INFINITESIMAL DERIVED FOLIATIONS

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Abstract. We introduce a notion of *infinitesimal derived foliation*. We prove it is related to the classical notion of infinitesimal cohomology, and satisfies some formal integrability properties. We also provide some hints on how infinitesimal derived foliations compare to our previous notion of derived foliations of [Toë20, TV].

Contents

Introduction

The object of this short note is to present a notion of *infinitesimal derived foliations* on general schemes and stacks over bases of arbitrary characteristics (see Definition 2.1). This notion is slightly different from the one exposed in [Toë20] but it is based on similar ideas. By definition, an infinitesimal derived foliation on X consists of a perfect complex E (the cotangent complex of the foliation) together with an action of $G_0 = \Omega_0\mathbb{G}_a$, the loop group of the additive group, on the total space $\mathbb{V}(E[-1])$ of the shift $E[-1]$, compatible with the natural gradings. This is very similar to the definition in [Toë20], except that the group $BKer$ is replaced here by G_0, and the shift by 1 is replaced by a shift by -1. These changes may sound purely aesthetic choices, but outside of characteristic zero these two notions of derived foliations are, in fact, different.

A major advantage of infinitesimal derived foliations is that they satisfy several formal integrability results (see Corollary 4.2) that do *not* hold for derived foliations. The key point here is that the natural cohomology theory related to infinitesimal derived foliations is (Hodge-completed derived) infinitesimal cohomology, whereas derived foliations are related to (Hodge-completed derived) de Rham cohomology. This relation with infinitesimal cohomology makes it possible to formally integrate any (smooth) infinitesimal derived foliation by a smooth formal groupoid. In a way, infinitesimal derived foliations can be thought of as derived foliations endowed with extra conditions/structures. This resembles the difference between foliations and *restricted foliations* in characteristic $p > 0$ (see for instance [Eke87, Prop. 2.3] as well as [Miy87, Def. 3.6]).

The first section of this paper contains results on *derived affine stacks*, that are a derived extension of affine stacks of [Toe06]. We present in particular algebraic models for those, using a certain model category of cosimplicial-simplicial commutative algebras, which might be of independent interests. These models will be particularly useful in our last section where we construct the red-shift construction for graded derived affine stacks (see §5). The second section presents the general definition of infinitesimal derived foliations and their basic properties. In section §3 we study the cohomology of infinitesimal derived foliations and prove that it recovers the usual notion of infinitesimal cohomology when applied to the tautological foliation. This cohomology comparison is then used to construct, for any smooth infinitesimal derived foliation $\mathcal{F}$ on a smooth scheme X, a smooth formal groupoid $G_{\mathcal{F}} \to X$, and we show that $\mathcal{F} \mapsto G_{\mathcal{F}}$ produces an equivalence of categories. This result has to be related with [Eke87, Prop. 2.3], and in a way, is one possible generalization of Ekedahl's result. Finally, in the very last section, we study the relations between infinitesimal derived foliations and our original notion of derived foliations (see [Toë20, TV]). Unfortunately the results here are not optimal, and we leave several open questions on the way. We think that there should exists a forgetful functor from infinitesimal derived foliations to derived foliations, obtained by some application of the red-shift construction, but we could not find a completely satisfying construction. The reader should consider this last part as a suggestion for future research.

As a final comment, this note will appear in an extended form as a chapter of a book in preparation on the subject ([TV]).

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1. Derived affine stacks

1.1. **Cosimplicial-simplicial algebras.** We let denote by sCR be the category of simplicial commutative rings. It is endowed with its usual *model category* structure for which equivalences and fibrations are defined on the underlying simplicial sets.

Similarly, we denote by $csCR$ the category of cosimplicial objects in sCR, or equivalently of cosimplicial-simplicial commutative rings. Its objects will be denoted by A^*_*, where the superscript will refer to the cosimplicial direction and subscript to the simplicial direction. The category $csCR$ can be endowed with a *tensored* and *cotensored* structure over the category $sSet$ of simplicial sets. We warn the reader here that there are several natural such structures and that the choice we make here is important for the sequel. For a simplicial set K and an object A of $csCR$, we define the cotensored structure by A^K by the formula

$$(A^K)^q_p := (A^q_p)^{K_q}$$

for $[p] \in \Delta^{op}$ and $[q] \in \Delta$. The tensored structure $K \otimes A$, for $A \in csCR$ and $K \in sSets$, is defined by the usual adjunction formula $Hom(K \otimes A, B) \simeq Hom(A, B^K)$, for arbitrary $B \in csCR$. Similarly, we have a canonical simplicial enrichment for which the simplicial Homs are denoted by $\underline{Hom}$ and are defined by

$$\underline{Hom}(A, B)_p := Hom_{csCR}(A, B^{\Delta^p}).$$

By considering the natural embedding $Mod_{\mathbb{Z}} \subset C(\mathbb{Z})$, from $\mathbb{Z}$-modules to cochain complexes, any object $A \in csCR$ can be considered, by forgetting the multiplicative structure, as a cosimplicial-simplicial object in $C(\mathbb{Z})$. We denote by $Tot^\pi(A)$ the complex equals to $\prod_{q-p=n} A^q_p$ in cohomological degree n and whose differential is given by the alternated sums of the faces and cofaces of the cosimplicial-simplicial diagram. Equivalently, A can be first turned into a bi-complex, by individual totalisations along the cosimplicial and the simplicial directions, and $Tot^\pi(A)$ can then be realized as the product-total complex of this bicomplex. Note also that $Tot^\pi(A)$ is a model for the homotopy limit-colimit of A

$$Tot^\pi(A) \simeq \mathrm{holim}_{[p]\in\Delta} \left(\mathrm{hocolim}_{[q]\in\Delta^{op}} A^q_p\right) \in C(\mathbb{Z}).$$

Definition 1.1. *A morphism $A \to B$ in $csCR$ is a* completed quasi-isomorphism *if the induced morphism $Tot^\pi(A) \to Tot^\pi(B)$ is a quasi-isomorphism of complexes.*

We warn the reader that even though the categories $csCR$ and $scCR$, of cosimplicial-simplicial commutative rings and of simplicial-cosimplicial rings, are equivalent, the notion of completed quasi-isomorphisms is not the same as the notion of weak equivalences used for instance in [BCN21].

Theorem 1.2. *The category $csCR$ is endowed with a model category structure for which the weak equivalences are the completed quasi-isomorphism, and the fibrations are the epimorphisms. This model category structure is moreover cofibrantly generated and is a simplicial model category for the above mentioned simplicial enrichment.*

Proof. As fibrations[1] and equivalences are defined via the functor Tot^π, the proof of the theorem consists of transferring the projective model structure on $C(\mathbb{Z})$ to $csCR$ via the functor Tot^π. For this we apply the path object argument (see for instance [BM03, §2.6]).

[1]Note that $f : A \to B$ in $csCR$ is surjective iff $Tot^\pi(f)$ is.

The functor Tot^π is a right adjoint whose left adjoint sends a complex $E \in C(\mathbb{Z})$ to the free commutative ring over the cosimplicial-simplicial module given by

$$(p, q) \mapsto (C^*(\Delta^p) \otimes C_*(\Delta^q) \otimes E)^0 / Im\, d^{-1},$$

where $C^*(\Delta^p)$ and $C_*(\Delta^q)$ are the cohomology and homology complexes of the standard simplex Δ^p and Δ^q. This left adjoint will be denoted by $L\phi$, as being the composition of the free commutative ring functor $L : csMod \to csCR$, with the functor $\phi : C(\mathbb{Z}) \to csMod$ defined by the above formula. Clearly, the functor Tot^π preserves small objects (but does not commute with filtered colimits in general). In order to apply [BM03, §2.6] we thus simply have to prove that $csCR$ possesses a fibrant replacement functor and that all fibrant object possesses a path object. But all objects are fibrant by definition, and we are thus reduced to show the existence of path objects. For this we use the following lemma.

