $\mathcal{F}$ is integrable.

Proof. This is an application of Malgrange's theorem on Frobenius with singularities, see [Mal77]. Indeed, by [TV23, Cor. 1.5.4] we already know that the truncation K of $\mathcal{F}$ is a formally integrable differential ideal at each point, and thus case B of [Mal77, Thm. 3.1] can be applied. We thus get that, after localizing on X, there exists a holomorphic map $f : X \to S$, where S is another complex manifold, such that f integrates K, i.e. $f^*(\Omega^1_S) = K \subset \Omega^1_X$. We claim that f also integrates $\mathcal{F}$. Indeed, we consider $\mathcal{O}_{\mathcal{F}} \subset \mathcal{O}_X$ the subring of functions which are killed by the relative de Rham differential $d : \mathcal{O}_X \to \Omega^1_X/K$. By the codimension 2 condition, we know that the natural morphism from derived de Rham cohomology along $\mathcal{F}$ to naive de Rham cohomology of K induces an isomorphism of sheaves of holomorphic rings (see [TV23, Prop. 3.1.5])

$$H^0_{dR}(\mathcal{F}) \longrightarrow H^0_{dR,naive}(K) = \mathcal{O}_{\mathcal{F}}.$$

In particular, as $\widehat{C}^*_{DR}(\mathcal{F})) = |\mathbf{DR}_{\mathcal{F}}|$ is the realization of $\mathbf{DR}_{\mathcal{F}}$, and because $H^i(|\mathbf{DR}_{\mathcal{F}}|) \simeq 0$ for all $i < 0$, we have a canonical morphism of holomorphic graded mixed cdga's

$$\mathcal{O}_{\mathcal{F}} \longrightarrow |\mathbf{DR}_{\mathcal{F}}| \to \mathbf{DR}_{\mathcal{F}}.$$

This implies that the sheaf of holomoprhic graded mixed $\mathbf{DR}_X$-cdga's $\mathbf{DR}_{\mathcal{F}}$ descend to a sheaf of holomorphic graded mixed $\mathbf{DR}_{\mathcal{O}_X/\mathcal{O}_{\mathcal{F}}}$-cdga's (where $\mathbf{DR}_{\mathcal{O}_X/\mathcal{O}_{\mathcal{F}}}$ is the relative derived de Rham algebra of the inclusion of sheaves of holomorphic rings $\mathcal{O}_{\mathcal{F}} \to \mathcal{O}_X$), as shown by the next lemma.

Lemma 4.2. *There exists a commutative square of sheaves of holomorphic graded mixed cdga's on X*

$$\begin{array}{ccc} \mathbf{DR}_X & \longrightarrow & \mathbf{DR}_{\mathcal{F}} \\ \uparrow & & \uparrow \\ \mathbf{DR}_{\mathcal{O}_{\mathcal{F}}} & \longrightarrow & \mathcal{O}_{\mathcal{F}}, \end{array}$$

where $\mathcal{O}_{\mathcal{F}} \to \mathbf{DR}_{\mathcal{F}}$ is the natural morphism described above, and $\mathbf{DR}_{\mathcal{O}_{\mathcal{F}}} \to \mathbf{DR}_X$ is the morphism induced from the inclusion of sheaves of holomorphic rings $\mathcal{O}_{\mathcal{F}} \to \mathcal{O}_X$.

Proof of the lemma. We have two possible composed morphisms $\mathbf{DR}_{\mathcal{O}_{\mathcal{F}}} \to \mathbf{DR}_{\mathcal{F}}$. These are morphisms of holomorphic graded mixed cdga's. By the universal property of $\mathbf{DR}_{\mathcal{O}_{\mathcal{F}}}$ (see

Proposition 2.2), in order to check that these two morphisms are homotopic it is enough to show that they induce the same morphism of holomorphic rings in weight 0. But by construction these two morphisms are equal to the canonical inclusion $\mathcal{O}_{\mathcal{F}} \to \mathcal{O}_X$. □

By the previous lemma, we have thus constructed a natural morphism of sheaves of holomoprhic graded mixed cdga's on X

$$\mathbf{DR}_{\mathcal{O}_X/\mathcal{O}_{\mathcal{F}}} = \mathbf{DR}_X \otimes_{\mathbf{DR}(\mathcal{O}_{\mathcal{F}})} \mathcal{O}_{\mathcal{F}} \longrightarrow \mathbf{DR}_{\mathcal{F}}.$$

We claim that this morphism is an equivalence. Indeed, it is enough to check that it induces an equivalence on the associated graded, or, equivalently by multiplicativity, on the weight 1 parts. However, in weight 1 the above morphism is the canonical morphism $\mathbb{L}_{\mathcal{O}_X/\mathcal{O}_{\mathcal{F}}} \to \mathbb{L}_{\mathcal{F}}$, where $\mathbb{L}_{\mathcal{O}_X/\mathcal{O}_{\mathcal{F}}}$ is the holomorphic relative cotangent complex of $\mathcal{O}_{\mathcal{F}} \subset \mathcal{O}_X$, and the morphism above is induced by the holomorphic derivation $\mathcal{O}_X \to \mathbf{DR}_{\mathcal{F}}^{(1)}$ which vanishes on $\mathcal{O}_{\mathcal{F}}$. To see that this morphism $\mathbb{L}_{\mathcal{O}_X/\mathcal{O}_{\mathcal{F}}} \to \mathbb{L}_{\mathcal{F}}$ is an equivalence we can work locally on X, and thus assume that the differential ideal K is integrable by a morphism $f : X \to S$ with S a smooth manifold. In this case, [Mal77, Thm. 2.1.1] shows that $\mathcal{O}_{\mathcal{F}} \simeq f^{-1}(\mathcal{O}_S)$, and thus $\mathbb{L}_{\mathcal{O}_X/\mathcal{O}_{\mathcal{F}}}$ is identified with the coherent sheaf Ω^1_f of relative 1-forms on X over S. The fact that $\Omega^1_f \simeq \mathbb{L}_{\mathcal{F}}$ then follows from the very definition of the fact that f integrates the differential ideal K.

To summarize, we have seen that under the hypothesis of the theorem $\mathbf{DR}_{\mathcal{F}}$ is of the form $\mathbf{DR}_{\mathcal{O}_X/\mathcal{O}_{\mathcal{F}}}$. Moreover, locally on X, $\mathbf{DR}_{\mathcal{O}_X/\mathcal{O}_{\mathcal{F}}}$ is the relative de Rham algebra $\mathbf{DR}_{X/S}$ for a morphism $f : X \to S$ integrating the differential ideal K. This shows in particular that f also integrates $\mathcal{F}$ as wanted. □

The following existence and uniqueness results are specific to the analytic setting and does not have an algebraic counterpart (see however 5.3 in the proper case). They are very close to theorem 4.1, and themselves are again consequences of the results of [Mal77].

Propositon 4.3. *Let X be a complex manifold and $K \subset \Omega^1_X$ be a differential ideal such that K is a vector bundle.*

1. *If the coherent sheaf Ω^1_X/K is a vector bundle in codimension 2, then there exists at most one, up to equivalence, transversally smooth and rigid derived enhancement for K.*
2. *If the coherent sheaf Ω^1_X/K is a vector bundle in codimension 3, then there exists a unique, up to equivalence, transversally smooth and rigid derived enhancement for K.*

Proof. (1) This is a consequence of the proof of the previous theorem 4.1. Indeed, during its proof we have shown that if $\mathcal{F} \in \mathcal{F}ol(X)$ is a transversally smooth and rigid derived enhancement of K, then we must have $\mathbf{DR}_{\mathcal{F}} \simeq \mathbf{DR}_{\mathcal{O}_X/\mathcal{O}_{\mathcal{F}}}$, for $\mathcal{O}_{\mathcal{F}} \subset \mathcal{O}_X$ the holomorphic subring of functions killed by the de Rham differential $d : \mathcal{O}_X \to \Omega^1_X/K$. This shows indeed that K determines $\mathcal{F}$ and thus implies the uniqueness of $\mathcal{F}$.

(2) The reasoning is the same as the proof of theorem 4.1. We let $\mathcal{O}_{\mathcal{F}} \subset \mathcal{O}_X$ be the holomorphic subring of functions killed by the de Rham differential $d : \mathcal{O}_X \to \Omega^1_X/K$. We set $\mathbf{DR}_{\mathcal{F}} := \mathbf{DR}_{\mathcal{O}_X/\mathcal{O}_{\mathcal{F}}}$ be the relative holomorphic de Rham algebra of the inclusion $\mathcal{O}_{\mathcal{F}} \subset \mathcal{O}_X$.

It is endowed with the canonical morphism $\mathbf{DR}_X \to \mathbf{DR}_{\mathcal{F}}$, and we claim that as such it is a transversally smooth and rigid derived enhancement of K. For this we use case (A) of [Mal77, Thm. 3.1] which ensures that K is locally integrable. Again by [Mal77, Thm. 2.1.1], this shows that $\mathbf{DR}_{\mathcal{F}}$ is locally of the form $\mathbf{DR}_{X/S}$ for some morphism $f : X \to S$ with S smooth. This clearly implies that the sheaf of graded mixed $\mathbf{DR}_X$-cdga $\mathbf{DR}_{\mathcal{F}}$ satisfies the conditions of being a transversally smooth and rigid derived foliation, and that the corresponding differential ideal is equal to K. □

5. Analytification of derived foliations: GAGA theorem

Let $\mathbf{dSt}_{\mathbb{C}}$ be the ∞-category of derived stacks over $\mathbf{Spec}\,\mathbb{C}$. We consider the full sub-∞-category $\mathbf{dSt}_{\mathbb{C}}^{\mathrm{afp}} \subset \mathbf{dSt}_{\mathbb{C}}$, consisting of derived stacks which are locally almost of finite presentation over $\mathbb{C}$. These are the derived stacks that can be obtained as colimits of objects of the form $\mathbf{Spec}\,A$ with A a connective cdga which is almost of finite presentation over $\mathbb{C}$. The canonical embedding $\mathbf{dSt}_{\mathbb{C}}^{\mathrm{afp}} \subset \mathbf{dSt}_{\mathbb{C}}$ can also be identified with the left Kan extension of the Yoneda embedding restricted to derived affine schemes locally of almost of finite presentation $\mathbf{dAff}_{\mathbb{C}}^{\mathrm{afp}} \subset \mathbf{dAff}_{\mathbb{C}} \subset \mathbf{dSt}_{\mathbb{C}}$. In particular, an alternative description of $\mathbf{dSt}_{\mathbb{C}}^{\mathrm{afp}}$ is as the ∞-category of (hypercomplete) stacks on the ∞-site $\mathbf{dAff}_{\mathbb{C}}^{\mathrm{afp}}$ (where the topology is the étale topology).