Lemma 1.3. *For any simplicial set K and any $A \in csCR$, there exists a canonical quasi-isomorphism*

$$Tot^\pi(A^K) \simeq Tot^\pi(A)^K,$$

where for $E \in C(\mathbb{Z})$ and $K \in sSet$ we define

$$E^K := \mathrm{holim}_{[p]\in\Delta} E^{K_p}.$$

Proof of the lemma. We remind that for any functor $F : \Delta \times \Delta \to \mathbf{dg}$ (or more generally any bi-cosimplicial object in a complete ∞-category), the homotopy limit of F can be computed, up to a quasi-isomorphism, either component-wise, or diagonally

$$\mathrm{holim}_{[p]\in\Delta} F(p, p) \simeq \mathrm{holim}_{[p]\in\Delta} \mathrm{holim}_{[q]\in\Delta} F(p, q).$$

We apply this to the functor sending $([p], [q])$ to the complex $N(A^q_*)^{K_p}$, where $N : sMod \to dg$ is the normalization construction. Computing the homotopy limit diagonally yields $Tot^\pi(A^K)$, whereas the double homotopy limit construction yields $Tot^\pi(A)^K$. $\square$

Lemma 1.3 now implies the existence of path objects. Indeed, for $A \in csCR$ an explicit path object is given by $A \to A^{\Delta^1} \to A^{\partial\Delta^1} = A \times A$. The fact that the constant map morphism $A \to A^{\Delta^1}$ is a completed quasi-isomorphism follows from the lemma 1.3. The fact that $A^{\Delta^1} \to A^{\partial\Delta^1} = A \times A$ is a fibration simply follows from the fact that it is a levelwise epimorphism as this can be checked easily using the explicit formula for A^K.

In order to finish the proof of the theorem, we are left to showing that the simplicial enrichement is compatible with the model category structure. In other words, we must show that for any cofibration of simplicial sets $i : K \hookrightarrow L$ and any fibration $f : A \to B$ in $csCR$, the induced morphism

$$f^i : A^L \longrightarrow B^L \times_{B^K} A^K$$

is a fibration, which is also an equivalence if i is a trivial cofibration or f is a trivial fibration. The fact that the morphism f^i is a fibration is clear because fibrations are epimorphisms and because of the explicit formula for the exponentiation by a simplicial set in $csCR$. Finally, the fact that f^i is also a weak equivalence when i or f is so simply follows from Lemma 1.3. □

Definition 1.4. *The* ∞-category of cs-rings *is the* ∞*-category obtained from* $csCR$ *by inverting the completed quasi-isomorphisms. It is denoted by* $\mathbf{csCR}$ *(or by* $\mathbf{csCR}_{\mathbb{U}}$ *if one wants to specify cosimplicial-simplicial rings belonging to a given universe* $\mathbb{U}$*).*

Remark 1.5. We note also that the model structure on $csCR$ can be equivalently be obtained by first constructing a model structure on $csMod_{\mathbb{Z}}$, the category of cosimplicial-simplicial abelian groups, where again the equivalences and fibrations are defined via the Tot^{π} functor. The model structure on $csCR$ can then be obtained by transferring along the forgetful functor $csCR \to csMod_{\mathbb{Z}}$, with left adjoint given by the free commutative cosimplicial-simplicial ring.

Remark 1.6. To finish, we mention that it is possible to a construct an ∞-functor Tot^{π} : $\mathbf{csCR} \longrightarrow LSym - Alg$, from our ∞-category of cosimplicial-simplicial commutative rings to the ∞-category of *derived rings*, namely modules over the $LSym$-monad. Here we denote by $LSym$ the monad associated to the derived commutative operad as in [BCN21, Ex. 3.71]. This ∞-functor simply sends A to $\lim_q A^q_*$, where the limit is taken in $LSym - Alg$ and A^q_* is considered as a connective $LSym$-algebra and thus as an object in $LSym - Alg$. This ∞-functor is the right adjoint of an adjunction, and preserves free objects over perfect complexes. It is likely that Tot^{π} is an equivalence of ∞-categories when restricted to nice enough objects.

1.2. **Spectrum of cosimplicial-simplicial rings.** For a fixed commutative simpicial ring $k \in sCR$, we can work relatively over k and define the model category $k - csCR$ of cosimplicial-simplicial commutative k-algebras, and its associated ∞-category $k - \mathbf{csCR}$. As usual, we have a natural equivalence of ∞-categories

$$k - \mathbf{csCR} \simeq k/\mathbf{csCR},$$

where k is considered as an object in $\mathbf{csCR}$ which is constant is the cosimplicial direction.

We consider $dAff_k := (k - sCR)^{op}$, the model category of derived affine k-schemes, which is defined to be the opposite category of that of simplicial commutative k-algebras. Remind from [TV08] that it can be endowed with the fpqc model topology, and that we can consider the model category of (hyper-complete) stacks $dAff_k^{\sim,fpqc}$. There are here some set-theoretical issues, that can be solved, as usual, by fixing two universes $\mathbb{U} \in \mathbb{V}$. By definition $k - sCR$ refers here to $\mathbb{U}$-small simplicial k-algebras, and $dAff_k^{\sim,fpqc}$ is then the category of functors $k - sCR \to sSet_{\mathbb{V}}$ to $\mathbb{V}$-small simplicial sets.

We then consider a functor

$$\mathbf{Spec}^{\Delta} : k - csCR_{\mathbb{V}}^{op} \longrightarrow dAff_k^{\sim,fpqc},$$

from $\mathbb{V}$-small cosimplicial-simplicial commutative k-algebras to $dAff_k^{\sim,fpqc}$. It is defined by sending $A \in k - csCR_{\mathbb{V}}$ to the functor $\mathbf{Spec}^{\Delta} A : sCR \to sSet_{\mathbb{V}}$ defined by

$$\mathbf{Spec}^{\Delta} A : B \mapsto \underline{Hom}(A, B) = Hom_{csCR}(A, B^{\Delta^{\bullet}}),$$

where $\underline{Hom}$ are the simplicial Hom's of the simplicial enrichement described in the previous section, and B is considered as an object in $k-csCR_{\mathbb{V}}$ by viewing it as constant in the cosimplicial direction.

Propositon 1.7. *The functor $\mathbf{Spec}^{\Delta} : k-csCR_{\mathbb{V}}^{op} \longrightarrow dAff_k^{\sim,fpqc}$ is right Quillen, and the induced functor on ∞-categories*

$$\mathbf{Spec}^{\Delta} : k-\mathbf{csCR}_{\mathbb{U}} \longrightarrow \mathbf{dSt}_k$$

is fully faithful. The essential image of $\mathbf{Spec}^{\Delta}$ is the smallest full sub-∞-category of $\mathbf{dSt}_k$ containing the objects $K(\mathbb{G}_a, n)$ for various n, and which is stable by $\mathbb{U}$-small limits.

Proof. This is proven in a very similar manner than [Toe06, Cor. 2.2.3]. The left adjoint to $\mathbf{Spec}^{\Delta}$, denoted by $\mathcal{O}$, sends a representable presheaf $h^B = Hom(B,-) : k-sCR \to sSet$ to $B \in k-csCR$, which is considered as constant in the cosimplicial direction. It also sends an object of the form $K \times h^B$, for a simplicial set $K \in sSet$, to $B^K \in k-csCR$. Finally, it is uniquely defined by these properties together with the requirement that it sends colimits in $dAff_k^{\sim,fpqc}$ to limits in $k-csCR$.

To prove that $\mathbf{Spec}^{\Delta}$ is right Quillen we use that $dAff_k^{\sim,fpqc}$ is a left Bousfield localization of the levelwise projective model structure on the category of simplicial presheaves $Fun(k-sCR_{\mathbb{U}}, sSet)$ obtained by inverting fpqc-local equivalences and equivalences in sCR (see [TV08]). Therefore, by general facts about Bousfield localizations, it is enough to show that $\mathbf{Spec}^{\Delta}$ is a right Quillen functor for the levelwise projective model structure, and moreover that for any cofibrant $A \in k-csCR$, $\mathbf{Spec}^{\Delta}(A)$ is a fibrant object in $dAff_k^{\sim,fpqc}$. The first of these statements easily follows from the fact that $k-csCR$ is a simplicial model category in which every object is fibrant, and thus that if $A \to A'$ is a (trivial) cofibration in $k-csCR$, for any $C \in k-sCR$ the morphism $\underline{Hom}(A', C) \to \underline{Hom}(A, C)$ is a (trivial) fibration of simplicial sets. For the second of these statements, let $A \in k-csCR$ be a cofibrant object. Any equivalence $B \to B'$ in $k-sCR$ obviously induces an equivalence in $k-csCR$ when considered both B and B' as constant in the cosimplicial direction, and thus $\underline{Hom}(A, B) \to \underline{Hom}(A, B')$ is an equivalence of simplicial sets. This shows that $\mathbf{Spec}^{\Delta}(A)$ is fibrant when the topology is the trivial topology. Finally, if $B = \mathrm{holim}_i B_i$ is a homotopy limit in $k-sCR$, it is also a homotopy limit in $k-csCR$, and thus the natural morphism $\underline{Hom}(A, B) \to \mathrm{holim}_i \underline{Hom}(A, B_i)$ is an equivalence. As the fpqc model topology is subcanonical, this implies that $\mathbf{Spec}^{\Delta}(A)$ is indeed a stack for the fpqc topology, and thus fibrant as an object in $dAff_k^{\sim,fpqc}$ (see [TV08]).

To finish the proof of the proposition, let $A \in k-csCR_{\mathbb{U}}$ and $X = \mathbf{Spec}^{\Delta}(A)$. By definition of the simplicial structure on $k-csCR$, the functor X is the geometric realization of the simplicial object $q \mapsto h^{A_*^q}$, where $h^{A_*^q} : k-sCR \to sSet$ is corepresented by $A_*^q \in k-sCR$. In other words, we have $\mathbf{Spec}^{\Delta}(A) \simeq \mathrm{hocolim}_q h^{A_*^q}$. Therefore, the adjunction morphism

$$A \longrightarrow \mathcal{O}(\mathbf{Spec}^{\Delta}(A))$$

is the canonical morphism $A \to \mathrm{holim}_q A_*^q$ in $Ho(k-csCR_{\mathbb{U}})$. This canonical morphism is obviously an equivalence, showing the fully faithfulness property in the proposition. It is then formal that the essential image of $\mathbb{R}\mathbf{Spec}^{\Delta}$ is stable by limits. It also contains the objects $K(\mathbb{G}_a, n)$, as these are the images of $L\phi(\mathbb{Z}[-n])$, where $L\phi : C(\mathbb{Z}) \to k - csCR$ is the left adjoint to Tot^{π}. Finally, as $k - csCR$ is cofibrantly generated, any object is equivalent to a I-cell object, where I is the image by $L\phi$ of the generating cofibrations in $C(\mathbb{Z})$. The images of I by $\mathbf{Spec}^{\Delta}$ are equivalent to morphisms of the form $* \to K(\mathbb{G}_a, n)$, so that any object is the essential image of $\mathbb{R}\mathbf{Spec}^{\Delta}$ lies in the smallest sub-∞-category containing the $K(\mathbb{G}_a, n)$ and stable by limits. □

Definition 1.8. *The ∞-category of* (k-linear) derived affine stacks *is the essential image of* $\mathbb{R}\mathbf{Spec}^{\Delta} : k - \mathbf{csCR}_{\mathbb{U}}^{op} \hookrightarrow \mathbf{dSt}_k$. *It is denoted by* $\mathbf{dChAff}_k$.

The above notions and results also have *graded versions*, for which the details are left to the reader. We denote by $k - csCR^{gr}$ the category of graded cosimplicial-simplicial commutative k-algebras. We endow this category with the model category structure for which equivalences and fibrations are defined by the graded Tot functor to graded complexes $Tot^{\pi} : k - scCR^{gr} \to C(\mathbb{Z})^{gr}$, defined by taking the Tot^{π} of each graded component. The proof of the existence of this model category structure follows the same lines as in the non-graded case. The corresponding ∞-category will be denoted by $k - \mathbf{csCR}^{gr}$.

We consider the strict multiplicative group $\mathbb{G}_m^{st} \in \mathbf{dAff}_k^{\sim,fpqc}$, given by the functor sending $B \in k - sCR$ to the group B_0^* of invertible elements in the ring B_0 of 0-simplices in B. This is a group object in the model category $\mathbf{dAff}_k^{\sim,fpqc}$, which is not a fibrant object (it does not preservers equivalences in B), but its image in $\mathbf{dSt}_k$ is the usual multiplicative group scheme $\mathbb{G}_m$. Indeed, $\mathbb{G}_m^{st} = h^{k[t,t^{-1}]}$ is corepresented by $k[t, t^{-1}]$, and, as shown in [TV08], a fibrant model for h^A is the derived scheme $\mathbf{Spec}\, A : B \mapsto Map(A, B)$. In particular, we have that the model category $\mathbb{G}_m^{st} - \mathbf{dAff}_k^{\sim,fpqc}$, of $\mathbb{G}_m^{st}$-equivariant objects in $\mathbf{dAff}_k^{\sim,fpqc}$, is a model for the ∞-category $\mathbb{G}_m - \mathbf{dSt}_k$, of $\mathbb{G}_m$-equivariant derived stacks, called *graded derived k-stacks*.

The graded version of the spec functor is the functor

$$\mathbf{Spec}^{\Delta,gr} : (k - csCR^{gr})^{op} \longrightarrow \mathbb{G}_m^{st} - \mathbf{dAff}_k^{\sim,fpqc},$$

sending $A \in k - csCR^{gr}$ to $\underline{Hom}(A, -)$, endowed with the natural $\mathbb{G}_m^{st}$-action coming from the grading on A. More explicitly, for $B \in k - sCR$, the set of q-simplices of $\mathbf{Spec}^{\Delta,gr}(A)(B)$ is the set of morphisms $Hom(A^q, B)$, which is endowed with a B_0^*-action as follows. For $b \in B_0^*$ and $f : A^q \to B$, we define $bf : A^q \to B$ by the formula $(bf)(x) = b^n.f(x)$ for a homogeneous element x of degree n. As in the non-graded case, $\mathbf{Spec}^{\Delta,gr}$ is a right Quillen functor, and the induced ∞-functor

$$\mathbb{R}\mathbf{Spec}^{\Delta,gr} : (k - \mathbf{csCR}_{\mathbb{U}}^{gr})^{op} \longrightarrow \mathbb{G}_m - \mathbf{dSt}_k$$

is fully faithful. Its essential image is the smallest sub-∞-category containing all the objects $K(\mathbb{G}_a^{\chi}, n)$, for $n \geq 0$ and $\chi \in \mathbb{Z}$, where $\mathbb{G}_m$ acts on $\mathbb{G}_a$ with weight $\chi \in \mathbb{Z}$, and which is stable by $\mathbb{U}$-small limits.

Definition 1.9. *The ∞-category of* graded derived affine k-stacks *is the essential image of* $\mathbb{R}\mathbf{Spec}^{\Delta,gr} : (k - \mathbf{csCR}_{\mathbb{U}}^{gr})^{op} \hookrightarrow \mathbb{G}_m - \mathbf{dSt}_k$. *It is denoted by* $\mathbf{dChAff}_k^{gr}$.

We finish this section by recalling the notion of *linear derived stack*, already introduced and studied in [Mon21]. We first notice that the Tot functor $Tot^{\pi} : csMod_k \longrightarrow C(\mathbb{Z})$ admits a natural lax monoidal structure. Therefore, it gives rise to an ∞-functor

$$Tot^{\pi} : \mathsf{csMod}_k \longrightarrow \mathbf{dg}_k,$$

where csMod_k is the ∞-category of cosimplicial-simplicial k-modules. This ∞-functor turns out to be an equivalence of ∞-categories, and we denote by $\phi : \mathbf{dg}_k \to \mathsf{csMod}_k$ its inverse.

For any object $E \in \mathbf{dg}_k$, we consider $\phi(E) \in \mathsf{csMod}_k$, and define

$$Sym^{\Delta}(E) := L\phi(E) \in csCR^{gr}$$

the free cosimpicial-simplicial commutative ring generated by $\phi(E)$ (same notation as in the proof of Theorem 1.2). Being a free commutative ring, $Sym^{\Delta}(E)$ comes natural equipped with a graduation, and thus is considered as an object in the ∞-category $\mathbf{csCR}^{gr}$.

Definition 1.10. *The k-*linear derived stack associated to a complex $E \in \mathbf{dg}_k$, *is the graded derived affine k-stack defined by*

$$\mathbb{V}(E) := \mathbb{R}\mathbf{Spec}^{\Delta,gr}\left(Sym^{\Delta}(E)\right) \in \mathbb{G}_m - \mathbf{dSt}_k.$$

Note that by construction the functor of points of $\mathbb{V}(E)$ sends $B \in k{-}sCR$ to $Map_{\mathbf{dg}_k}(E, N(B))$, where $N(B)$ is the normalisation of B as a simplicial k-module. The $\mathbb{G}_m$-action is then the natural action of invertible elements of B on B itself.

It is proven in [Mon21] that the ∞-functor sending E to $\mathbb{V}(E)$ is fully faithful when restricted to bounded above k-dg-modules E, and in particular for perfect k-dg-modules E.

2. Infinitesimal derived foliations

As before, we fix k a base simplicial commutative ring.