We consider the analytification ∞-functor

$$(-)^h : \mathbf{dSt}_{\mathbb{C}}^{\mathrm{afp}} \to \mathbf{dSt}^h$$

from the ∞-category of derived stacks locally almost of finite presentation to the ∞-category of derived analytic stacks. It is obtained as the left Kan extension of the following composition of ∞-functors

$$\mathbf{dAff}_{\mathbb{C}}^{\mathrm{afp}} \longrightarrow \mathbf{dAff}^h \longrightarrow \mathbf{dSt}^h$$

where the first ∞-functor is the analytification, sending $\mathbf{Spec}\,A$ to $\mathbf{Spec}^h\,A^h$, where A^h is the holomorphic cdga generated by A, while the second ∞-functor is the Yoneda embedding. The analtyfication ∞-functor $(-)^h : \mathbf{dSt}_{\mathbb{C}}^{\mathrm{afp}} \to \mathbf{dSt}^h$ is the left adjoint of the restriction ∞-functor, sending $X \in \mathbf{dSt}^h$ to derived stack defined by $A \mapsto X(A^h)$. This restriction ∞-functor will be denoted by r so that we have an adjunction

$$(-)^h : \mathbf{dSt}_{\mathbb{C}}^{\mathrm{afp}} \leftrightarrows \mathbf{dSt}^h : r.$$

Moreover, the right adjoint r is a geometric morphism of ∞-topos, or equivalently $(-)^h$ commutes with finite limits. To avoid confusions with the various $(-)^h$ notations involved (for instance for graded mixed cdga's), we will denote this adjunction by

$$(-)^h =: u^{-1} : \mathbf{dSt}_{\mathbb{C}}^{\mathrm{afp}} \leftrightarrows \mathbf{dSt}^h : u_* := r.$$

Remind that to both ∞-topoi $\mathbf{dSt}_{\mathbb{C}}^{\mathrm{afp}}$ and $\mathbf{dSt}^h$ we have associated the canonical stacks (in ∞-categories) of almost perfect derived foliations, denoted by $\mathcal{F}ol^{\mathrm{ap}}$ in the algebraic setting, and simply by $\mathcal{F}ol$ in the holomorphic setting. These can be compared by means of the analytification ∞-functor u^{-1} as in the following proposition.

Propositon 5.1. *There exists a canonical morphism of derived analytic stacks*

$$u^{-1}(\mathcal{F}ol^{\mathrm{ap}}) \longrightarrow \mathcal{F}ol.$$

Proof. By adjunction it is enough to construct a morphism $\mathcal{F}ol^{\mathrm{ap}} \to u_*(\mathcal{F}ol)$. For an object $\mathbf{Spec}\, A \in \mathbf{dAff}^{\mathrm{afp}}_{\mathbb{C}}$, we consider the analytification construction $(-)^h : \mathcal{F}ol^{\mathrm{ap}}(X) \to \mathcal{F}ol(X^h)$, sending a graded mixed cdga B with $B^{(0)} \simeq A$ to B^h, which is a holomorphic graded mixed cdga with $(B^h)^{(0)} \simeq A^h$ satisfying the conditions to define a holomorphic derived foliations on X^h. This is clearly functorial in A, and thus defines a morphism $\mathcal{F}ol^{\mathrm{afp}} \to u_*(\mathcal{F}ol)$ as required. □

The above proposition is needed in order to construction the analytification for derived foliations. Indeed, for $X \in \mathbf{dSt}^{\mathrm{afp}}_{\mathbb{C}}$, we define an ∞-functor

$$(-)^h : \mathcal{F}ol^{\mathrm{ap}}(X) \to \mathcal{F}ol(X^h)$$

by means of the analytification ∞-functor followed by the morphism of Proposition 5.1

$$\mathcal{F}ol^{\mathrm{ap}}(X) \simeq Map(X, \mathcal{F}ol^{\mathrm{ap}}) \to Map(X^h, u^{-1}(\mathcal{F}ol^{\mathrm{ap}})) \to Map(X^h, \mathcal{F}ol) \simeq \mathcal{F}ol(X^h).$$

Definition 5.2. *For $X \in \mathbf{dSt}^{\mathrm{afp}}_{\mathbb{C}}$ the ∞-functor defined above*

$$(-)^h : \mathcal{F}ol^{\mathrm{ap}}(X) \to \mathcal{F}ol(X^h)$$

is called the analytification ∞-functor for derived foliations*.*

The GAGA theorem for derived foliations can now be stated as follows.

Theorem 5.3. *Let X be a proper derived algebraic Deligne-Mumford stack. Then the ∞-functor*

$$(-)^h : \mathcal{F}ol^{\mathrm{ap}}(X) \to \mathcal{F}ol(X^h)$$

from almost perfect algebraic derived foliations on X to almost perfect holomorphic derived foliations on X^h, is an equivalence of ∞-categories.

Proof. We start by enlarging quite a bit the ∞-categories $\mathcal{F}ol^{\mathrm{ap}}(X)$ and $\mathcal{F}ol(X^h)$. For this, let C_X be the ∞-category of sheaves of graded mixed cdga's B on X_{et} together with an augmentation $B \to \mathcal{O}_X$ (of graded mixed cdga's for the trivial graded mixed structure on $\mathcal{O}_X$) and satisfying the following conditions:

(1) The augmentation $B \to \mathcal{O}_X$ induces an equivalence on weight zero parts $B^{(0)} \simeq \mathcal{O}_X$.
(2) The negative weight pieces of B vanish: $B^{(i)} \simeq 0$ for all $i < 0$.
(3) For all i, the $\mathcal{O}_X$-module $B^{(i)}$ is almost perfect.

Similarly, we have C_{X^h} the ∞-category of sheaves of holomorphic graded mixed cdga's B on X^h_{et} together with an augmentation $B \to \mathcal{O}_{X^h}$ (of holomorphic graded mixed cdga's for the trivial graded mixed structure on $\mathcal{O}_{X^h}$) and satisfying the following conditions:

(1) The augmentation $B \to \mathcal{O}_{X^h}$ induces an equivalence on weight zero parts $B^{(0)} \simeq \mathcal{O}_{X^h}$.
(2) The negative weight pieces of B vanish: $B^{(i)} \simeq 0$ for all $i < 0$.
(3) For all i, the $\mathcal{O}_{X^h}$-module $B^{(i)}$ is almost perfect.

As a first observation we have the following finiteness result of cotangent complexes.

Lemma 5.4. *Let $B \in C_X$ (resp. $B \in C_{X^h}$), then the graded cotangent complex $\mathbb{L}_B$ is such that each individual weight piece $\mathbb{L}_B^{(i)}$ is almost perfect over $\mathcal{O}_X$ and is zero for $i < 0$.*

Proof of the lemma. We give the proof for objects in C_X, the holomorphic case is treated similarly.

First of all this is a statement about the underlying graded cdga's so we can forget the mixed structure. We construct a "graded cell decomposition" of B in the usual way, for which cells in a given dimension will be parametrized by an almost perfect $\mathcal{O}_X$-modules. We construct, by induction, a sequence of sheaves of graded $\mathcal{O}_X$-cdga's

$$C(0) = \mathcal{O}_X \longrightarrow C(1) \longrightarrow \dots \longrightarrow C(n) \longrightarrow C(n+1) \longrightarrow \dots \longrightarrow B \tag{1}$$

such that $C(n) \to B$ induces an equivalence in weights less than or equal to n, and for each n there exists a push-out square of sheaves of graded cdga's

$$\begin{array}{ccc} C(n) & \longrightarrow & C(n+1) \\ \uparrow & & \uparrow \\ Sym_{\mathcal{O}_X}(E(n)) & \longrightarrow & \mathcal{O}_X \end{array}$$

with $E(n)$ an almost perfect $\mathcal{O}_X$-module pure of weight $n+1$. Assuming that $C(n) \to B$ has been constructed, we consider the induced morphism $C(n)^{(n+1)} \to B^{(n+1)}$ on the weight $(n+1)$ pieces, and we denote by $E(n)$ is homotopy fiber, which is a graded $\mathcal{O}_X$-module pure of weight $n+1$. The graded $\mathcal{O}_X$-cdga $C(n+1)$ is then defined by the push-out above. Because the morphism $K(n) \to B$ is canonically homotopic to zero, the morphism $C(n) \to B$ factors canonically through $C(n+1) \to B$. Finally, $C(n)$ and $C(n+1)$ coincide in weight less than n, so $C(n+1) \to B$ induces an equivalence in weight less than n. Moreover, the weight n part of $C(n+1)$ is by definition equivalent to the cone of $E(n) \to C(n)^{(n+1)}$, and thus is sent by an equivalence to $B^{(n+1)}$ as required.

The existence of the sequence C does imply the lemma, as $\mathbb{L}_B$ will coincide with $\mathbb{L}_{C(n)}$ in weight less than n. Moreover, by induction on n and from the push-out square 1, $\mathbb{L}_{C(n)}$ is seen to be a graded perfect $C(n)$-module for all n, and thus it is graded almost perfect over $\mathcal{O}_X$. □

A second observation is that C_X (resp. C_{X^h}) contains $\mathcal{F}ol^{\mathrm{ap}}(X)$ (resp. $\mathcal{F}ol(X^h)$) as a full sub-∞-category.