We let $\mathbb{G}_a$ be the additive group scheme over k, and consider $G_0 := \Omega_0\mathbb{G}_a$ its derived loop scheme based at the origin. This is a derived affine group scheme over k, and as an affine derived scheme is given by $G_0 \simeq \mathbf{Spec}\, k[\epsilon_{-1}]$, where $k[\epsilon_{-1}]$ is the free commutative simplicial k-algebra over one generator ϵ in homotopical degree 1. The standard $\mathbb{G}_m$-action on $\mathbb{G}_a$ of weight 1 induces a $\mathbb{G}_m$-action on G_0 corresponding to the grading on $k[\epsilon_{-1}]$ for which ϵ_{-1} is of weight -1. We then form the semi-direct product $\mathcal{H}_{\pi} := \mathbb{G}_m \ltimes G_0$, which is a derived affine group scheme over k.

Before going further, let us mention that representations of the group $\mathcal{H}_{\pi}$ are precisely the graded mixed complexes for which the mixed structure is of cohomological degree 1 (as opposed of the usual conventions taken for instance in [PTVV13, Toë20, TV] where the cohomological degree is -1). More precisely, we have an equivalence of symmetric monoidal ∞-categories

$$\mathsf{QCoh}(B\mathcal{H}_{\pi}) \simeq +\epsilon - \mathbf{dg}_k^{gr},$$

where $+\epsilon-\mathbf{dg}_k^{gr}$ is the symmetric monoidal ∞-category of graded mixed complexes with mixed structure of cohomological degree 1. The ∞-category of usual graded mixed complexes with mixed structure of degree -1 is denoted by $\epsilon-\mathbf{dg}_k^{gr}$, and is itself given by representations of the group stack $\mathcal{H}=\mathbb{G}_m\ltimes BKer$ of [MRT22]. It is well known that there exists a symmetric monoidal equivalence of ∞-categories

$$\mathsf{RS}:\epsilon-\mathbf{dg}_k^{gr}\simeq +\epsilon-\mathbf{dg}_k^{gr}$$

called the *red-shift self equivalence*. The ∞-functor RS sends a graded complex $\oplus_n E^{(n)}$ to $\oplus_n E^{(n)}[-2n]$, and thus transforms a mixed structure $E^{(n)}\to E^{(n+1)}[-1]$ of cohomological degree -1 to a mixed structure $E^{(n)}[-2n]\to E^{(n+1)}[-2n-1]\simeq E^{(n+1)}[-2n+2][1]$ of cohomological degree 1. To summarize, we have a commutative square of symmetric monoidal equivalences

$$\begin{array}{ccc} \mathsf{QCoh}(B\mathcal{H}) & \xrightarrow{\sim} & \mathsf{QCoh}(B\mathcal{H}_\pi) \\ \sim\downarrow & & \downarrow\sim \\ \epsilon-\mathbf{dg}_k^{gr} & \xrightarrow[\mathsf{RS}]{} & +\epsilon-\mathbf{dg}_k^{gr}. \end{array}$$

It is proven in [MRT22] that $\mathsf{QCoh}(B\mathcal{H})$ is equivalent, as a symmetric monoidal ∞-category, to the ∞-category $\widehat{\mathbf{dg}_k^{fil}}$ of complete filtered complexes. This equivalence is given by the Tate realization $|-|^t$ sending a graded mixed complex $\oplus_n E^{(n)}$ to $\oplus_{i\leq 0}\prod_{p\geq i}E^{(p)}[-2p]$ endowed with the total differential, sum of the mixed structure and the cohomological differential. Similarly, $\mathsf{QCoh}(B\mathcal{H}_\pi)$ is equivalent to $\widehat{\mathbf{dg}_k^{fil}}$ by means of the corresponding Tate realization $|-|^t$, sending $\oplus_n E^{(n)}$ to $\mathrm{colim}_{i\leq 0}(\prod_{p\geq i}E^{(p)})$ with the total differential.

For any derived stack $X\in\mathbf{dSt}_k$, we can form $\mathcal{L}_\pi^{gr}(X/k):=\underline{\mathbf{Map}}(G_0,X)$, the derived Hom stack (relative to k) from G_0 to X. It comes equipped with a natural action of $\mathcal{H}$, where G_0 acts on itself by translations and $\mathbb{G}_m$ acts by the grading on G_0. Together with this $\mathcal{H}_\pi$-action, $\mathcal{L}_\pi^{gr}(X)$ is the cofree $\mathcal{H}_\pi$-equivariant derived stack generated by X endowed with the trivial $\mathbb{G}_m$-action.

Definition 2.1. *Let $X=\mathbf{Spec}\,A$ be a derived affine scheme over k. A (perfect)* infinitesimal derived foliation on X (relative to k) *consists of a $\mathcal{H}_\pi$-equivariant derived stack $\mathcal{F}$ together with a $\mathcal{H}_\pi$-equivariant morphism $p:\mathcal{F}\to X$, such that, as a $\mathbb{G}_m$-equivariant stack over X, $\mathcal{F}$ is a perfect linear derived affine stack (i.e. of the form $\mathbb{V}(E)$ for E a perfect complex on X).*

The infinitesimal derived foliations on X (relative to k) form an ∞-category $\mathcal{F}ol^\pi(X/k)$, which by definition is a full sub-∞-category of $(\mathcal{H}_\pi-\mathbf{dSt}_k)/\mathcal{L}_\pi^{gr}(X/k)$, the $\mathcal{H}_\pi$-equivariant derived stacks over $\mathcal{L}_\pi^{gr}(X/k)$. The construction $X\mapsto\mathcal{F}ol^\pi(X/k)$ can be promoted to an ∞-functor $\mathcal{F}ol^\pi(-/k):\mathbf{dAff}_k^{op}\to\mathbf{Cat}_\infty$, by defining pull-backs as follows: for a morphism $f:X\to Y$ of affine derived k-schemes, we define f^* to be induced by the base change

$$(\mathcal{H}_\pi-\mathbf{dSt}_k)/\mathcal{L}_\pi^{gr}(Y/k)\longrightarrow(\mathcal{H}_\pi-\mathbf{dSt}_k)/\mathcal{L}_\pi^{gr}(X/k),$$

along the induced morphism $\mathcal{L}^{gr}_\pi(f/k) : \mathcal{L}^{gr}_\pi(X/k) \to \mathcal{L}^{gr}_\pi(Y/k)$. As the ∞-functor $\mathcal{F}ol^\pi(-/k)$ clearly is a hyperstack for the fpqc topology, it can be uniquely extended to a colimit preserving ∞-functor

$$\mathcal{F}ol^\pi(-/k) : \mathbf{dSt}_k \to \mathbf{Cat}_\infty^{op}.$$

Let X be a derived affine k-scheme, and $\mathcal{F} \in \mathcal{F}ol^\pi(X/k)$ be an infinitesimal derived foliation on X. By [Mon21], the linear stack $\mathcal{F} \to X$ is of the form $\mathbb{V}(\mathbb{L}_{\mathcal{F}/k}[-1])$, for some uniquely defined perfect complex $\mathbb{L}_{\mathcal{F}/k}$ on X.

Definition 2.2. *With the notations above, the complex $\mathbb{L}_{\mathcal{F}/k}$ is called the* cotangent complex of $\mathcal{F}$ (relative to k)*. It is also simply denoted by $\mathbb{L}_\mathcal{F}$ if k is clear from the context.*

When X is a general derived stack and $\mathcal{F} \in \mathcal{F}ol^\pi(X/k)$, it is also possible to define the cotangent complex $\mathbb{L}_{\mathcal{F}/k}$, as a quasi-coherent complex over X. This construction is not straightforward, as cotangent complexes in the sense above are not stable by pull-backs and thus do not glue naively on stacks. When X is moreover a derived Artin stack, the cotangent complex $\mathbb{L}_{\mathcal{F}/k}$ is a perfect complex over X. This global aspects of cotangent complexes will not be studied in details in this note, and we refer to [TV] for more on the subject. However, the special case where X is a derived Deligne-Mumford stack presents no difficulties, as the cotangent complex of Definition 2.2 are stable by étale pull-backs, and thus will glue globally on X. This applies in particular to the case where X is any derived scheme.

Let X be a derived affine scheme of finite presentation over k. The ∞-category $\mathcal{F}ol^\pi(X/k)$ admits a final object, namely $\mathcal{F} = \mathcal{L}^{gr}_\pi(X/k)$. It is denoted by $*_{X/k}$, and its cotangent complex is naturally equivalent to $\mathbb{L}_{X/k}$, the cotangent complex of X. Indeed, it is easy to see, simply by contemplating the functors of points and using the very definition of the cotangent complex, that $\underline{\mathbf{Map}}(G_0, X) \simeq \mathbb{V}(\mathbb{L}_{X/k}[1])$. Similarly, the initial object in $\mathcal{F}ol^\pi(X/k)$ exists, is denoted by $0_{X/k}$, and its cotangent complex is 0. It corresponds to the constant-map morphism $\mathcal{F} = X \longrightarrow \mathcal{L}^{gr}_\pi(X/k)$.