Lemma 5.5. *The natural forgetful ∞-functors*

$$\mathcal{F}ol^{\mathrm{ap}}(X) \longrightarrow C_X \qquad \mathcal{F}ol(X^h) \longrightarrow C_{X^h}$$

are fully faithful. Their essential images consists of $B \in C_X$ (resp. $B \in C_{X^h}$) such that $Sym_{\mathcal{O}_X}(B^{(1)}) \simeq B$ (resp. $Sym_{\mathcal{O}_{X^h}}(B^{(1)}) \simeq B$) as graded cdga's.

Proof of the lemma. This simply follows by unraveling the various definitions and the universal property of the de Rham algebras. We give the argument for C_X, the case of C_{X^h} being totally similar.

For two derived foliations $\mathcal{F}$ and $\mathcal{F}'$ in $\mathcal{F}ol^{\mathrm{ap}}(X)$, corresponding to sheaves of graded mixed $\mathbf{DR}_X$-cdga's $\mathbf{DR}_{\mathcal{F}}$ and $\mathbf{DR}_{\mathcal{F}'}$, the mapping space $Map(\mathcal{F},\mathcal{F}')$ is by definition the homotopy fiber, taken at the identity of $\mathcal{O}_X$, of the natural morphism

$$Map_{\epsilon-\mathbf{cdga}_X^{gr}}(\mathbf{DR}_{\mathcal{F}'},\mathbf{DR}_{\mathcal{F}}) \longrightarrow Map_{\epsilon-\mathbf{cdga}_X^{gr}}(\mathbf{DR}_X,\mathbf{DR}_{\mathcal{F}}) \simeq \mathbf{Map}_{\mathbf{cdga}_X}(\mathcal{O}_X,\mathbf{DR}_{\mathcal{F}}^{(0)}) \simeq Map_{\mathbf{cdga}_X}(\mathcal{O}_X,\mathcal{O}_X)$$

(where we have denoted by $\epsilon-\mathbf{cdga}_X^{gr}$ and $\mathbf{cdga}_X$ the ∞-categories of sheaves of graded mixed cdga's and of cdga's on X_et). In the same manner, when $\mathbf{DR}_{\mathcal{F}}$ are considered as augmented towards $\mathcal{O}_X$ via their natural augmentation to $\mathbf{DR}_{\mathcal{F}}^{(0)} \simeq \mathcal{O}_X \simeq \mathbf{DR}_{\mathcal{F}'}^{(0)}$, their mapping space in C_X is the homotopy fiber, taken at the identity, of the natural morphism

$$Map_{\epsilon-\mathbf{cdga}_X^{gr}}(\mathbf{DR}_{\mathcal{F}'},\mathbf{DR}_{\mathcal{F}}) \longrightarrow Map_{\epsilon-\mathbf{cdga}_X^{gr}}(\mathbf{DR}_{\mathcal{F}'},\mathbf{DR}_{\mathcal{F}}^{(0)}) \simeq Map_{\mathbf{cdga}_X}(\mathbf{DR}_{\mathcal{F}'}^{(0)},\mathbf{DR}_{\mathcal{F}}^{(0)}) \simeq Map_{\mathbf{cdga}_X}(\mathcal{O}_X,\mathcal{O}_X).$$

These two morphisms are easily seen to be equivalent, and thus so are their homotopy fibers. This shows that the ∞-functors of the lemma are fully faithful. Similarly, if $B \to \mathcal{O}_X$ is an object in C_X, choosing an inverse of the equivalence $B^{(0)} \simeq \mathcal{O}_X$ provides a morphism $\mathcal{O}_X \to B^{(0)}$, and thus a morphism $\mathbf{DR}_X \to B$ by the universal property of de Rham alegbras. The graded mixed cdga B, together with this morphism from $\mathbf{DR}_X$ clearly defines an object in $\mathcal{F}ol^{\mathrm{ap}}(X)$, whose image by the forgetful ∞-functor is equivalent to B, showing the essential surjectivity. □

The analytification ∞-functor $(-)^h : \mathcal{F}ol^{\mathrm{ap}}(X) \to \mathcal{F}ol(X^h)$ extends to $(-)^h : C_X \to C_{X^h}$, simply by sending a graded mixed cdga to the corresponding holomorphic graded mixed cdga by the analytification construction. Concretely, this sends a sheaf of graded mixed cdga B on X_{et} to the sheaf (on X_{et}^h) of graded cdga $\mathcal{O}_{X^h} \otimes_{u^{-1}(\mathcal{O}_X)} u^{-1}(B)$ endowed with its canonical mixed structure and its natural augmentation to $\mathcal{O}_{X^h}$. To prove the theorem, it is therefore enough to prove that this extended ∞-functor $(-)^h : C_X \to C_{X^h}$ is an equivalence of ∞-categories.

For this, we consider the forgetful ∞-functor to graded cdga's by forgetting the mixed structures. We let C_X^o be the ∞-category of sheaves of graded cdga's B on X_{et} together with an augmentation $B \to \mathcal{O}_X$ and satisfying the graded analogues of the previous three conditions: $B^{(0)} \simeq \mathcal{O}_X$, $B^{(i)}$ are zero for $i < 0$ and are almost perfect for $i > 0$. Similarly we have $C_{X^h}^o$ consisting of sheaves of graded cdga's B on X_{et}^h together with an augmentation $B \to \mathcal{O}_{X^h}$ and satisfying the graded analogues of the previous three conditions: $B^{(0)} \simeq \mathcal{O}_{X^h}$, $B^{(i)}$ are zero for $i < 0$ and are almost perfect for $i > 0$. We have forgetful ∞-functors $C_X \to C_X^o$ and

$C_{X^h} \to C^o_{X^h}$ which commute with the analytification functors

$$\begin{array}{ccc} C_X & \xrightarrow{(-)^h} & C_{X^h} \\ \downarrow & & \downarrow \\ C^o_X & \xrightarrow[(-)^h]{} & C^o_{X^h}. \end{array} \tag{2}$$

Lemma 5.6. *The analytification ∞-functor*

$$(-)^h : C^o_X \to C^o_{X^h}$$

is an equivalence of ∞-categories.

Proof of the lemma. This is a simple application of the GAGA theorem for almost perfect complexes of proper derived Artin stack of [Por19]. Indeed, GAGA implies that the analytification $(-)^h : \mathsf{APerf}(X) \to \mathsf{APerf}(X^h)$ is an equivalence (note that APerf is denoted by Coh^- in [Por19]). We deduce that it also induces an equivalence on the ∞-categories of $\mathbb{N}$-graded objects $\mathsf{APerf}^{\mathbb{N}}(X) \to \mathsf{APerf}^{\mathbb{N}}(X^h)$. As this equivalence is moreover an equivalence of symmetric monoidal ∞-categories, we get an induced equivalence on commutative algebra objects. Finally, considering commutative algebra augmented towards $\mathcal{O}_X$ and $\mathcal{O}_{X^h}$ we get the statement of the lemma. □

In order to deduce the theorem from the lemma we use the Barr-Beck theorem of [Lur22]. Indeed, the forgetful ∞-functors $C_X \to C^o_X$ and $C_X \to C_{X^h}$ are both monadic. The left adjoint to $C_X \to C^o_X$ sends an augmented sheaf of graded algebras $B \to \mathcal{O}_X$ to $Sym_B(\mathbb{L}_B[1])$, the graded de Rham algebra of B endowed with its total grading and its natural mixed structure induced by the de Rham differential. The weight 0 part of $Sym_B(\mathbb{L}_B[1])$ clearly is $\mathcal{O}_X$, and so we get an object in C_X. Similarly, the forgetful ∞-functor $C_{X^h} \to C^o_{X^h}$ has a left adjoint given by the graded holomorphic de Rham algebra construction. Because cotangent complexes commute with analytification, we have that the commutative square 2 is adjointable and the adjoint square

$$\begin{array}{ccc} C_X & \xrightarrow{(-)^h} & C_{X^h} \\ \uparrow & & \uparrow \\ C^o_X & \xrightarrow[(-)^h]{} & C^o_{X^h} \end{array}$$

naturally commutes (the canonical natural transformation between the two compositions is an equivalence). Therefore, the equivalence $(-)^h : C^o_X \simeq C^o_{X^h}$ preserves the monads defining C_X and C_{X^h} and we have that the induced ∞-functor $(-)^h : C_X \to C_{X^h}$ is an equivalence as desired. □

6. Existence of leaf space and holonomy

In this section we will restrict ourselves to the following specific setting. We let X be a separated and connected complex manifold. Let $\mathcal{F} \in \mathcal{F}ol(X/\mathbb{C})$ be a derived foliation on X which is transversally smooth and rigid. We will study the existence of an analytic leaf space for $\mathcal{F}$. Namely we would like to contract all the leaves to points and produce the quotient space. This quotient space does not exist in general, and we start by studying extra conditions on $\mathcal{F}$ allowing its existence.

6.1. **Some conditions.** We start by discussing some conditions on $\mathcal{F}$ which will later on ensure the existence of a leaf space.

Codimension 2. Because $\mathcal{F}$ is assumed transversally smooth and rigid the cotangent complex $\mathbb{L}_{\mathcal{F}}$ is a length two complex of vector bundles on X of the form $\mathcal{N}^*_{\mathcal{F}} \to \Omega^1_{X/\mathbb{C}}$, where $\mathcal{N}^*_{\mathcal{F}}$ is the conormal bundle of $\mathcal{F}$. The *codimension* 2 *condition* for $\mathcal{F}$ states that there exists a closed analytic subset $Z \subset X$, of codimension 2 or higher, such that $\mathbb{L}_{\mathcal{F}}$ restricted to $U = X - Z$ is a vector bundle. Equivalently, $\mathcal{N}^*_{\mathcal{F}} \to \Omega^1_{X/\mathbb{C}}$ is injective, and $\mathcal{N}^*_{\mathcal{F}}$ is a sub-bundle when restricted to U. In particular, the perfect complex $\mathbb{L}_{\mathcal{F}}$ is equivalent to a single coherent sheaf in degree zero, namely $\Omega^1_{X/\mathbb{C}}/\mathcal{N}^*_{\mathcal{F}}$, which is a vector bundle outside of Z. The locus outside which $\mathbb{L}_{\mathcal{F}}$ is a vector bundle, is a closed subset of X, and is called the *singular set of* $\mathcal{F}$. Hence, the codimension 2 condition holds for $\mathcal{F}$ iff its singular locus is of codimension ≥ 2.