3. Infinitesimal cohomology

Let X be a derived Artin stack of finite presentation over k, and $\mathcal{F} \in \mathcal{F}ol^\pi(X/k)$ be an infinitesimal derived foliation on X. We define the ∞-category of quasi-coherent complexes along $\mathcal{F}$ as follows. We first consider $\mathsf{QCoh}([\mathcal{F}/\mathcal{H}_\pi])$, the ∞-category of quasi-coherent complexes on the quotient stack $[\mathcal{F}/\mathcal{H}_\pi]$ (which is also, by definition, the ∞-category of $\mathcal{H}_\pi$-equivariant quasi-coherent complexes on $\mathcal{F}$). The ∞-category $\mathsf{QCoh}^\mathcal{F}(X)$ is defined as the full sub-∞-category of $\mathsf{QCoh}([\mathcal{F}/\mathcal{H}_\pi])$ whose objects E are graded free: the pull-back of E on $[\mathcal{F}/\mathbb{G}_m]$ is of the form $p^*(E_0)$ where $p : [\mathcal{F}/\mathbb{G}_m] \to X$ is the natural projection.

Definition 3.1. *With the notations above, the* ∞-category of quasi-coherent complexes along $\mathcal{F}$ *is $\mathsf{QCoh}^\mathcal{F}(X)$. For $E \in \mathsf{QCoh}^\mathcal{F}(X)$,* the infinitesimal cohomology of X along $\mathcal{F}$ with coefficients in E *is defined to be*

$$\widehat{C}^*_{\mathrm{inf}}(\mathcal{F}, E) := q_*(E) \in \mathbf{dg}_k$$

where $q : [\mathcal{F}/\mathcal{H}_\pi] \longrightarrow \mathbf{Spec}\, k$ is the structural map.

The definition above will be justified by our next result, stating that when $\mathcal{F} = *_{X/k}$ and $E = \mathcal{O}_X$, $\widehat{C}^*_{\inf}(\mathcal{F}, E)$ coincides with (completed derived) infinitesimal cohomology. For this, recall that if $X \to \mathbf{Spec}\, k$ is a smooth affine scheme, we have the infinitesimal stack $X_{\inf}$ (relative to k), defined by the functor sending $A \in k - \mathsf{sCR}$ to $X(A_{red})$ (where $A_{red} = \pi_0(A)_{red}$ by definition). In characteristic zero, the functor $X_{\inf}$ is often denoted by X_{DR} ([Sim96]). We prefer to avoid using the notation X_{DR} in non-zero characteristic, and use $X_{\inf}$ instead, as the cohomology of $X_{\inf}$ computes infinitesimal cohomology and not de Rham cohomology. As any arbitrary derived stack, $X_{\inf}$ possesses a cohomology with coefficients in the structure sheaf $\mathcal{O}$, denoted by $C^*(X_{\inf}, \mathcal{O})$.

Theorem 3.2. *Let $X \to \mathbf{Spec}\, k$ be a smooth affine scheme (and assume k is discrete for simplicity). There exists an equivalence in $\mathbf{dg}_k$*

$$\phi_X : C^*(X_{\inf}, \mathcal{O}) \simeq \widehat{C}^*_{\inf}(*_{X/k}, \mathcal{O}_X).$$

Proof. We first define the morphism ϕ_X. For this we use the canonical morphism $p : X \to X_{\inf}$, induced by $X(A) \to X(A_{red})$ coming from the canonical projection $A \to A_{red}$. It induces a natural morphism of $\mathcal{H}_\pi$-equivariant derived stacks $\mathcal{L}^{gr}_\pi(X) \to \mathcal{L}^{gr}_\pi(X_{\inf})$. By the very definition of $X_{\inf}$, and because $G_0 = \Omega_0 \mathbb{G}_a$ is a derived affine scheme with trivial reduced sub-scheme, we see that the canonical morphism $X_{\inf} \to \mathcal{L}^{gr}_\pi(X_{\inf})$ is in fact an ($\mathcal{H}_\pi$-equivariant) equivalence of derived stacks. We thus obtain a natural $\mathcal{H}_\pi$-equivariant morphism $\mathcal{L}^{gr}_\pi(X) \to X_{\inf}$, or equivalently a morphism of derived stacks

$$q_X : [\mathcal{L}^{gr}_\pi(X)/\mathcal{H}_\pi] \to X_{\inf},$$

which is clearly functorial in X. It induces, by pull-back, a morphism between the cohomology complexes with coefficients in $\mathcal{O}$

$$\phi_X := q_X^* : C^*(X_{\inf}, \mathcal{O}) \to \widehat{C}_{\inf}(*_{X/k}, \mathcal{O})$$

which is again functorial in X.

To prove that ϕ_X is an equivalence, we proceed by descent along $p : X \to X_{\inf}$. As X is smooth, the morphism p is an epimorphism of derived stacks (infinitesimal criterion for smoothness). The simplicial nerve X_* of p, that comes equipped with its augmentation $X_* \to X_{\inf}$, is an hypercovering. In particular, the canonical morphism

$$C^*(X_{\inf}, \mathcal{O}) \longrightarrow \lim_{[n]\in\Delta} \mathcal{O}(X_n)$$

is a quasi-isomorphism. Note that X_n is naturally isomorphic to the formal completion of X^n along the diagonal embedding $X \hookrightarrow X^n$, and thus $C^*(X_n, \mathcal{O})$ is the discrete algebra $\mathcal{O}(X_n)$ of functions on the formal scheme X_n.

On the other hand, the fact that X is smooth also implies that $X \to [\mathcal{L}^{gr}_\pi(X)/\mathcal{H}_\pi]$ is an epimorphism, because $\mathcal{L}^{gr}_\pi(X) \simeq \mathbb{V}(\Omega^1_{X/k}[1]) = B\mathbb{T}_X$ is the classifying stack of the tangent bundle of X. We thus consider the commutative diagram of derived stacks with $\mathcal{H}_\pi$-actions

$$\begin{array}{ccc} X & \xrightarrow{id} & X \\ \downarrow & & \downarrow \\ \mathcal{L}^{gr}_\pi(X) & \xrightarrow[q_X]{} & X_{\inf}. \end{array}$$

This diagram induces a morphism of simplicial objects on the nerves of the vertical morphisms. The nerve of the morphism $X \to \mathcal{L}^{gr}_\pi(X)$ can be described as the simplicial object $[n] \mapsto \mathcal{L}^{gr}_\pi(X/X^n)$, where X sits inside X^n via the diagonal embedding, and $\mathcal{L}^{gr}_\pi(X/X^n)$ is the relative mapping stack from G_0 to X relative to X^n. At the simplicial level n, the morphism induced on the nerve is thus an $\mathcal{H}_\pi$-equivariant morphism of stacks over S

$$q_{X,n} : \mathcal{L}^{gr}_\pi(X/X^n) \longrightarrow X_n,$$

(where X_n is the formal completion of X^n along its diagonal).

Lemma 3.3. *The morphism $q_{X,n}$ induces by pull-back a quasi-isomorphism of complexes*

$$C^*(X_n, \mathcal{O}) \simeq C^*([\mathcal{L}^{gr}_\pi(X/X^n)/\mathcal{H}_\pi], \mathcal{O}).$$

Proof of Lemma 3.3. By descent we can easily restrict to the case where $X = \mathbf{Spec}\, A$ is a smooth affine scheme over $S = \mathbf{Spec}\, k$. In this case, $C^*(X_n, \mathcal{O})$ identifies canonically with $\hat{A}^{\otimes_k n}$, the formal completion of $A^{\otimes_k n}$ along the augmentation $A^{\otimes_k n} \to A$ induced by the multiplication map. Similarly, $\mathcal{L}^{gr}_\pi(X/X^n)$ is canonically identified with the smooth affine scheme $\mathbf{Spec}\, Sym_A((\Omega^1_{A/k})^{n-1})$. Therefore, the morphism $\phi_{X,n}$ is a morphism of discrete commutative algebras

$$\phi_{A,n} : \hat{A}^{\otimes_k n} \longrightarrow |Sym_A((\Omega^1_{A/k})^{n-1})|^t,$$

where the right hand side is the Tate realization of the graded mixed object $Sym_A((\Omega^1_{A/k})^{n-1})$ induced by the $\mathcal{H}_\pi$-action on $\mathcal{L}^{gr}_\pi(X/X^n)$. The morphism $\phi_{A,n}$ is compatible with the canonical augmentations to A, and thus is a morphism of complete filtered commutative algebras. Therefore, in order to prove it is an isomorphism it is enough to show that it induces an isomorphism

$$gr\phi_{A,n} : Gr^*(\hat{A}^{\otimes_k n}) \longrightarrow Sym_A((\Omega^1_{A/k})^{n-1})$$

on the associated graded objects. Since A is a smooth k-algebra, the left hand side is canonically isomorphic to $Sym_A(I_n/I_n^2)$, where I_n is the augmentation ideal. Moreover, we have the usual natural isomorphism of A-modules $I_n/I_n^2 \simeq (\Omega^1_{A/k})^{n-1}$, and thus the morphism $gr\phi_{A,n}$ can be identified with a graded endomorphism of the graded algebra $Sym_A((\Omega^1_{A/k})^{n-1})$. As it is compatible with augmentations to A it is thus determined by an A-linear endomorphism of $(\Omega^1_{A/k})^{n-1}$. By functoriality in A and in the simplicial direction n, we see moreover that it is determined by a functorial endormorphism the A-module $\Omega^1_{A/k}$, and thus must be the multiplication by an element $\lambda \in k$. To conclude the lemma, we have to show that $\lambda = 1$.

By functoriality, it is easy to reduce to the case $A = k[T]$, a polynomial ring, and by base change we can even assume that $k = \mathbb{Z}$. In order to check that $\lambda = 1$, we can base change from $\mathbb{Z}$ to $\mathbb{Q}$, and thus even reduce to the case $A = \mathbb{Q}[T]$ and $k = \mathbb{Q}$. As we are now in characteristic zero, $C^*([\mathcal{L}^{gr}_\pi(X/X^n)/\mathcal{H}_\pi]$ identifies canonically with the completed de Rham complex $\widehat{C}^*_{DR}(X/X^n)$. Indeed, the red shift equivalence sends the graded mixed cdga $\mathcal{O}(\mathcal{L}^{gr}_\pi(X/X^n))$ to $\mathcal{O}(\mathcal{L}^{gr}(X/X^n))$. In such terms, the morphism $\phi_{X,n}$ is now the natural isomorphism relating derived de Rham cohomology of a closed embedding with functions on the formal completion (see [CPT+17]). □

Lemma 3.3 concludes the proof of Theorem 3.2, as we have a commutative diagram

$$\begin{array}{ccc} C^*(X_{\mathrm{inf}}, \mathcal{O}) & \xrightarrow{\phi_X} & \widehat{C}_{\mathrm{inf}}(*_X, \mathcal{O}_X) \\ \sim\downarrow & & \downarrow\sim \\ \lim_{[n]} \mathcal{O}(X_n) & \xrightarrow[\sim]{} & \lim_{[n]} \widehat{C}_{\mathrm{inf}}(*_{X/X^n}, \mathcal{O}_X). \end{array}$$

□

As a direct corollary of Theorem 3.2, we obtain that, for a general derived Artin stack X, $\widehat{C}_{\mathrm{inf}}(*_{X/k}, \mathcal{O})$ computes the *completed derived infinitesimal cohomology of* X (relative to k). Indeed, it is easy to see the ∞-functor $X \mapsto \mathcal{O}(\mathcal{L}^{gr}_\pi(X/k))$, from affine derived schemes to graded mixed complexes, is equivalent to its extension by sifted colimits from smooth X's. For $A \in \mathsf{sCR}$, where each A_n is a smooth algebra over k, we thus have

$$\widehat{C}_{\mathrm{inf}}(*_X, \mathcal{O}_X) \simeq \mathrm{colim}_{[n]} C^*((Spec\, A_n)_{\mathrm{inf}}, \mathcal{O}),$$

where the colimit is computed in the ∞-category of *complete* filtered complexes. The right hand side of this equivalence is, by definition, the completed derived infinitesimal cohomology of X, and is denoted by $\mathbb{R}\hat{C}^*(X_{\mathrm{inf}}, \mathcal{O})$. This notion is extended to any derived Artin stack locally of finite presentation over k by the usual gluing formula

$$\mathbb{R}\hat{C}^*(X_{\mathrm{inf}}, \mathcal{O}) := \lim_{U \to X} \mathbb{R}\hat{C}^*(U_{\mathrm{inf}}, \mathcal{O})$$

where the limit runs over affine U locally of finite presentation.

Corollary 3.4. *For a general derived Artin stack X of finite presentation over k, $\widehat{C}_{\mathrm{inf}}(*_{X/k}, \mathcal{O})$ recovers the completed derived infinitesimal cohomology of X relative to k*

$$\widehat{C}_{\mathrm{inf}}(*_{X/k}, \mathcal{O}) \simeq \mathbb{R}\hat{C}^*(X_{\mathrm{inf}}, \mathcal{O}).$$

4. Integrability by formal groupoids

We believe that infinitesimal derived foliations are formally integrable in a very strong sense which we do not investigate in this paper in full generality. Rather, we prove here a special case, showing that the ∞-category of smooth infinitesimal derived foliations over a k-scheme X is in fact *equivalent* to the category of *formally smooth formal groupoids* over X.

Let X be a smooth k-scheme and $\mathcal{F} \in \mathcal{F}ol^{\pi}(X/k)$. We say that $\mathcal{F}$ is *smooth* if $\mathbb{L}_{\mathcal{F}}$ is a vector bundle over X. In this case, we consider $X \times_{\mathcal{F}} X$, endowed with its natural $\mathcal{H}_{\pi}$-action and with its canonical projection to X. Note that, as a graded derived stack, $X \times_{\mathcal{F}} X$ is of the form $\mathbb{V}(\mathbb{L}_F)$, and thus is a the total space of the tangent bundle $\mathbb{T}_{\mathcal{F}}$ of $\mathcal{F}$. Also, the natural projection $X \times_{\mathcal{F}} X \to X$ is $\mathcal{H}_{\pi}$-equivariant. As a result, $X \times_{\mathcal{F}} X$ is a new object in $\mathcal{F}ol^{\pi}(X)$, that will be simply denoted by $\Omega_X \mathcal{F}$, and called the *loop space* of $\mathcal{F}$. Being the fiber product of the canonical morphism $0_X \to \mathcal{F}$, the loop space $\Omega_X \mathcal{F}$ comes equipped with a natural structure of a groupoid object in $\mathcal{F}ol^{\pi}(X)$, acting on the object 0_X.

Propositon 4.1. *The full sub-∞-category of $\mathcal{F}ol^{\pi}(X/k)$ formed by objects $\mathcal{F}$ whose cotangent complex is of the form $V[1]$ for V a vector bundle on X, is naturally equivalent to the category of formal schemes Z with an identification $Z_{red} \simeq X$, which are locally equivalent on X to $X \times \widehat{\mathbb{A}}^d$ where $d = rank(V)$.*

Proof. To each $\mathcal{F}$ as in the proposition, we associate the sheaf of infinitesimal cohomology $\widehat{C}_{\inf}(\mathcal{F})$ on X. By assumptions on the cotangent complex of $\mathcal{F}$, $\widehat{C}_{\inf}(\mathcal{F})$ is a sheaf of discrete complete filtered commutative algebras, augmented to $\mathcal{O}_X$ and its associated graded object is isomorphic to $\mathcal{O}_X[[t_1, \ldots, t_d]]$. Therefore, $Z_{\mathcal{F}} := \mathsf{Spf}(\widehat{C}_{\inf}(\mathcal{F}))$ is a formal scheme with a canonical identification $Z_{red} \simeq X$ which is locally equivalent to $X \times \widehat{\mathbb{A}}^d$.