When the codimension 2 condition is satisfied for $\mathcal{F}$, we will also say, synonymously, that $\mathcal{F}$ *is smooth in codimension* 2.

Flatness. For any point $x \in X$, we know that there exists a formal leaf $\hat{\mathcal{L}}_x(\mathcal{F})$ passing through x (see e.g. [TV23, §1.6]). This formal leaf is a formal moduli problem given by a dg-Lie algebra whose underlying complex is $\mathbb{T}_{\mathcal{F},x}[-1]$, the fiber of the tangent complex of $\mathcal{F}$ at x shifted by -1. As $\mathcal{F}$ is transversally smooth and rigid, $\mathbb{T}_{\mathcal{F},x}[-1]$ is cohomologicaly concentrated in degrees $[1,2]$, and therefore corresponds to a pro-representable formal moduli problem. So there exists a pro-artinian connective local cdga $\underline{A} = "\lim" A_i$ such that $\hat{\mathcal{L}}_x(\mathcal{F}) \simeq \mathsf{Spf}\underline{A}$ is the formal spectrum of A.

The *flatness condition* states that, if $A = \lim_i A_i$ is the realization of $\underline{A}$, then $H^i(A) = 0$ for all $i < 0$. Equivalently, this means that $\hat{\mathcal{L}}_x(\mathcal{F})$ is flat over $\mathbb{C}$. As A is naturally quasi-isomorphic to the Chevalley-Eilenberg complex of the dg-Lie algebra $\mathbb{T}_{\mathcal{F},x}[-1]$, the flatness condition can also be stated as the following vanishing for dg-Lie algebra cohomology

$$H^i(\mathbb{T}_{\mathcal{F},x}[-1], \mathbb{C}) \simeq 0 \qquad \forall\, i < 0.$$

Local connectedness. The following local connectedness condition only makes sense under the codimension 2 condition. Indeed, we already know that when $\mathcal{F}$ is smooth in codimension 2, then it is locally integrable on X (see theorem 4.1). Therefore, there exists a basis B for the topology of X, and for all open $U \in B$, a holomorphic map $f : U \to \mathbb{C}^d$, such that $\mathcal{F}_{|U} \simeq f^*(0)$.

Then, the *local connectedness coinditon* states that B and the holomorphic maps f as above can be moreover chosen so that the non-empty fibers of f are all connected. This condition is of topological nature and is not always satisfied. When $d = 1$, it is always satisfied as f is smooth in codimension 2 and thus it is well known that the Milnor fibers of f are connected (see [KM75]). When $d > 1$, there are examples of holomorphic maps f such that the fibers close to a singular fiber are connected or not depending on the direction along which we approach the singular value. A simple example is given by $f : \mathbb{C}^3 \to \mathbb{C}^2$ given by $f(x, y, z) = (x^2 - y^2 z, y)$ (see [Sab83, Introduction]).

Our existence theorem below (Theorem 6.2) requires *all three conditions*, i.e. codimension 2, flatness and local connectedness, to be satisfied. Before going further we would like to make some comments on these conditions.

Remark 6.1.
(1) As already reminded before, the codimension 2 condition implies that $\mathcal{F}$ is locally integrable (see Theorem 4.1). So, for any given point $x \in X$, there is an open $x \in U$ and a holomorphic map $f : U \to \mathbb{C}^d$ such that $f^*(0) \simeq \mathcal{F}_{|U}$. This implies that the formal leaf $\hat{\mathcal{L}}_x(\mathcal{F})$ is the formal completion of $f^{-1}(f(x))$ at the point x, where f^{-1} denotes here the derived fiber of f. Since formal completions are faithfully flat, flatness of $\hat{\mathcal{L}}_x(\mathcal{F})$ implies that $f^{-1}(f(x))$ is flat over $\mathbf{Spec}\,\mathbb{C}$ locally at x. Therefore, the flatness condition implies that the derived fibers are all flat over $\mathbb{C}$, and thus that f must be a flat morphism. The converse is obviously true. Therefore, we see that under the codimension 2 condition, flatness is equivalent to the fact that the all the local holomorphic maps integrating $\mathcal{F}$ are flat holomorphic maps.
(2) If $\mathcal{F}$ is moreover smooth, and thus is given by an integrable subbundle of $\mathbb{T}_X$, then all three conditions above are automatically satisfied. Indeed, the local holomorphic maps f must be smooth and thus its fibers are themselves smooth. In particular, the local connectedness is true because f are submersions. Moreover, the formal leaves are the formal completions of smooth sub-varieties (the fibers of f), and or thus all equivalent to formal affine spaces $\widehat{\mathbb{A}}^{n-d}$ (where $n = dimX$), and thus are flat.

6.2. **Existence theorem.**

Theorem 6.2. *Let X be a connected and separated (i.e. Hausdorff) complex manifold and $\mathcal{F} \in \mathcal{F}ol(X/\mathbb{C})$ be a derived foliation that is flat, smooth in codimension 2 and locally connected. There exists a smooth and 1-truncated Deligne-Mumford analytic stack $X /\!\!/ \mathcal{F}$, together with a holomorphic morphism $\pi : X \to X /\!\!/ \mathcal{F}$ with the following properties.*

(1) *The morphism π is flat, surjective, and we have $\pi^*(0) \simeq \mathcal{F}$.*
(2) *The Deligne-Mumford stack $X /\!\!/ \mathcal{F}$ is quasi-separated, effective and is of dimension d equal to the rank of $\mathcal{N}^*_{\mathcal{F}}$ (i.e. the codimension of $\mathcal{F}$).*
(3) *For any global point $x : * \to X /\!\!/ \mathcal{F}$ the fiber of π at x*

$$\pi^{-1}(x) := * \times_{X /\!\!/ \mathcal{F}} X$$

is a connected and separated analytic space.

(4) *For any global point $x : * \to X /\!\!/ \mathcal{F}$, the isotropy group H_x of $X /\!\!/ \mathcal{F}$ at x acts freely and properly on $\pi^{-1}(x)$.*

(5) *The morphism π is* relatively connected *in the sense of topos theory i.e. the pull-back ∞-functor $\pi^{-1} : St(X /\!\!/ \mathcal{F}) \to St(X)$ is fully faithful when restricted to 0-truncated objects (i.e. sheaves of sets).*

Before giving a proof of Theorem 6.2, we explain some of the terms used in in (2). First of all, *quasi-separated* here means that the diagonal of $X /\!\!/ \mathcal{F}$ is a separated morphism. In more concrete terms, if the stack $X /\!\!/ \mathcal{F}$ is presented as an étale groupoid object $G \rightrightarrows U$, with U a disjoint union of Stein manifolds, then the smooth analytic space G must be separated. In terms of such a presentation, *effectiveness* of $X /\!\!/ \mathcal{F}$ means that the action of G on U by germs of biholomorphic maps is faithful (see [Car19]). As shown in [Car19], any Deligne-Mumford stack Y possesses a universal effective quotient Y^{eff}, and by construction $X /\!\!/ \mathcal{F}$ will be defined as the universal effective quotient of a more general construction (called the *monodromy* stack), that we will study in a future work.

For later use, we set the following terminology.

Definition 6.3. *Let X be a complex manifold, $\mathcal{F}$ be a derived foliation on X, and assume the conditions of the theorem 6.2 hold.*

(1) *The stack $X /\!\!/ \mathcal{F}$ is called the* leaf space of the derived foliation *$\mathcal{F}$.*

(2) *For any $x \in X$ the quotient analytic space $\pi^{-1}(\pi(x))/H_{\pi(x)}$ is called the* maximal leaf passing through *x.*

(3) *The $H_{\pi(x)}$-covering $\pi^{-1}(\pi(x)) \to \pi^{-1}(\pi(x))/H_{\pi(x)}$ is called the* holonomy covering*, and the group $H_{\pi(x)}$ the* holonomy group *of $\mathcal{F}$ at x.*

The maximal leaf will be denoted by $\mathcal{L}^{max}_x(\mathcal{F})$ and the holonomy group $Hol_{\mathcal{F}}(x)$. The holonomy covering of $\mathcal{L}^{max}_x(\mathcal{F})$ will be denoted by $\widetilde{\mathcal{L}}^{max}_x(\mathcal{F})$.

We are now ready to start the proof of Theorem 6.2.

Construction of $X /\!\!/ \mathcal{F}$. We let $\mathcal{B}$ be the set of all open subsets $U \subset X$ such that there exists a flat holomorphic function $f : U \to \mathbb{C}^d$ with connected fibers, and with $f^*(0) \simeq \mathcal{F}_{|U}$. As f_U is flat it has an open image $V \subset \mathbb{C}^d$, and it factors as $U \to V \subset \mathbb{C}^d$. For each $U \in \mathcal{B}$, we fix once and for all $f_U : U \to V \subset \mathbb{C}^d$, where V is open, and f_U is flat, surjective with connected fibers, and such that $f^*(0) \simeq \mathcal{F}_{|U}$. Our conditions on $\mathcal{F}$ imply that the set $\mathcal{B}$ forms a basis for the topology of X.

Before going further, we note that $\mathcal{B}$ consists of all opens $U \subset X$ with the required properties, so the only choices that have been made here are the choices of the functions f_U. In order to control the impact of such choices, we will need the following result (Lemma 6.4). By the codimension 2 condition we know that the perfect complex $\mathbb{L}_{\mathcal{F}}$ is a coherent sheaf, denoted by $\Omega^1_{\mathcal{F}} \simeq H^0(\mathbb{L}_{\mathcal{F}})$. Therefore, the de Rham differential for $\mathcal{F}$ induces a morphism $dR_{\mathcal{F}} : \mathcal{O}_U \to \Omega^1_{\mathcal{F}}$

of of sheaves on U. Since $f^*(0) \simeq \mathcal{F}$, this de Rham differential can be identified with the relative de Rham differential $dR_{U/V} : \mathcal{O}_U \to \Omega^1_{U/V}$. In particular, the composite morphism $f^{-1}(\mathcal{O}_V) \to \mathcal{O}_U \to \Omega^1_{\mathcal{F}}$ is zero.

Lemma 6.4. *The natural morphism*

$$f^{-1}(\mathcal{O}_V) \to Ker(dR_{\mathcal{F}})$$

of sheaves of rings on U, is an isomorphism.

Proof of Lemma 6.4. Because f is flat and surjective, the induced morphism $f^{-1}(\mathcal{O}_V) \to \mathcal{O}_U$ is a monomorphism of sheaves. Therefore, $f^{-1}(\mathcal{O}_V) \to Ker(dR_{\mathcal{F}})$ is also a monomorphism. It remains to show that a local section of $Ker(dR_{\mathcal{F}})$, locally descend to a local holomorphic function on V.

Sub-Lemma 6.5. *Let $f : X \to Y$ be a flat surjective morphism of complex manifolds which is smooth in codimension 2. If a map $u : Y \to \mathbb{C}$ is such that $f \circ u : X \to \mathbb{C}$ is holomorphic, then u is holomorphic.*

Proof of sublemma 6.5. Let $U_0 \subset X$ be the open on which f is smooth, so that $Z = X - U_0$ is a closed analytic subset of codimension ≥ 2. The image of U_0 by f is an open $V_0 \subset Y$, because f is flat, and V_0 is dense by Sard's theorem. As $f \circ u : U_0 \to V_0$ is a holomorphic submersion, it has local sections, and thus u is holomorphic when restricted to V_0. Moreover, f being flat and surjective implies that the topology of Y is the quotient of the topology on X by the map f. This shows that u is also continuous on Y. The application u is continuous on Y and holomorphic on a dense open, and thus it is holomorphic on all of Y. □

Now, let s be local section of $Ker(dR_{\mathcal{F}})$ defined on an open $U' \subset U$. By the local connectivity condition, we can assume, possibly shrinking U', that the restriction of f on U' has connected fibers. The codimension 2 condition implies that the canonical map from derived de Rham cohomology of U over V coincides with the naive de Rham cohomology in degree 0. In particular, $dR_{\mathcal{F}}(s) = 0$ implies that s determines a natural element in $H^0_{dR}(U'/V)$, the derived de Rham cohomology of U' relative to V. We can write this as $s \in |\mathbf{DR}(U/V)|(U')$, where $\mathbf{DR}(U/V)$ is the sheaf of graded mixed cdga's of relative de Rham theory of U' over V. Note that the sheaf $|\mathbf{DR}(U/V)|$ is naturally a module over $f^{-1}(\mathcal{O}_V)$. Moreover, if $y \in V$ is a point, corresponding to a morphism of sheaves of rings $f^{-1}(\mathcal{O}_V) \to \mathbb{C}$ then we have an equivalence of sheaves of cdga's

$$|\mathbf{DR}(U/V)| \otimes_{f^{-1}(\mathcal{O}_V)} \mathbb{C} \simeq |\mathbf{DR}(U/V) \otimes_{f^{-1}(\mathcal{O}_V)} \mathbb{C}| \simeq j_*(|\mathbf{DR}(U_y/\mathbb{C})|),$$

where $j : U_y \hookrightarrow U$ is the inclusion of the fiber $U_y = f^{-1}(y)$, and where the first equivalence follows from the fact that $\mathbb{C}$ is a perfect $f^{-1}(\mathcal{O}_V)$-module (and thus the tensor operation $(-) \otimes_{f^{-1}(\mathcal{O}_V)} \mathbb{C}$ commutes with limits and with the realization functor $|-|$). To summarize, the restriction of s on U'_y, lies in derived de Rham cohomology $H^0_{dR}(U'_y/\mathbb{C})$. As U'_y is connected, and by the comparison between derived de Rham cohomology and Betti cohomology ([Bha12, Cor.

4.27] and [Har75, Theorem IV.1.1]), we have that $H^0_{dR}(U'_y/\mathbb{C}) \simeq \mathbb{C}$, and thus the restriction of s on U'_y is a constant function. As $f : U' \to V$ is flat with connected fibers, the function s descends to an application defined on the open $V' = f(U') \subset V$, which is moreover holomorphic by the sublemma 6.5. This shows that s comes from a local section of $f^{-1}(\mathcal{O}_V)$ as required. □

Lemma 6.4 has the following important corollary, showing that the choices of the local morphisms $f : U \to V$ are essentially unique, up to a unique local biholomorphism.

Corollary 6.6. *Let $f : U \to V$ and $g : U \to V'$ be two flat and surjective holomorphic morphisms of smooth manifolds with $f^*(0) \simeq g^*(0)$. Assume that V' is isomorphic to an open in $\mathbb{C}^d$. If the fibers of f are connected Then, there exists a unique étale holomorphic morphism $\alpha : V \simeq V'$ such that $\alpha f = g$.*

Proof Corollary 6.6. We apply Lemma 6.4 to the local coordinate functions of $\mathbb{C}^d$ restricted to V', and thus get a unique factorization $\alpha : V \to V'$ such that $\alpha f = g$. Because f is surjective, α must be unique. Finally, as $f^*(0) \simeq g^*(0)$, by comparing the cotangent complexes we get that α must be étale. □

We come back to the base $\mathcal{B}$ of the topology of X. Recall that $\mathcal{B}$ consists of all the open subsets $U \subset X$ such that there exists a flat holomorphic function $f : U \to \mathbb{C}^d$ with connected fibers, and with $f^*(0) \simeq \mathcal{F}_{|U}$. The set $\mathcal{B}$ is ordered by inclusions, and as such will be considered as category. We define a functor

$$\phi : \mathcal{B} \to \mathbb{C}Man^d_{et},$$

from B to the category of complex manifolds of dimension d and étale holomorphic morphisms between them. On objects, the functor ϕ sends U to the space V, the base of the morphism $f_U : U \to V$. If $U' \subset U$ is an inclusion of opens in $\mathcal{B}$, with morphisms $f_U : U \to V$ and $f_{U'} U' \to V'$. Corollary 6.6 implies that there exists a unique étale factorization $\alpha : V' \to V$ such that $f_{U'}\alpha = (f_U)_{|U'}$. This provides, for each open inclusion $U' \subset U$ in $\mathcal{B}$, a natural morphism $\phi(U') = V' \to \phi(U) = V$ in $\mathbb{C}Man^d_{et}$. This completes the definition of the functor ϕ.

By [Car19], we know that smooth Deligne-Mumford stacks are stable under arbitrary colimits of diagrams of étale morphisms. We therefore consider the colimit of the functor ϕ, and take its universal effective quotient as defined in [Car19]

$$X /\!\!/ \mathcal{F} := (\mathrm{colim}_{U \in \mathcal{B}} V)^{eff}.$$

By construction $X /\!\!/ \mathcal{F}$ is an effective smooth Deligne-Mumford stack of dimension d. Moreover, as $\mathcal{B}$ is a basis for the topology of X, we have that $X \simeq \mathrm{colim}_{U \in \mathcal{B}} U$. The local morphisms $\{f_U : U \to V\}_{U \in \mathcal{B}}$ therefore define the projection

$$\pi : X \simeq \mathrm{colim}_{U \in B} U \to \mathrm{colim}_{U \in \mathcal{B}} V \to (\mathrm{colim}_{U \in \mathcal{B}} V)^{eff} = X /\!\!/ \mathcal{F}.$$

The morphism π is flat surjective and $\pi^*(0) \simeq \mathcal{F}$. By construction, the morphism $\pi : X \to X /\!\!/ \mathcal{F}$, restricted to an open $U \in \mathcal{B}$ factors as $\pi_{|U} : U \xrightarrow{f_U} V \xrightarrow{p_V} X /\!\!/ \mathcal{F}$, where

p_V is the composition of the natural morphism $V \to \mathrm{colim}_{U\in B} V$, with the canonical projection $\mathrm{colim}_{U\in B} V \to X /\!\!/ \mathcal{F}$ to the effective quotient. The morphism p_V is étale, and f_U is flat, so we see that $\pi_{|U}$ is a flat morphism for all $U \in \mathcal{B}$. As $\mathcal{B}$ is a basis for the topology of X, this shows that π is flat.

To see that π is surjective, it is enough to notice that the universal effective quotient map is always surjective (because it does not the change the objects), and moreover that $\coprod_{U\in\mathcal{B}} V \to co\lim_{U\in\mathcal{B}} V$ is surjective. As a result, we see that the composition $\coprod_{U\in\mathcal{B}} U \to X \to X /\!\!/ /\mathcal{F}$ is a surjective morphism, and thus that π is surjective.

Finally, let us consider $\pi^*(0)$. We have two derived foliations on X, $\pi^*(0)$ and $\mathcal{F}$. For all $U \in \mathcal{B}$, we know that $f_U^*(0) \simeq \mathcal{F}_{|U}$, but as $p_V : V \to X /\!\!/ \mathcal{F}$ is étale, we have that $f_U^*(0) \simeq \pi^*(0)_{|U}$. Therefore, the two derived foliations $\pi^*(0)$ and $\mathcal{F}$ coincides on each open U. This implies in particular that the corresponding differential ideals $K_{\mathcal{F}}$ and $K_{\pi^*(0)}$, agree on each U, and thus globally agree on X. This means that $\mathcal{F}$ and $\pi^*(0)$ are two transversally smooth and rigid derived enhancements of the same differential ideals $K_{\mathcal{F}}$. It follows from the uniqueness statement of derived enhancements in codimension 2 (Proposition 6.10) that $\mathcal{F} \simeq \pi^*(0)$ as required.

Quasi-separatedness. The quasi-separatedness property for $X /\!\!/ \mathcal{F}$ follows from the general lemma below.