The fact that the construction $\mathcal{F} \mapsto Z_{\mathcal{F}}$ induces the equivalence of the proposition simply follows from the fact that the Tate realization induces an equivalence between graded mixed complexes and complete filtered complexes (see [TV20, Prop. 1.3.1]). ◻

By Proposition 1.7, for any smooth infinitesimal derived foliation $\mathcal{F}$, its loop space $\Omega_X \mathcal{F}$ can be realized as a groupoid object in formal schemes acting on X

$$Z_{\Omega_X \mathcal{F}} \longrightarrow X \times X.$$

The construction $\mathcal{F} \mapsto Z_{\Omega_X \mathcal{F}}$ provides an functor from the category of smooth infinitesimal derived foliations on X, and the category of formally smooth formal groupoids over X. We can produce a functor in the other direction, as follows. Starting from a formally smooth formal groupoid $G \to X \times X$, we can consider its nerve $G_* \to X$, which is a simplicial object in formal schemes. Taking functions on G_* provides a cosimplicial complete filtered commutative algebra $\mathcal{O}(G_*)$ whose associated graded is the cosimplicial algebra of functions on the simplicial scheme BV, where $V = e^*(\Omega^1_{G/X})$ is the vector bundle of invariant relative forms on G. Using the equivalence between graded mixed complexes and complete filtered complexes of [TV20, Prop. 1.3.1], we find that $\mathcal{O}(G_*)$ can be realized as a cosimplicial object in the category of graded commutative algebras endowed with a compatible action on G_0. Passing then to *Spec*, we get a simplicial diagram of affine schemes endowed with a $\mathcal{H}_{\pi}$-action, whose underlying simplicial diagram (obtained by forgetting the action) is the usual simplicial scheme BV. Taking geometric realization we get a $\mathcal{H}_{\pi}$-action on the derived stack $\mathbb{V}(V[1])$, which defines a smooth infinitesimal derived foliations $\mathcal{F}$ on X.

We can then subsume the previous discussion in the following corollary.

Corollary 4.2. *The category of smooth infinitesimal derived foliations on X is equivalent to the category of formally smooth formal groupoids on X.*

We note that the above corollary marks a major difference between derived foliations of [Toë20, TV] and infinitesimal derived foliations. Indeed, any finite dimensional Lie algebra L over a field k, defines a smooth derived foliation over k, by considering $Sym(L^\vee[1])$ endowed with the mixed structure coming from the Chevalley-Eilenberg differential. However, it is not true that any Lie algebra can be integrated to a smooth formal scheme in general. On the contrary, the above corollary, when applied to $X = \mathbf{Spec}\, k$ with k a field, states that smooth infinitesimal derived foliations over k form a category equivalent to smooth formal groups over k. Any such smooth infinitesimal derived foliation is of the form $\mathbb{V}(V[-1])$ for V a finite dimensional k-vector space V. Therefore, the data of an infinitesimal derived foliation structure on $\mathbb{V}(V[-1])$ seems to be related to that of a *partition Lie algebra structure* in the sense of [BCN21]. The precise comparison is out of the scope of this note, but some recent results of J. Fu ([Fu]) make us strongly believe this is possible.

5. Towards a comparison between infinitesimal derived foliations and derived foliations

5.1. **Red shift.** We have already mentioned the red-shift equivalence $\mathsf{RS} : \epsilon-\mathbf{dg}_k^{gr} \simeq +\epsilon-\mathbf{dg}_k^{gr}$. We will now lift this to an endofunctor on the level of derived affine stacks endowed with actions of $\mathcal{H}$ and $\mathcal{H}_\pi$. For this, we use our cosimplicial-simplicial models for graded derived affine stacks. We treat the absolute case of derived affine stacks over $\mathbb{Z}$, the realtive situation follows from considering graded affine stacks over a given affine derived scheme $X = \mathbf{Spec}\, k$ (trivially graded). For technical reasons we will actually invert 2, i.e. most of the time work over $\mathbb{Z}[1/2]$.

We denote by $\mathbb{Z}[u]$ the cosimplicial commutative ring obtained by denormalization from the free commutative algebra over one generator in homotopical degree 0 (endowed with the 0 differential and considered as a non-negatively graded commutative dg-algebra). In [MRT22], it is proven that $\mathbf{Spec}^\Delta(\mathbb{Z}[u]) \simeq BS^1_{gr}$, the classifying stack of the *graded circle*, also called the *graded infinite projective space*. We consider $\mathbb{Z}[u]$ as an object in $csCR^{gr}$ constant in the simplicial direction where u is of weight 1. Similarly, we denote by $\mathbb{Z}[v]$ the free simplicial commutative ring generated by one variable in homotopical degree 2. We consider $\mathbb{Z}[v]$ as an object in $csCR^{gr}$ constant in the cosimplicial direction where v is of weight -1. Finally, we set

$$\mathbb{Z} < u, v > := \mathbb{Z}[v] \times_{\mathbb{Z}} \mathbb{Z}[u] \in csCR^{gr},$$

which is a cosimplicial-simplicial commutative $\mathbb{Z}$-graded ring.

For two commutative graded rings A and B, we define their *convolution product* $A \odot B$, which is a new commutative graded ring with

$$(A \odot B)^{(n)} = A^{(n)} \otimes B^{(n)}$$

with the natural componentwise multiplication. This can be extended to a convolution product for two objects in $csCR^{gr}$ by applying this construction levelwise, both in the simplicial and cosimplicial directions.

Definition 5.1. *The red shift endofunctor* $\mathsf{RS} : csCR^{gr} \to csCR^{gr}$ *is defined by*

$$\mathsf{RS}(A) := A \odot (\mathbb{Z} < u, v >).$$

It is easy to see that $Tot^{\pi}(\mathsf{RS}(A))$ is a graded complex which is naturally quasi-isomorphic to $\mathsf{RS}(Tot^{\pi}(A))$. Therefore, RS preserves weak equivalence in $csCR^{gr}$, and thus induces a well defined ∞-functor $\mathsf{RS} : \mathbf{csCR}^{gr} \to \mathbf{csCR}^{gr}$ covering the red-shift self equivalence by the functor Tot^{π}

$$\begin{array}{ccc} \mathbf{csCR}^{gr} & \xrightarrow{\mathsf{RS}} & \mathbf{csCR}^{gr} \\ {\scriptstyle Tot^{\pi}}\downarrow & & \downarrow{\scriptstyle Tot^{\pi}} \\ \mathbf{dg}_k^{gr} & \xrightarrow[\mathsf{RS}]{} & \mathbf{dg}_k^{gr}. \end{array}$$

We note however that RS does not induce a self equivalence of $\mathbf{csCR}^{gr}$, and for instance does not preserve free objects, as opposed for instance to the corresponding situation with E_{∞}-algebras. In fact the behaviour of RS with respect to free objects is quite subtle, and the authors do not claim to fully understand the situation.

Note also that $\mathsf{RS}(A)$ is only well behaved when A is only $\mathbb{Z}_{\geq}0$-graded (or $\mathbb{Z}_{\leq 0}$-graded). For instance, $\mathsf{RS}(\mathbb{Z}[t, t^{-1}])$ is equivalent to $\mathbb{Z} < v, u >$, for which u is the image of t and v the image of t^{-1}. However, by definition $uv = 0$, and thus the multiplicative structure on $\mathsf{RS}(\mathbb{Z}[t, t^{-1}])$ is somehow degenerate. Trying to use $\mathbb{Z}[u, u^{-1}]$ is not a solution here, as the presence of divided powers in the homotopy of commutative simplicial rings would force us to work over $\mathbb{Q}$.

Using Proposition 4.1, we can then consider RS as an endofunctor of the ∞-category of graded derived affine stacks $\mathsf{RS} : \mathbf{dChAff}^{gr} \to \mathbf{dChAff}^{gr}$. The convolution construction $- \odot B$ for $B \in csCR^{gr}$ is lax monoidal in the sense that there is a natural morphism $(A \odot B) \otimes (A' \odot B) \to (A \otimes A') \odot B$. This lax monoidal structure is monoidal if and only if the multiplication maps $B^{(p)} \otimes B^{(q)} \to B^{(p+q)}$ are completed quasi-isomorphisms of cosimplicial-simplicial modules (i.e. becomes quasi-isomorphismes ater applying Tot^{π}). In the case $B = \mathbb{Z} < u, v >$ this is never the case, as $uv = 0$. However, if A is $\mathbb{Z}_{\geq 0}$-graded, we have $A \odot \mathbb{Z} < u, v > \simeq A \odot \mathbb{Z}[u]$, and clearly $\mathbb{Z}[u]^{(p)} \otimes \mathbb{Z}[v]^{(q)} \to \mathbb{Z}[u]^{(p+q)}$ is a completed quasi-isomorphism as its image by Tot^{π} is equivalent to the canonical isomorphism $\mathbb{Z}[-2p] \otimes \mathbb{Z}[-2q] \simeq \mathbb{Z}[-2p - 2q]$.