Lemma 6.7. *Any smooth effective Deligne-Mumford stack is quasi-separated.*

Proof of Lemma 6.7. Let Y be an effective Deligne-Mumford stack and $U \to Y$ an étale atlas with U a disjoint union of Stein manifolds (hence U is separated). The nerve of $U \to Y$ defines an étale groupoid $G = U \times_Y U$ acting on U. The quasi-separatedness of Y is then equivalent to the fact that the analytic space G is separated. So, we are left to prove that G is separated.

Let $\mathsf{Haf}(U)$ be the Haefliger groupoid on U. As a set, $\mathsf{Haf}(U)$ consists of triplets (x, y, u), with x and y points in U and u is a germ of holomorphic isomorphism from x to y. A basis for the topology on $\mathsf{Haf}(U)$ is defined as follows. Fix $V \subset U$ and $W \subset U$ two opens and $u : V \simeq W$ and isomorphism. Associated to V, W and u we have a subset $\mathsf{Haf}(U)_{V,W,u} \subset \mathsf{Haf}(U)$ consisting of all points $(x, u(x), u_x)$, where $x \in V$, $u(x) \in W$ its image, and u_x the germ of isomorphism at x from x to $u(x)$. By definition, a basis for the topology on $\mathsf{Haf}(U)$ consists of all $\mathsf{Haf}(U)_{V,W,u}$ where (V, W, u) varies. The groupoid structure on $\mathsf{Haf}(U)$ is the obvious one: the source (resp. target) map sends (x, y, u) to x (resp. y), and the composition is given by composing germs of isomorphisms.

Since we are working in the holomorphic context, the space $\mathsf{Haf}(U)$ is automatically separated, as two holomorphic maps agreeing on an open subset agree globally (on the connected components meeting this open).

Finally, there is a canonical morphism of groupoids over U, $\psi : G \to \mathsf{Haf}(U)$. To a point $u \in G$, with source x and target y, we associate $\psi(u)$ the germs of isomorphisms defined by the diagram

$$U \longleftarrow G \longrightarrow U \ ,$$

which, G being an étale groupoid, induces an isomorphism on germs

$$U_x \xleftarrow{\simeq} G_u \xrightarrow{\simeq} U_y .$$

The isomorphism $U_x \simeq U_y$ obtained this way defines $\psi(u)$. Now, the property of Y being effective is equivalent to the fact that the morphism ψ is a monomorphism. So $G \hookrightarrow \mathsf{Haf}(U)$ is here an injective holomorphic map. As $\mathsf{Haf}(U)$ is separated, this implies that G is also separated. $\square$

Actions of isotropy groups. Let $x : * \to X /\!\!/ \mathcal{F}$ be a point in $X /\!\!/ \mathcal{F}$, and H_x be the isotropy group at x. We have a commutative diagram with cartesian squares

$$\begin{array}{ccccc} \pi^{-1}(x) & \longrightarrow & \pi^{-1}(BH_x) \simeq [\pi^{-1}(x)/H_x] & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \bullet & \longrightarrow & BH_x & \longrightarrow & X /\!\!/ \mathcal{F}. \end{array}$$

The canonical morphism $BH_x \to X /\!\!/ \mathcal{F}$ is a monomorphism, and thus $[\pi^{-1}(x)/H_x] \to X$ is also a monomorphism. This implies that the stack $[\pi^{-1}(x)/H_x]$ is 0-truncated and thus is an analytic space. Moreover, as X is separated, so is $[\pi^{-1}(x)/H_x]$. Therefore, H_x acts freely and properly on $\pi^{-1}(x)$.

Relative connectedness. Let us denote by $Sh(Z) \subset St(Z)$ the full sub-category of 0-truncated objects (for some stack Z), i.e. the category of sheaves of sets. By construction, the functor $\pi^{-1} : Sh(X /\!\!/ \mathcal{F}) \to Sh(X)$ is obtained as a functor on limit categories

$$\lim_{U \in \mathcal{B}} f_U^{-1} : Sh(X /\!\!/ \mathcal{F}) \simeq \lim_{U \in B} Sh(V) \to \lim_{U \in B} Sh(U) \simeq Sh(X).$$

Therefore, in order to prove that π^{-1} is fully faithful, it is enough to prove that $f_U^{-1} : Sh(V) \to Sh(U)$ is fully faithful. But this follows easily from the fact that f_U is a flat surjective morphism with connected fibers, as shown by the next lemma.

Lemma 6.8. *Let $f : U \to V$ be an open and surjective morphism for which the topology of V is the quotient of the topology of U (i.e. a subset $T \subset V$ is open if and only if $f^{-1}(T) \subset U$ is open). Then, if f has connected fibers, for any sheaf of sets E on V the canonical morphism*

$$E \to f_* f^{-1}(E)$$

is an isomorphism.

Proof of Lemma 6.8. Let us represent E by its espace étalé $p : E \to V$. As the statement is local on V, it is enough to show that $E(V) \to f^{-1}(E)(U)$ is bijective. Note that $E(V)$ is the set of continuous sections of $E \to V$, and $f^{-1}(E)(U)$ is the set of continuous maps $s : U \to E$ such that $ps = f$. The map $E(V) \to f^{-1}(E)(U)$ simply sends a section $s : V \to E$ to $sf : U \to E$. As f is surjective, this map is clearly injective. Moreover, if $s : U \to E$ is an element $f^{-1}(E)(U)$, we can restrict s to the fiber of f at a point $v \in V$. This provides a continuous map $f^{-1}(v) \to E_v$, where E_v is the fiber of E at v endowed with the discrete topology. As $f^{-1}(v)$ is connected we see that the restriction of s along all the fibers of f is a constant morphism. Because f is a topological quotient, this implies that $s : U \to E$ descend to a continuous map $V \to E$, and thus s is the image of an element in $E(V)$. $\square$

An étale groupoid description. It is possible to describe the stack $X /\!\!/ \mathcal{F}$ by a (more or less explicit) étale groupoid acting on $W := \coprod_{U \in \mathcal{B}} V$. Recall that we have chosen for any $U \in \mathcal{B}$ a smooth surjective holomoprhic map with connected fibers $f : U \to V$ which integrates $\mathcal{F}_{|U}$. We consider all pairs of embeddings

$$U_1 \supset U_0 \subset U_2$$

of opens $V_i \in \mathcal{B}$. By Corollary 6.6, there is a corresponding pair of étale morphisms

$$V_1 \xleftarrow{b} V_0 \xrightarrow{s} V_2.$$

Therefore, for any $v \in V_0$, we deduce a germ of isomorphism $\alpha(v)$ between $(V_1, s(v))$ and $(V_2, b(v))$. When v varies inside V_0, we get a family of elements $\{(s(v), b(v), \alpha(v))\}_{v \in V_0}$ in $\mathsf{Haf}(W)$ in the Haefliger groupoid on $W = \coprod_{U \in \mathcal{B}} V$.

Lemma 6.9. *Let $G \to W$ be the étale groupoid obtained as the nerve of the canonical morphism $W \to X /\!\!/ \mathcal{F}$. Then, G is isomorphic, as a groupoid over W, to the subgroupoid of $\mathsf{Haf}(W)$ generated by the elements $(s(v), b(v), \alpha(v))$ for all possible choices of $U_1 \supset U_0 \subset U_2$ in $\mathcal{B}$ and of $v \in V_0$.*

Proof of Lemma 6.9. This follows from a more general formula for the effective part of a colimit of étale morphisms. Let $V_* : I \to \mathbb{C}Man^d_{et}$ be a diagram of smooth complex manifolds of dimension d with étale transition maps. Let $X = (\mathrm{colim} U_i)^{eff}$, and $W = \coprod V_i \to X$ be the canonical map. Then, the nerve $W \to X$ is isomorphic, as a groupoid over W, to the subgroupoid of $\mathsf{Haf}(W)$ generated by all the germs of isomorphisms generated by diagrams of the form $V_j \longleftarrow V_i \longrightarrow V_k$, image of $j \longleftarrow i \longrightarrow k$ in I by V_*. This last general fact is obvious from to the universal property of colimits and of the effective part. $\square$

Connectedness of the leaves and their holonomy coverings. We first prove that the leaves are connected. For this, let x and y be two points in X such that $\pi(x)$ and $\pi(y)$ are isomorphic in $X /\!\!/ \mathcal{F}$. By the construction of $X /\!\!/ \mathcal{F}$ as a colimit, we know that there exists a finite sequence of open inclusions in $\mathcal{B}$

$$U_1 \supset U_{1,2} \subset U_2 \supset U_{2,3} \subset U_3 \dots U_{n-1} \supset U_{n-1,n} \subset U_n,$$

with points $x_{i,j} \in U_{i,j}$, and such that

(1) $x \in U_1$ and $y \in U_n$
(2) $f_1(x_{1,2}) = f_1(x)$ and $f_n(x_{n-1,n}) = f_n(y)$
(3) $f_i(x_{i-1,i}) = f_i(x_{i,i+1})$

where $f_i : U_i \to V_i$ is our chosen map integrating $\mathcal{F}_{|U_i}$. As the fibers of f_i are all connected, it is possible to find path γ_i joining $x_{i-1,i}$ with $x_{i,i+1}$ and such that $f_i(\gamma_i)$ is constant (for all $i \neq 1, n$). In the same manner, we can choose a path α from x to $x_{1,2}$ with $f_1(\alpha)$ constant, and β from $x_{n-1,n}$ to y with $f_n(\beta)$ constant. The concatenation of the γ_i's with α and β provides a path $\gamma : [0,1] \to X$, from x to y and such that, for all $t \in [0,1]$, the objects $\pi(\gamma(t))$ are all isomorphic. By definition, this means that γ factors through the image of the monomorphism $\mathcal{L}_x^{max}(\mathcal{F}) \hookrightarrow X$, so that it is a continuous path in $\mathcal{L}_x^{max}(\mathcal{F})$ joining x and y. This implies that $\mathcal{L}_x^{max}(\mathcal{F})$ is path connected.