Lemma 5.2. *Let X and Y be two graded derived affine stacks which are both $\mathbb{Z}_{\geq 0}$-graded. Then, the natural morphism* $\mathsf{RS}(X \times Y) \to \mathsf{RS}(X) \times \mathsf{RS}(Y)$ *is an equivalence.*

Another special case where RS preserves products is given in the following Lemma. It follows from the easy observation that the image by Tot^{π} of the multiplication $\mathbb{Z}[v]^{(-1)} \otimes \mathbb{Z}[v]^{(-1)} \to \mathbb{Z}[v]^{(-2)}$ is equivalent to the multiplication by 2 (because $\mathbb{Z}[v]$ has divided powers)

$$\times 2 : \mathbb{Z}[2] \otimes \mathbb{Z}[2] \simeq \mathbb{Z}[4] \to \mathbb{Z}[4].$$

Lemma 5.3. *Let X and Y be two graded derived affine stacks whose weights are concentrated in $[-1,0]$. Then, the natural morphism $\mathsf{RS}(X\times Y)\to \mathsf{RS}(X)\times \mathsf{RS}(Y)$ is an equivalence when restricted to $\mathbf{Spec}\,\mathbb{Z}[1/2]$.*

Remind from [MRT22, Toë20] the group object $BKer = S^1_{gr} \in \mathbf{dChAff}^{gr}$. As a graded derived affine stack its weights are concentrated in $[-1,0]$, so Lemma 5.3 applies, and we deduce that that $\mathsf{RS}(BKer)$ is another group object in $\mathbf{dChAff}^{gr}$ (at least over $\mathbf{Spec}\,\mathbb{Z}[1/2]$). As a graded affine stack, it is clearly of the form $\mathbf{Spec}(\mathbb{Z}[1/2][\epsilon_{-1}])\simeq G_0$. As the group structure on G_0 is essentially unique (see [MRT22]), we moreover deduce that $\mathsf{RS}(BKer)\simeq G_0$ as group objects in $\mathbf{dChAff}^{gr}$.

We have thus seen that the image of the group $BKer$ by RS is the group G_0. We believe that the functor RS can also be promoted to a functor on the level of equivariant objects. This is not a formal statement, as RS does not commute with finite products in general. The precise construction is outside of scope of this note and we leave this as an open question for the future.

Question 5.4. *Can the ∞-functor RS be extended to*

$$\mathsf{RS} : BKer - \mathbf{dChAff}^{gr+} \longrightarrow G_0 - \mathbf{dChAff}^{gr+},$$

in such a way that the cofree object X^{BKer} is sent to $\mathsf{RS}(X)^{G_0}$ (at least over $\mathbf{Spec}\,\mathbb{Z}[1/2]$) ?

Remark 5.5. The above question is a coherence problem. It is indeed possible to prove easily that for any $X\in \mathbf{dChAff}^{gr+}$, there exists a canonical morphism $\mathsf{RS}(X^{BKer})\to \mathsf{RS}(X)^{G_0}$ of graded derived affine stacks. Therefore, a $BKer$-action on X, induces a morphism $X\to X^{BKer}$, and therefore, via RS, a morphism $\mathsf{RS}(X)\to \mathsf{RS}(X^{Bker})\to \mathsf{RS}(X)^{G_0}$. The problem here is to control the higher coherences in order to promote the previous morphism to an action of G_0 on $\mathsf{RS}(X)$.

5.2. **Infinitesimal structures on derived foliations.** We have now everything we need in order to compare infinitesimal derived foliations and our original definition of derived foliations, assuming that we have a positive answer to Question 5.4.

Let $\mathcal{F}\to X$ be a derived foliation on a derived k-scheme X in the sense of [Toë20]. Remind that it consists of a linear derived stack $\mathbb{V}(\mathbb{L}_{\mathcal{F}}[1])$, for $\mathbb{L}_{\mathcal{F}}$ a perfect complex on X, together with an action of the graded group $BKer$. By applying the red-shift functor, we get a graded derived affine stack $\mathsf{RS}(\mathbb{V}(\mathbb{L}_{\mathcal{F}}[1]))$ together with an action of G_0. The complex of functions on $\mathsf{RS}(\mathbb{V}(\mathbb{L}_{\mathcal{F}}[1]))$ is the graded complex $\mathsf{RS}(Sym_{\mathcal{O}_X}(\mathbb{L}_{\mathcal{F}}[1]))$ and thus receive a canonical morphism $\mathbb{L}_{\mathcal{F}}[-1]\to \mathsf{RS}(Sym_{\mathcal{O}_X}(\mathbb{L}_{\mathcal{F}}[1]))$. By the universal property of the Sym construction, we thus get a canonical morphism of graded stacks $\mathsf{RS}(\mathbb{V}(\mathbb{L}_{\mathcal{F}}[1]))\to \mathbb{V}(\mathbb{L}_{\mathcal{F}}[-1]))$.

Definition 5.6. *Let $\mathcal{F}$ be a derived foliation on X as above. An* infinitesimal structure on $\mathcal{F}$ *consists of an extension of the G_0-action along the morphism $\mathsf{RS}(\mathbb{V}(\mathbb{L}_{\mathcal{F}}[1]))\to \mathbb{V}(\mathbb{L}_{\mathcal{F}}[-1]))$.*

By definition, an infinitesimal structure on $\mathcal{F}$ determines an infinitesimal derived foliation $\mathcal{F}^{\pi}$, given by the G_0-action on $\mathbb{V}(\mathbb{L}_{\mathcal{F}}[-1]))$. We can thus define the infinitesimal cohomology of $\mathcal{F}$ by taking infinitesimal cohomology of $\mathcal{F}^{\pi}$ in the sense of our Definition 3.1. Therefore, the first consequence of the data of an infinitesimal structure is the possibility to define infinitesimal cohomology. Note that with the definition above, there is a canonical morphism

$$\widehat{C}_{\mathrm{inf}}(\mathcal{F}^{\pi},\mathcal{O}) \longrightarrow \widehat{C}_{DR}(\mathcal{F},\mathcal{O})$$

from the infinitesimal cohomology of $\mathcal{F}^{\pi}$ to the derived de Rham cohomology of $\mathcal{F}$ introduced in [Toë20]. This is a generalization of the well known comparison morphism between infinitesimal and crystalline cohomology. This can be enhanced to a pull-back functor, from $\mathsf{QCoh}(\mathcal{F}^{\pi})$ to $\mathsf{QCoh}(\mathcal{F})$. Moreover, the differential operators construction of [TV20] can also be performed for $\mathcal{F}$ and $\mathcal{F}^{\pi}$. This provides two different sheaves of filtered dg-algebras on X, the crystalline differential operators $\mathcal{D}_{\mathcal{F}}$ along $\mathcal{F}$ and the Grothendieck differential operators $\mathcal{D}^{\infty}_{\mathcal{F}^{\pi}}$ along $\mathcal{F}^{\pi}$. These are two filtered dg-algebras, with a natural morphism $\mathcal{D}_{\mathcal{F}} \longrightarrow \mathcal{D}^{\infty}_{\mathcal{F}^{\pi}}$, whose induced morphism on the associated graded objects is $Sym(\mathbb{T}_{\mathcal{F}}) \longrightarrow Sym^{pd}(\mathbb{T}_{\mathcal{F}})$, from the symmetric algebra (in the sense of cosimplicial-simplicial models of our §1) to its PD-completion.

We also believe that it should be possible to prove that a derived foliation that admits an infinitesimal structure is automatically *formally integrable*. In fact, it seems that when it exists, $\mathcal{F}^{\pi}$ completely recovers $\mathcal{F}$ as follows. The graded derived affine stack $\mathsf{RS}(\mathbb{V}(\mathbb{L}_{\mathcal{F}}[1]))$ can be obtained as the PD completion of $\mathbb{V}(\mathbb{L}_{\mathcal{F}}[-1]))$ along its zero section, and it is very reasonable to expect that the RS construction induces an equivalence between graded derived affine stacks and graded derived affine stacks with PD structures. In particular, the $G_0 = \mathsf{RS}(BKer)$-action on $\mathsf{RS}(\mathbb{V}(\mathbb{L}_{\mathcal{F}}[1]))$ should comes from a unique $BKer$-action on $\mathbb{V}(\mathbb{L}_{\mathcal{F}}[1])$. What we are describing here is probably the existence of a fogertful functor, from infinitesimal derived foliations to derived foliations, even though we are not able, at the moment, to provide the details of this construction. This forgetful functor seems very closely related to the forgetful functor from partition Lie algebras to Lie algebras, and recent results of J. Fu ([Fu]) indicate that infinitesimal derived foliations could be essentially the same thing as (perfect) partition Lie algebroids. The fact that partition Lie algebras are related to formal moduli problem then explains the fact that infinitesimal derived foliations have nice formal integrability properties.

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