It remains to show that the H_x-covering $\widetilde{\mathcal{L}}_x^{max}(\mathcal{F}) \to \mathcal{L}_x^{max}(\mathcal{F})$ is connected, or equivalentlty that the corresponding classifying morphism $\pi_1(\mathcal{L}_x^{max}(\mathcal{F}), x) \to H_x$ is a surjective morphism of groups. This is proven using the same argument as before with $x = y$. Indeed, an element $h \in H_x$ is determined by a finite sequence of open inclusions in $\mathcal{B}$

$$U_1 \supset U_{1,3} \subset U_2 \supset U_{2,3} \subset U_3 \dots U_{n-1} \supset U_{n-1,n} \subset U_n,$$

with points $x_{i,j} \in U_{i,j}$, and such that

(1) $x \in U_1 \cap U_n$
(2) $f_1(x_{1,2}) = f_1(x)$ and $f_n(x_{n-1,n}) = f_n(x)$
(3) $f_i(x_{i-1,i}) = f_i(x_{i,i+1})$.

We have seen that there exists a loop γ in X, pointed at $x \in X$, such that $\pi(\gamma(t))$ is independent of $t \in [0,1]$, up to isomorphism, in $X /\!\!/ \mathcal{F}$. The image of γ by the morphism $\pi_1(\mathcal{L}_x^{max}(\mathcal{F}), x) \to H_x$ is h by construction.
This finishes the proof of Theorem 6.2. □

We finish this section with the following statement, concerning the functoriality of the leaf space construction, and thus of its characterization via a universal property. We leave the details to the reader.

Propositon 6.10. *Let $f : X \to Y$ be a morphism between smooth, connected and separated complex manifolds, $\mathcal{F}_Y \in \mathcal{F}ol(Y)$, and $\mathcal{F}_X := f^*(\mathcal{F}_Y)$. We assume that both derived foliations $\mathcal{F}_Y$ and $\mathcal{F}_X$ are transversally smooth, flat, smooth in codimension 2 and locally connected. Then, there exists a unique, up to a unique isomorphism, morphism $u : X /\!\!/ \mathcal{F}_X \to Y /\!\!/ \mathcal{F}_Y$ such that the following diagram commutes up to isomorphism*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X /\!\!/ \mathcal{F}_X & \xrightarrow[u]{} & Y /\!\!/ \mathcal{F}_Y . \end{array}$$

The following lemma relates the leaf space with the base of a flat proper morphism with connected fibers.

Lemma 6.11. *Let $f : X \to S$ be a proper and flat holomorphic morphism with S connected. If the derived foliation $f^*(0_S)$ is locally connected, then the Stein factorization $X \to S' =$* **Spec** *$f_*(\mathcal{O}_X) \to S$ is such that $S' \to S$ is a finite étale covering. We have*

$$X /\!\!/ f^*(0_Y) \simeq S'.$$

In particular, if one fiber of f is connected, then all the fibers of S are (and $S' = S$), and we have a canonical equivalence $X /\!\!/ f^*(0_Y) \simeq S$.

Proof of the lemma. By connectivity we can pull-back to a curve $C \to S$, and thus assume that S is a smooth curve. As f is flat then so is $S' \to S$ (because it must be non-constant). At a ramification point of $S' \to S$, the Milnor fiber can not be connected, so $S' \to S$ has no ramification point and is thus étale.

As $S' \to S$ is étale, we have $f^*(0_S) \simeq g^*(O_{S'})$ where $f : X \to S'$ is the Stein factorization of f. By Proposition 6.10, we then have a canonical morphism

$$X /\!\!/ f^*(0_S) \longrightarrow S'.$$

This morphism is étale as it can be checked easily on cotangent complexes. It is moreover a monomorphism, as its fibers must be connected because they are covered by the fibers of g. This implies that all the holonomy groups are trivial and thus that $X /\!\!/ f^*(0_S)$ is a separated smooth analytic space. As $X \to S'$ is proper, we have that $X /\!\!/ f^*(0_S) \to S'$ is also proper, and thus an isomorphism. □

7. Reeb stability and algebraic integrability

Theorem 7.1. *Let X be a smooth, proper and connected complex algebraic variety, and $\mathcal{F} \in \mathcal{F}ol(X/\mathbb{C})$ be a transversally smooth derived foliation on X. We suppose that $\mathcal{F}$ is flat, smooth in codimension 2, and locally connected. We denote by X^h and $\mathcal{F}^h$ the analytifications of X and $\mathcal{F}$. Then, the following three conditions are equivalent.*

1. *The analytic Deligne-Mumford stack $X^h /\!\!/ \mathcal{F}^h$ is algebraizable and proper.*
2. *There exists a smooth and proper effective Deligne-Mumford stack M, and a flat surjective morphism $p : X \to M$ with connected fibers such that $p^*(0) \simeq \mathcal{F}$.*
3. *There exists one point $x \in X^h$, such that $\mathcal{L}_x^{max}(\mathcal{F}^h)$, the maximal leaf passing through x, is compact, and its holonomy group $Hol_{\mathcal{F}^h}(x)$ is finite (or equivalently the holonomy covering $\widetilde{\mathcal{L}_x^{max}(\mathcal{F}^h)}$ is compact).*

Proof. (1) $\Rightarrow$ (2) We consider a smooth and proper Deligne-Mumford stack M with $M^h \simeq X^h /\!\!/ \mathcal{F}^h$. By GAGA the morphism $\pi : X^h \to M^h$ algebraizes to a morphism $p : X \to M$. Moreover, by the GAGA theorem for perfect derived foliations (Theorem 5.3), we know that the analytification ∞-functor $\mathcal{F}ol(X) \to \mathcal{F}ol(X^h)$ is an equivalence of categories. From $\pi^*(0) \simeq \mathcal{F}^h$, and from the compatibility of the analytification with pull-backs, we deduce that $p^*(0) \simeq \mathcal{F}$. Moreover, as $\pi = p^h$ has connected fibers, so does p.

(2) ⇒ (3) By the universal property of the leaf spaces (Proposition 6.10), we have a canonical identification $M^h \simeq X^h /\!\!/ \mathcal{F}^h$. In particular, the maximal leaves of $\mathcal{F}^h$ are automatically the analytification of the fibers of p, and thus compact, by hypothesis. In the same way, the holonomy groups of $\mathcal{F}^h$ are the isotropy groups of M, and thus are finite because M is proper.

(3) ⇒ (1) This implication is the main content of the theorem, and its proof will be somehow long. Let $x \in X$ be a closed point such that $\mathcal{L}_x^{max}(\mathcal{F}^h)$ is compact. The monomorphism $\mathcal{L}_x^{max}(\mathcal{F}^h) \hookrightarrow X^h$ is thus a closed immersion, and by GAGA $\mathcal{L}_x^{max}(\mathcal{F}^h)$ arises as the analytification of a closed subscheme $X_x \subset X$. We denote by $\widehat{X_x}$ the formal completion of X along X_x.

Analogously, the monomorphism $BH_x \hookrightarrow X^h /\!\!/ /\mathcal{F}^h$ is a closed immersion. Indeed, $X^h \longrightarrow X^h /\!\!/ /\mathcal{F}^h$ being flat and surjective, it is enough (by flat descent) to show that the pull-back $BH_x \times_{X^h /\!\!/ /\mathcal{F}^h} X^h \hookrightarrow X^h$ is a closed immersion. But $BH_x \times_{X^h /\!\!/ /\mathcal{F}^h} X^h \simeq X_x^h \simeq \mathcal{L}_x^{max}(\mathcal{F}^h)$, and we have seen that it is closed in X^h by assumption. We can therefore consider $\widehat{BH_x}$, the formal completion of the stack $X^h /\!\!/ /\mathcal{F}^h$ along the closed substack BH_x. We thus get a diagram of formal stacks

$$\begin{array}{ccc} \widehat{X_x} & \longrightarrow & X \\ \downarrow & & \\ \widehat{BH_x}. & & \end{array} \tag{3}$$

The formal stack $\widehat{BH_x}$ is of the form $[\widehat{A}^d/H_x]$, and as H_x is finite we can even assume that the action of H_x is induced by a linear action on $\mathbb{A}^d$. The above diagram can be considered as a formal $\widehat{BH_x}$-point of the stack $Fin(X)$ of schemes finite over X, whose reduced BH_x-point is given by $X_x \to X \times BH_x$. Because the stack $Fin(X)$ is an algebraic stack locally of finite presentation, this formal point can be algebraized (by Artin algebraization theorem). Therefore, we can find a pointed smooth variety (S, s), which is an étale neighborhood of 0 in $\mathbb{A}^d$, with a compatible H_x-action (fixing s), a scheme Z with a flat and proper map $X \to [S/H_x]$, and a diagram

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \downarrow{\scriptstyle q} & & \\ [S/H_x]. & & \end{array} \tag{4}$$

The above diagram is such that the fiber Z at $BH_x = [s/H_x] \hookrightarrow [S/H_x]$ is isomorphic to X_x as a subscheme in X. Moreover, considering the formal completion of the diagram 4 at $X_x \subset Z$ recovers the diagram (3) above.

Lemma 7.2. *By possibly shrinking (S, s) to a smaller étale neighborhood of* 0 *in* $\mathbb{A}^d$*, we have* $f^*(\mathcal{F}^h) \simeq q^*(0)$ *as derived analytic foliations on* Z *(i.e. in* $\mathcal{F}ol(Z^h)$*).*

Proof of Lemma 7.2. To prove this lemma we use the uniqueness of derived enhancement of Proposition 6.10. For this, we need to prove that $f^*(\mathcal{F}^h)$ and $q^*(0)$ are both smooth in codimension 2, and then that the corresponding differential ideals in $\Omega^1_{Z^h}$ coincide.

We first notice that by construction the morpshim $f : Z \to X$ is formally étale around $X_x \hookrightarrow Z$, and thus, by shrinking S if necessary we can assume that f is an étale morphism. Therefore, it is clear that $f^*(\mathcal{F}^h)$ is smooth in codimension 2, as it is locally isomorphic to $\mathcal{F}$. Concerning $q^*(0)$, in order to prove that it is smooth in codimension 2 it is enough to show that $\mathbb{L}_{Z/[S/H_x]}$ is a vector bundle outside of a codimension 2 closed subset. However, the perfect complex $\mathbb{L}_{Z/[S/H_x]}$ and $f^*(\mathbb{L}_{\mathcal{F}})$ are quasi-isomorphic on the formal completion $\widehat{X_x}$. Because the stack of quasi-isomorphisms between two perfect complex is an Artin stack of finite presentation (see for instance [TV07]), we deduce that these two perfect complexes must be quasi-isomorphic locally around $X_x \subset Z$. Therefore, by shrinking S if necessary, we can assume that they are quasi-isomorphic on Z. This implies that $q^*(0)$ is also smooth in codimension 2.
Finally, we can use the same argument but now for the stack of quasi-isomorphisms between $\mathbb{L}_{Z/[S/H_x]}$ and $f^*(\mathbb{L}_{\mathcal{F}})$ compatible with the canonical morphism

$$\mathbb{L}_{Z/[S/H_x]} \longleftarrow \Omega^1_Z \longrightarrow f^*(\mathbb{L}_{\mathcal{F}}).$$

This shows that the two quotients of coherent sheaves

$$\Omega^1_Z \to H^0(\mathbb{L}_{Z/[S/H_x]}) \qquad \Omega^1_Z \to H^0(f^*(\mathbb{L}_{\mathcal{F}}))$$

are isomorphic, and thus their kernel must be equal. These kernels being precisely the differential ideals $K_{q^*(0)}$ and $K_{f^*(\mathcal{F})}$, we see that these must coincide. Therefore, all the conditions are met to apply Proposition 4.3 and we deduce the lemma. ☐

Since the morphism $f : Z \to X$ is étale, it is easy to see that $f^*(\mathcal{F})^h$ is not only smooth in codimension 2, but also flat and locally connected. We can thus apply functoriality of leaf spaces (Proposition 6.10) in order to see that there exists a commutative diagram in the analytic category

$$\begin{array}{ccc} Z^h & \xrightarrow{f} & X^h \\ \downarrow p & & \downarrow \pi \\ Z^h /\!\!/ f^*(\mathcal{F})^h & \xrightarrow[g]{} & X^g /\!\!/ \mathcal{F}^h \\ \downarrow r & & \\ [S^h/H_x]. & & \end{array}$$

The morphism r is here a finite étale cover, which can be identified with the relative connected components of $Z \to [S/H_x]$ (see Lemma 6.11). Since X_x is connected, we see that this finite étale cover must be trivial, and thus r is an equivalence. We therefore conclude the existence of commutative diagram

$$\begin{array}{ccc} Z^h & \xrightarrow{f} & X^h \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ [S^h/H_x] & \xrightarrow[g]{} & X^g /\!\!/ \mathcal{F}^h. \end{array}$$

But f is étale, and $q^*(0) \simeq \pi^*(0)$, a comparison of the cotangent complexes shows that g must also be étale.

As a result, for any $y \in S^h$, we have an étale morphism of connected analytic spaces

$$\{y\} \times_{[S^h/H_x]} Z^h \longrightarrow \widetilde{\mathcal{L}}^{max}_{f(y)}(\mathcal{F}).$$

However, because p is proper we deduce that the source is a proper analytic space, and that the above morphism is a finite étale cover, and thus a proper surjective morphism. In particular, $\widetilde{\mathcal{L}}^{max}_{f(y)}(\mathcal{F})$ must be compact. We thus conclude that for all $y \in X$, in the image of f, the maximal leaf $\mathcal{L}^{max}_y(\mathcal{F}) \subset X^h$ is compact and its holonomy H_y is finite. As f is an étale morphism, its image is a dense Zariski open subset in X, say $W \subset X$. Let $y \in X - U$ be a closed point, and let y_i be a sequence of closed points in X converging to y in the analytic topology. We may assume that the holonomy groups of the y_i are all trivial, as these form a Zariski dense open substack in $[S/H_x]$. We then consider the corresponding sequence of closed subschemes $\mathcal{L}^{max}_{y_i}(\mathcal{F}^h) \subset X$. By compactness of the Hilbert scheme, this sequence of subschemes possesses a limit $L \subset X$. Clearly π sends L to $BH_y \subset X^h /\!\!/ \mathcal{F}^h$, and thus L is contained in the maximal leaf $\mathcal{L}^{max}_y(\mathcal{F}^h)$. Moreover, y being the limit of y_i, we have $y \in L$. More generally, any point $y' \in \mathcal{L}^{max}_y(\mathcal{F})$ is a limit of points $y'_i \in \mathcal{L}^{max}_{y_i}(\mathcal{F}^h)$, and thus $y' \in L$. Therefore we have that L and $\mathcal{L}^{max}_y(\mathcal{F}^h)$ coincide set theoretically. This implies in particular that $\mathcal{L}^{max}_y(\mathcal{F}^h)$ is compact, that the monomorphism $\mathcal{L}^{max}_y(\mathcal{F}^h) \hookrightarrow X^h$ is a closed immersion, and by GAGA that $\mathcal{L}^{max}_y(\mathcal{F}^h)$ is algebraizable. We thus conclude that the maximal leaves $\mathcal{L}^{max}_y(\mathcal{F}^h)$ are compact and algebraizable for all $y \in X^h$.

It remains to show that the holonomy group $H_y = Hol_y(\mathcal{F}^h)$ is finite for all y. For this, we use the same techniques. We write the maximal leaf $\mathcal{L}^{max}_y(\mathcal{F}^h)$ as the limit, in the Hilbert scheme, of compact leaves with trivial holonomy $\mathcal{L}^{max}_{y_i}(\mathcal{F}^h)$. This limit can be arranged as a diagram of analytic spaces

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \\ S & & \end{array}$$

where S is a small holomorphic disk, Z is flat and proper with connected fibers over S, and $Z \to X \times S$ is a closed immersion. Let $Z_0 = \mathcal{L}^{max}_y(\mathcal{F}^h) \subset X$ be the fiber at the central point $0 \in S$, and $Z_t \subset X$ denotes the "generic" or Milnor fiber. The nearby cycles in dimension 0 form a constructible sheaf on X_0, and therefore any open cover U_i of X_0 in X can be refined to a cover such that each $U_i \cap X_t$ has a uniformly bounded number of connected components. By covering X_0 with U_i belonging to our basis $\mathcal{B}$ for the topology, we deduce the existence of an étale morphism $V \to X^h /\!\!/ \mathcal{F}^h$ covering the point y, and such that $\pi^{-1}(V) \cap \mathcal{L}^{max}_{y_i}(\mathcal{F}^h)$ possesses at most m connected components for some fixed integer m. Equivalently, let $G \rightrightarrows V$ be the étale groupoid induced on V, so that $[V/G]$ is an open substack in $X^h /\!\!/ \mathcal{F}^h$. Then, all the orbits of the points y_i are finite of order at most m in V. As a result, if $h \in G_y$ is a point in the stabilizer of $y \in V$, since all y_i's have trivial holonomies, we must have that h^m is the identity near y_i, for all i (for all i big enough such that h^m is defined at y_i). By analytic continuation we have that $h^m = id$. Now, let H_y be the holonomy group at a point y. We have seen that H_y is of finite exponent m. Moreover, as it is a subgroup of the group of germs of holonomorphic isomorphisms of $\mathbb{C}^d$ at 0, it must be finite (see [Per01, Lem. 2]).

To summarize, we have seen that all the maximal leaves are compact, and all the holonomy groups are finite. We still need to show that this implies that $X^h /\!\!/ \mathcal{F}^h$ is proper and algebraizable. We start by proving that $X^h /\!\!/ \mathcal{F}^h$ is a separated Deligne-Mumford stack, i.e. that its diagonal is a finite morphism. For this, we come back to the very beginning of the proof of (3) $\Rightarrow$ (1). Now that we know that all the maximal leaves are compact, and all holonomy groups are finite, we can run exactly the same argument starting with any point $x \in X$. We have seen that there exists a commutative diagram

$$\begin{array}{ccc} Z^h & \xrightarrow{f} & X^h \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ [S^h/H_x] & \xrightarrow[g]{} & X^h /\!\!/ \mathcal{F}^h. \end{array}$$

In this diagram p is a proper morphism, and moreover the natural morphism $Z^h \to [S^h/H_x] \times_{X^h /\!\!/ \mathcal{F}^h} X^h$ is étale. Properness of p implies that this étale morphism is moreover a finite covering. As a consequence $[S^h/H_x] \times_{X^h /\!\!/ \mathcal{F}^h} X^h \to [S^h/H_x]$ is proper. As the étale morphisms of the form $[S^h/H_x] \to X^h /\!\!/ \mathcal{F}^h$ cover the whole stack $X^h /\!\!/ \mathcal{F}^h$, we deduce that the morphism $\pi : X^h \to X^h /\!\!/ \mathcal{F}^h$ is a proper morphism.

We consider the nerve of π, which defines a proper and flat groupoid $G \rightrightarrows X^h$. The diagonal morphism $G \to X^h \times X^h$ is thus a proper and unramified morphism. It is thus a finite morphism. As $X^h \to X^h /\!\!/ \mathcal{F}^h$ is a flat covering, this implies that the diagonal of the stack $X^h /\!\!/ \mathcal{F}^h$ is a finite morphism, and thus that $X^h /\!\!/ \mathcal{F}^h$ is a separated Deligne-Mumford stack. Finally, as $X^h \to X^h /\!\!/ \mathcal{F}^h$ is proper and surjective, we deduce that $X^h /\!\!/ \mathcal{F}^h$ is also proper.

To finish, we need to prove that $X^h /\!\!/ \mathcal{F}^h$ is algebraizable. But this follows by considering the groupoid $G \rightrightarrows X^h$ which is the nerve of π as above. Since $G \to X^h \times X^h$ is finite, and X^h and G are both proper algebraic spaces, we know by GAGA that the groupoid $G \rightrightarrows X^h$ is the analytification of a proper flat algebraic groupoid $G' \rightrightarrows X$. We can consider the quotient stack $[X/G']$ of this proper flat groupoid (see for instance [Toe11]) which defines an algebraic Deligne-Mumford stack whose analytification is $X^h /\!\!/ \mathcal{F}^h$. □

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